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Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and
Non-Commutative Geometry in Condensed Matter Physics and Field Theory
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Supersymmetric Extension of Quantum Hall Effects

Quantum Hall Effects on Fuzzy Spheres

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Quantum Hall Effects on Fuzzy Spheres

ICTP 2005 (Mar) 1-4

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Motivation

Zhang and Hu's 4D QHE

Many interesting properties

- Noncommutative geometry
- Brane-like excitations
- Fractional statistics based on the second Hopf map
- Edge states etc

How should we proceed from here ?

(Does it suggest something to high energy physics ?)

In this talk, I present our trials to generalize it into
even higher dimensional systems and supersymmetric systems.

Even Higher Dimensional Quantum Hall Liquids

Mathematical foundation of QHE → Noncommutative geometry

Noncommutative geometry → incorporated into symplectic manifolds, such as Kahler manifolds.

Higher D. QH liquids may be constructed on higher D. Kahler manifolds.

Haldane's 2D QHE $\quad CP^1 = S_F^2 \quad SU(2)/U(1) = SO(3)/U(1)$

$\downarrow$

$CP^2 \quad SU(3)/U(2)$

$\downarrow$

Zhang and Hu's 4D QHE $\quad CP^3 = S_F^4 \quad SU(4)/U(3) = SO(5)/U(2)$

Karabali and Nair's SU(N) course $\swarrow$ $\searrow$ Hasebe and Kimura's SO(N) course

$CP^4 \quad S_F^6 \quad SU(5)/U(4) \neq SO(6)/U(3)$

$\downarrow \quad \downarrow$

$CP^5 \quad S_F^8 \quad SU(6)/U(5) \neq SO(8)/U(4)$

$\downarrow \quad \downarrow$

References

- ``Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres'',
  Phys.Lett. B602 (2004) 255, hep-th/0310274.
  by K. Hasebe and Y. Kimura
- ``Fuzzy Supersphere and Supermonopole'',
  Nucl.Phys. B709 (2004) 94, hep-th/0409230
  by K. Hasebe and Y. Kimura
- ``Supersymmetric Quantum Hall Effects on Fuzzy Superspheres'',
  hep-th/0411137
  by K. Hasebe
- ``Supersymmetric extension of Noncommutative Spaces, Berry Phase and Quantum Hall Effects'',
  to appear,
  by K. Hasebe

Plan

0. Introduction

1-1 Mathematical preparation

1-2 Higher dimensional quantum Hall effects on fuzzy spheres

2-1 Mathematical preparation for super case

2-2 Supersymmetric quantum Hall effects on fuzzy superspheres

3. Summary and Discussion

Fuzzy Spheres

In general, spheres $S^{2k} = SO(2k+1)/SO(2k)$ are not Kahler manifolds.

We need ``Kahler spheres''. They are what we call fuzzy spheres, which are given by

$$S_F^{2k} = SO(2k+1)/U(k)$$

P-M. Ho and S. Ramgoolam (2002)

$$dim(S_F^{2k}) = k(k+1) \neq dim(S^{2k}) = 2k$$

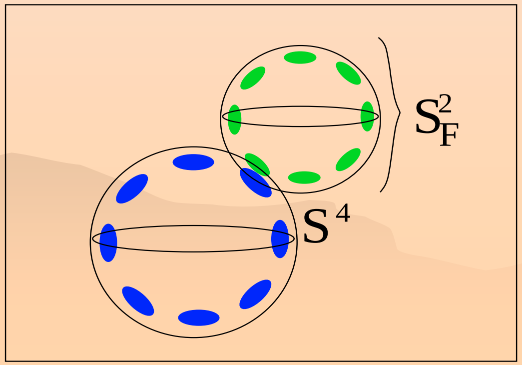
The number of dimensions of the ``Internal space''

$$= dim(S_F^{2k}) - dim(S^{2k}) = (k-1)k = dim(S_F^{2k-2})$$

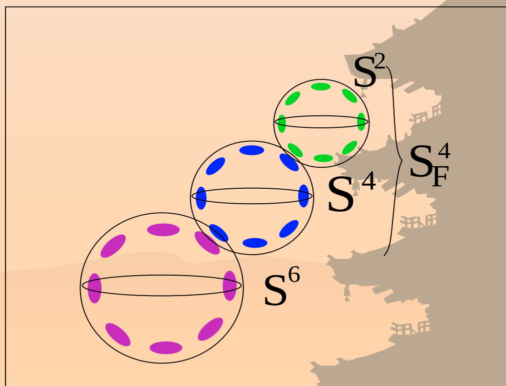
The S_F^{2k} ``contains'' $S_F^{2k-2}$ as its ``internal space''.

Y. Kimura (2002, 2003)

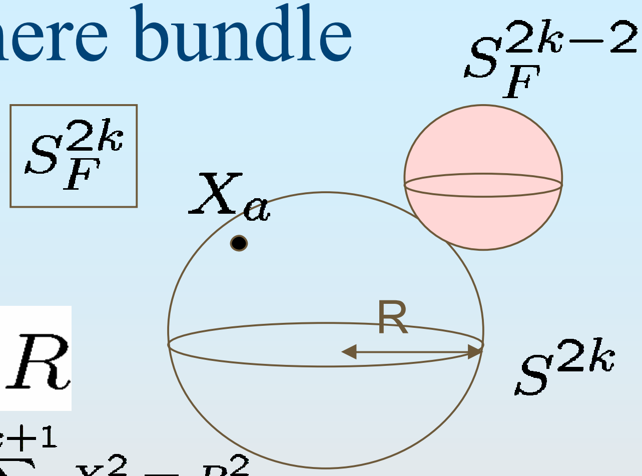
S_F^4



S_F^6



Connection of internal fuzzy sphere bundle



Map $\overset{S_F^{2k-2}}{S_F^{2k} \to S^{2k}}$:

$$\Psi \to \Psi^\dagger \Gamma_a \Psi = X_a/R$$

Γ_a : Gamma matrices in (2k+1) D. $\sum_{a=1}^{2k+1} X_a^2 = R^2$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{2^k} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{R+X_{2k+1}}{2R}} \\ \frac{1}{\sqrt{2R(R+X_{2k+1})}}(X_{2k} - iX_i\gamma_i) \end{pmatrix} \times \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{2^{k-1}} \end{pmatrix}$$

Connection of S_F^{2k-2} bundle

γ_i : Gamma matrices in (2k-1) D.

$$i\Psi^\dagger d\Psi = \psi_\alpha^*(A_a dX^a)_{\alpha\beta}\psi_\beta$$

$$A_\mu = \frac{1}{R(R+X_{2k+1})}\Sigma_{\mu\nu}X_\nu \ , \quad A_{2k+1} = 0$$

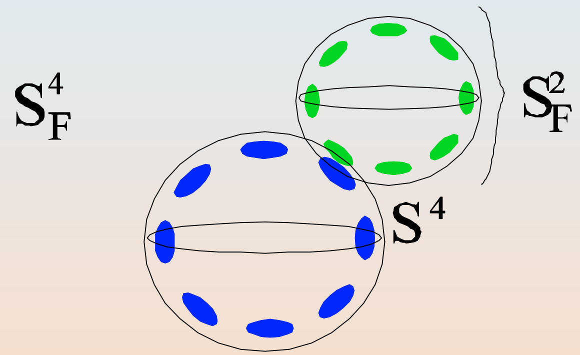
$\Sigma_{\mu\nu}$: SO(2k) generator $\mu = 1, 2, \cdots, 2k$

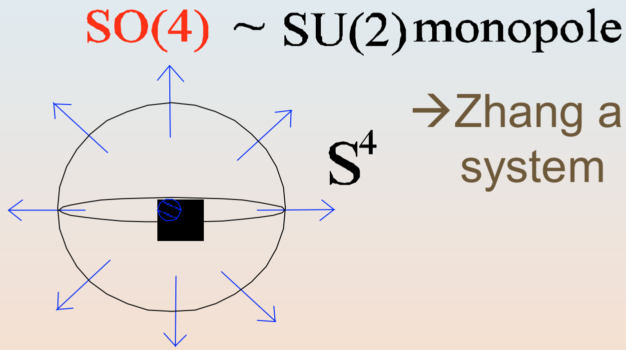
→ Gauge field of SO(2k) colored monopole

Physical Realization of Fuzzy Spheres

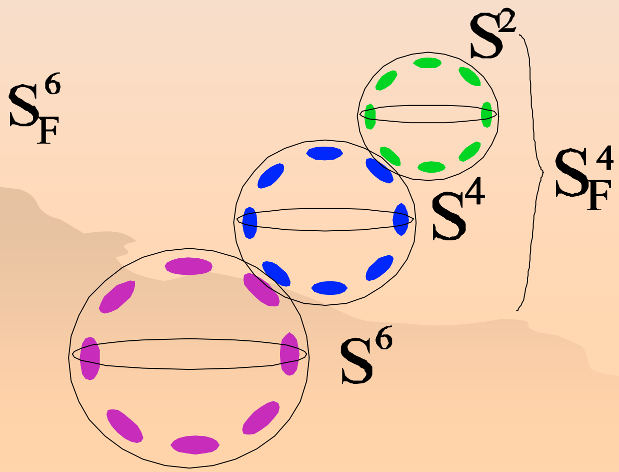
The connection of internal fuzzy sphere bundle S_F^{2k-2}

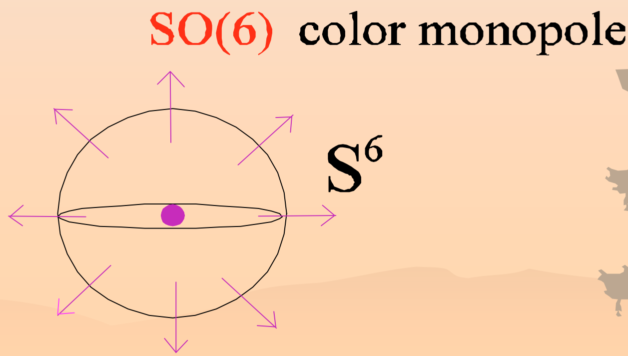
= The gauge field of $SO(2k)$ colored monopole.





→Zhang and Hu's 4D QH system





Higher dimensional QH systems

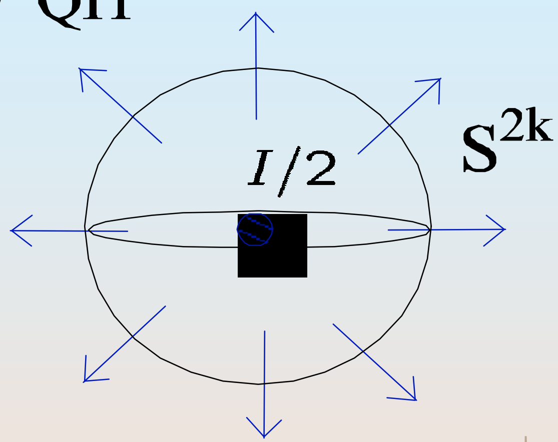
2k-D QH Hamiltonian

$$H = \frac{1}{2MR^2}\sum_{a>b=1}^{2k+1}\Lambda_{ab}^2$$

$$\Lambda_{ab} = -i(x_a D_b - x_b D_a),$$
$$D_a = \partial_a + iA_a$$

$$E_{LLL} = \omega/2, \quad \omega = B/M$$

2k-D QH



SO(2k)colored monopole

$$d(I) = \frac{(I+2k-1)!!}{(2k-1)!!(I-1)!!}\prod_{l=1}^{2k}\frac{(I+2l)!l!}{(I+l)!(2l)!} \approx I^{\frac{1}{2}k(k+1)}$$

$$\nu = m^{-\frac{1}{2}k(k+1)}$$

(ex) 4D QHE (k=2)

$$\nu = 1,\ 1/3^3,\ 1/5^3,\ \ldots$$

Fractionally charged excitations

$$q = m^{-\frac{1}{2}k(k+1)}$$

Ground state and its Incompressibility

$$\Psi_{Slator}(x_1, \cdots, x_N) = \epsilon_{\alpha_1,\cdots,\alpha_N} \Phi_{\alpha_1}(x_1) \cdots \Phi_{\alpha_N}(x_N)$$

Laughlin-like wavefunction

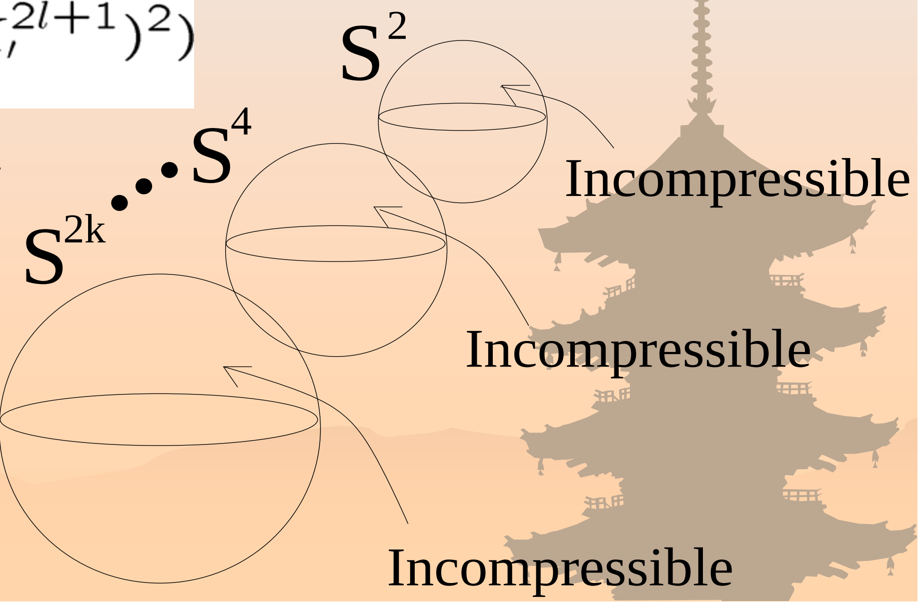
$$\Psi_{Llin}(x_1, \cdots, x_N) = \Psi^m_{Slator}(x_1, \cdots, x_N)$$

$$m \ : \ odd$$

Radial distribution function

$$g(X - X_l) \propto 1 - \prod_{l=1}^{k} \exp(-(X^{2l+1} - X_l^{2l+1})^2)$$

$$at \quad \nu = 1$$



``Anyonic'' Topological Membrane

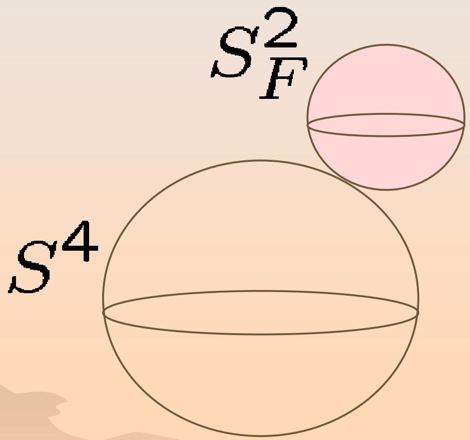
Generalization of Linking number in higher D based on ``Hopf'' map.

$$\pi_{4k-1}(S^{2k}) = Z \quad k = 1, 2, 3, \cdots .$$

Y-S. Wu and A. Zee (1988)

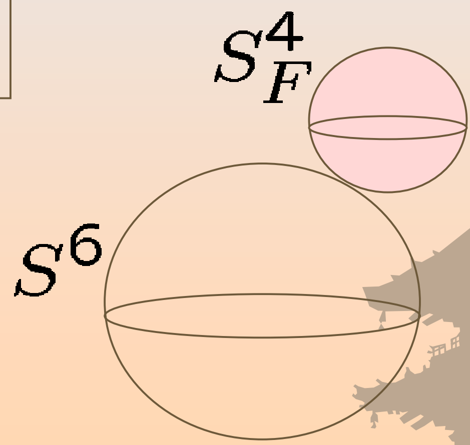
4 D QHE

$\boxed{S_F^4}$



6 D QHE

$\boxed{S_F^6}$



The topological (2k-2)-D. branes in 2k-D QHE become ``anyonic'' !

Noncommutative Algebra

$$A_\mu = \frac{1}{2\ell_B^2}\epsilon_{\mu\nu}X_\nu \to A_\mu = \frac{1}{2\ell_B^2}\Sigma_{\mu\nu}X_\nu = \frac{1}{2\ell_B^2}\eta^a_{\mu\nu}t^a X_\nu$$

$$\Sigma_{\mu\nu} \in SO(2k)$$

(Ex.) In 4D QHE $t^a \in SU(2)$, In 6D QHE $t^a \in SU(4)$.

Noncommutative algebra

$$[X_\mu, X_\nu] = i\ell_B^2\eta^a_{\mu\nu}t^a$$

Extension to even higher D. is possible !

(Spin) Hall current

$$I_\mu = -i[x_\mu, V] = -\ell_B^2\eta^a_{\mu\nu}t^a E_\nu$$

Hall orthogonality

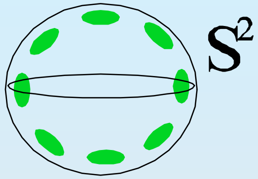
$$E_\mu I_\mu = 0$$

Dimensional Hierarchy

$l_B \sim R/\sqrt{I}$

S_F^2

2D QH

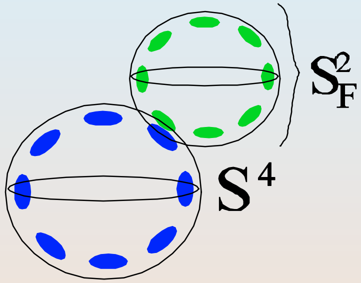


the number of 0-branes $\sim R^2/l_B^2 \sim I$

$\nu = 1/m$

S_F^4

4D QH

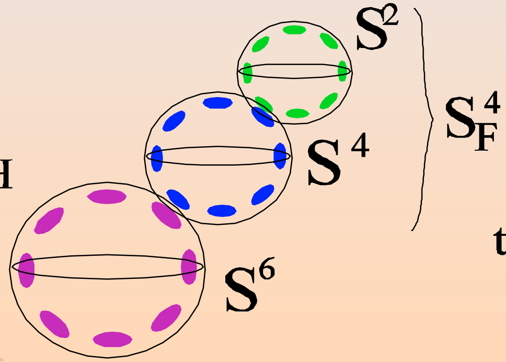


the number of 2-branes $\sim R^4/l_B^4 \sim I^2$

$\nu = 1/m \cdot 1/m^2 = 1/m^3$

S_F^6

6D QH



the number of 4-branes $\sim R^6/l_B^6 \sim I^3$

$\nu = 1/m \cdot 1/m^2 \cdot 1/m^3 = 1/m^6$

$$\nu = \frac{1}{m}\frac{1}{m^2}\cdots\frac{1}{m^k} = m^{-\frac{1}{2}k(k+1)}$$

➔ 2D QH construct higher D. QH !

➔ Brane world, Matrix theory !

Supersphere and OSp(1|2) algebra

Sphere $S^2 = SU(2)/U(1)$

Supersphere $S^{2|2} = OSp(1|2)/U(1)$

OSp(1|2) super Lie algebra $\{l_a, l_\alpha\}, \quad a = x, y, z \quad \alpha = \theta_1, \theta_2$

$$[l_a, l_b] = i\epsilon_{abc} l_c,$$

SU(2) subgroup

$$[l_a, l_\alpha] = \frac{1}{2}(\sigma_a)_{\beta\alpha} l_\beta,$$

$$\{l_\alpha, l_\beta\} = \frac{1}{2}(C\sigma_a)_{\alpha\beta} l_a, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(fundamental rep.)

$$l_a = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \quad l_{\theta_1} = \frac{1}{2}\begin{pmatrix} 0 & \tau_1 \\ \tau_2^t & 0 \end{pmatrix}, \quad l_{\theta_2} = \frac{1}{2}\begin{pmatrix} 0 & \tau_2 \\ -\tau_1^t & 0 \end{pmatrix}$$

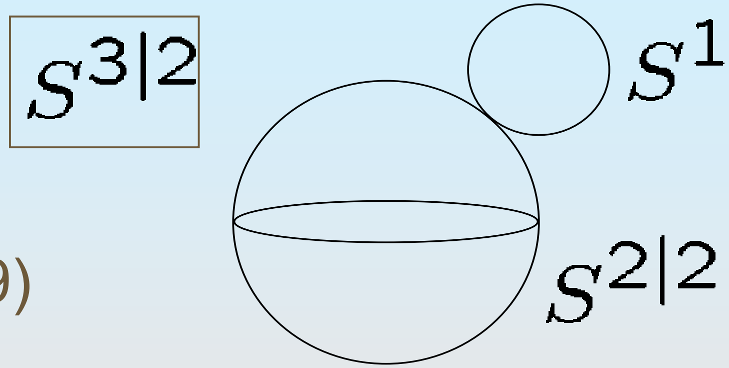
$$\tau_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Super 1-st Hopf map and Super monopole

Super 1-st Hop map

$$S^{3|2} \to S^{2|2}$$

G. Landi (1999)



$$\psi = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} \to (x_a, \theta_\alpha) = (2\psi^\ddagger l_a \psi, 2\psi^\ddagger l_\alpha \psi)$$

$$\psi^\ddagger = (u^*, v^*, -\eta^*)$$

$$\psi^\ddagger \psi = 1 \to x_a^2 + C_{\alpha\beta}\theta_\alpha\theta_\beta = 1 \quad (R = 1)$$

Super monopole connection

$$i\psi^\ddagger d\psi = A_a dx_a + A_\alpha d\theta_\alpha$$

Super Hopf spinor and Super monopole field

Super Hopf spinor

$$\psi = \begin{pmatrix} \sqrt{\frac{1+x_3}{2}}(1 - \frac{1}{4(1+x_3)}\theta C\theta) \\ \frac{x_1+ix_2}{\sqrt{2(1+x_3)}}(1 + \frac{1}{4(1+x_3)}\theta C\theta) \\ \frac{1}{\sqrt{1+x_3}}((1+x_3)\theta_1 + (x_1+ix_2)\theta_2) \end{pmatrix} \cdot e^{i\chi}$$

Super monopole gauge field

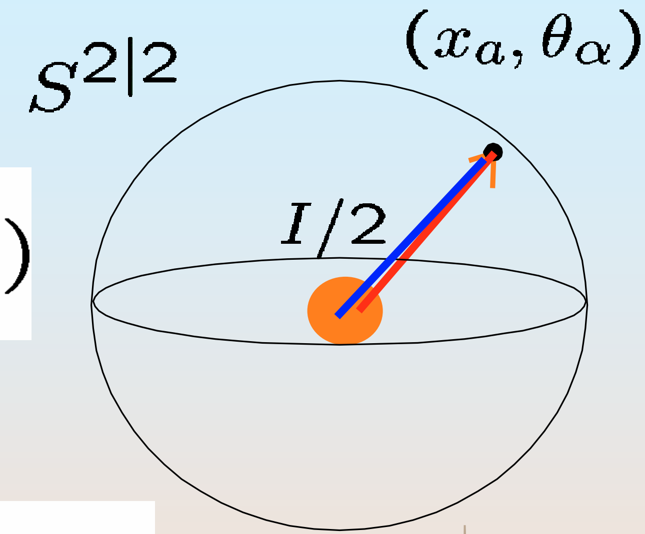
$$A_a = \frac{1}{2}\epsilon_{ab3}\frac{x_b}{1+x_3}(1 + \frac{2+x_3}{2(1+x_3)}\theta C\theta)$$

$$A_\alpha = \frac{1}{2}i(\sigma_a C)_{\alpha\beta}x_a\theta_\beta$$

Supersymmetric quantum Hall systems

Hamiltonian

$$H = \frac{1}{2MR^2}(\Lambda_a^2 + C_{\alpha\beta}\Lambda_\alpha\Lambda_\beta)$$



OSp(1|2) covariant ``angular momenta'' of the particle

$$\Lambda_a = -i\epsilon_{abc}x_b D_c + \frac{1}{2}\theta_\alpha(\sigma_a)_{\alpha\beta}D_\beta,$$
$$\Lambda_\alpha = \frac{1}{2}(C\sigma_a)_{\alpha\beta}x_a D_\beta - \frac{1}{2}\theta_\beta(\sigma_a)_{\beta\alpha}D_a$$

$$D_a = \partial_a + iA_a$$
$$D_\alpha = \partial_\alpha + iA_\alpha$$

OSp(1|2) generators

$$L_a = \Lambda_a + \frac{I}{2}x_a,$$
$$L_\alpha = \Lambda_\alpha + \frac{I}{2}\theta_\alpha$$

SO(3) angular momenta

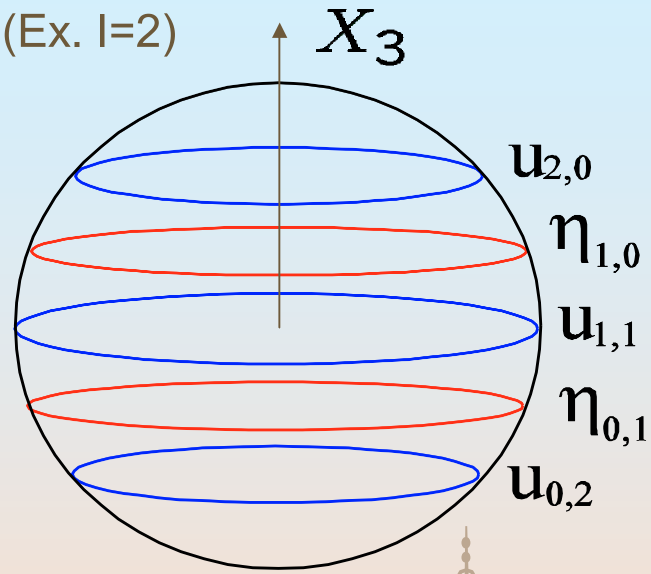
Supercharges

$$E_{LLL} = \omega/4, \quad \omega = B/M$$

Super monopole Harmonics

Super monopole Harmonics : OSp(1|2) I/2 rep.

(Ex. I=2)



$$u_{m_1,m_2} = \sqrt{\frac{I!}{m_1!m_2!}} u^{m_1} v^{m_2},$$

$$\eta_{n_1,n_2} = \sqrt{\frac{I!}{n_1!n_2!}} u^{n_1} v^{n_2} \eta.$$

Constraints

$$m_1 + m_2 = I \qquad X_3 = \frac{R}{I}(m_1 - m_2)$$

$$n_1 + n_2 = I - 1 \qquad X_3 = \frac{R}{I}(n_1 - n_2)$$

$$d(I) = (I+1) + (I) = 2I + 1 \propto 2I$$

(In the ordinary monopole case, $d(I) = I + 1 \propto I$)

Supersymmetric extension of Laughlin-Haldane wave function

Laughlin-Haldane wave function

: OSp(1|2) singlet without including complex variables

$$\nu = N/N_\Phi = 1/m$$

$$\Psi^{(m)} = \prod_{i<j}(\psi_i^t \mathcal{R} \psi_j)^m$$
$$= \prod_{i<j}(u_i v_j - v_i u_j - \eta_i \eta_j)^m$$

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Complex representation of OSp(1|2)

$$\tilde{l}_a = -l_a^*, \quad \tilde{l}_\alpha = C_{\alpha\beta} l_\beta$$

$$\tilde{l}_a = \mathcal{R}^t l_a \mathcal{R}, \quad \tilde{l}_\alpha = \mathcal{R}^t l_\alpha \mathcal{R}$$

Topological Excitations

Quasi-hole creation operator

$$A^{\ddagger} = \prod_{i=1}^{N} (b u_i - a v_i + \xi \eta_i)$$

$$\chi = \begin{pmatrix} a \\ b \\ \xi \end{pmatrix} \to (\Omega_a, \Omega_\alpha) = (2\chi^{\ddagger} l_a \chi, 2\chi^{\ddagger} l_\alpha \chi) \in S^{2|2}$$

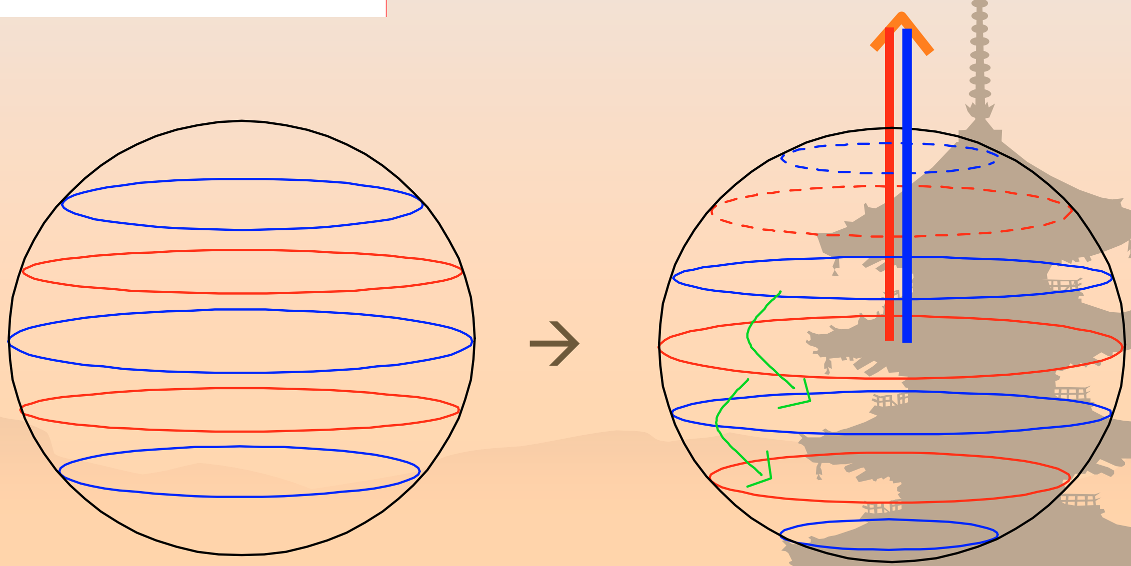
(Ex.) Quasi-hole at N.P.

$$\chi_{N.P.} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to (\Omega_a, \Omega_\alpha)_{N.P.} = (0,0,1,0,0) \quad \to \quad A^{\ddagger}_{N.P.} = \prod_{i=1}^{N} v_i$$

$$\delta N = \frac{1}{m} \delta N_{\Phi}$$

$$\to \quad q = 1/m$$

: Fractional charge



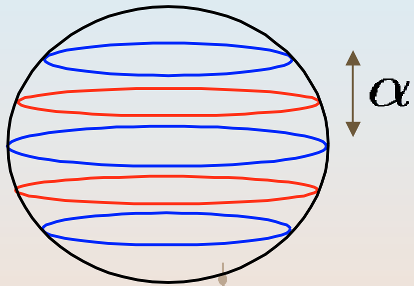
Noncommutative Algebra in Super QHE

In LLL

$$L_a, L_\alpha \to \frac{1}{\alpha} X_a, \frac{1}{\alpha} \Theta_\alpha \qquad \alpha = 2R/I$$

Supersymmetric Noncommutative Algebra,

$$[X_a, X_b] = i\alpha\epsilon_{abc}X_c$$
$$[X_a, \Theta_\alpha] = \tfrac{\alpha}{2}(\sigma_a)_{\beta\alpha}\Theta_\beta$$
$$\{\Theta_\alpha, \Theta_\beta\} = \tfrac{\alpha}{2}(C\sigma_a)_{\alpha\beta}X_a$$



Noncommutativity and Susy are naturally incorporated !

Super Hall current

$$I_a = -i[x_a, V] = \alpha\epsilon_{abc}x_b E_c - i\tfrac{\alpha}{2}(\sigma_a C)_{\alpha\beta}\theta_\alpha E_\beta$$
$$I_\alpha = -i[\theta_\alpha, V] = -i\tfrac{\alpha}{2}x_a(\sigma_a)_{\beta\alpha}E_\beta - i\tfrac{\alpha}{2}\theta_\beta(\sigma_a)_{\beta\alpha}E_a$$

Super Hall orthogonality

$$E_a I_a + C_{\alpha\beta}E_\alpha I_\beta = 0$$

Higher D. QH systems based on Fuzzy Spheres

D. of QH system	2D	4D	2k-D
Fuzzy Sphere	S_F^2	S_F^4	S_F^{2k}
Monopole	U(1)	SU(2)	SO(2k)
``Hopf'' map	$S^3 \to S^2$	$S^7 \to S^4$	$S^{4k-1} \to S^{2k}$
Topological object	0-brane (vortex)	2-brane	(2k-2)-brane
NC algebra	$[X_\mu, X_\nu] = i\ell_B^2 \epsilon_{\mu\nu}$	$[X_\mu, X_\nu] = i\ell_B^2 \eta^a_{\mu\nu}\sigma^a/2$	$[X_\mu, X_\nu] = i\ell_B^2 \eta^a_{\mu\nu} t^a$

Supersymmetric generalization !

D. of Super QH systems	2D	4D	2k-D
Fuzzy Supersphere	$S_F^{2\|2}$	?	?
Super Monopole	U(1)	SU(2) ?	SO(2k) ?
Hopf map	$S^{3\|2} \to S^{2\|2}$	?	?
Topological object	Vortex (0-brane)	?	?
NC algebra	$[X_a, X_b] = i\alpha\epsilon_{abc}X_c$ $[X_a, \Theta_\alpha] = \frac{\alpha}{2}(\sigma_a)_{\beta\alpha}\Theta_\beta,$ $[\Theta_\alpha, \Theta_\beta] = \frac{\alpha}{2}(C\sigma_a)_{\alpha\beta}X_a$	?	?

Discussion

Did we really obtain any hints to the understanding of the high energy physics from higher dimensional quantum Hall effects ?

Yes ! In, particular, it was shown that, with use of DBI action, higher dimensional fuzzy spheres in Matrix models can be identified as dielectric D-branes in colored monopole b.g.d.
(Y. Kimura, 2004)

Super QHE
→ Super matrix models, Supersymmetric Myers Effects ?

Any relation to real systems ? → Yes.
In a planer limit, the super QH system reduces to Pauli Hamiotonian with g=2 or Jaynes-Cummings model without interactions used in quantum optics. (K. Hasebe, to appear.)
There are many real systems that show supersymmtric properties.
Hopefully, the super QHE is relevant to these systems.

Higher dimensional generalization of super QHE
➔ Super 2-nd, 3-rd Hopf map,
higher dimensional fuzzy supersphere, supertwistor …