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Twistors, Landau Levels & Yang-Mills Amplitudes

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TWISTORS, LANDAU LEVELS & YANG-MILLS AMPLITUDES

TWISTOR APPROACH TO GAUGE THEORY AMPLITUDES
(TWISTOR STRING THEORY)
→ ATTEMPT TO RELATE TO LANDAU LEVELS

WHY IS THIS INTERESTING?

1. TWISTOR STRING THEORY: WEAK GAUGE COUPLING VERSION OF AdS/CFT ?

2. CALCULATION OF GAUGE THEORY AMPLITUDES

A. QUANTUM CHROMODYNAMICS, SU(3) GAUGE THEORY AMPLITUDES ARE NOT ONLY INTERESTING, THERE IS A REAL NEED FOR THEM.

$$\alpha_S = 0.120 \pm \underbrace{0.001}_{\text{expt}} \pm \underbrace{0.006}_{\text{theory}}$$

Theoretical Uncertainty can affect { hadronic background analysis @ LHC; unification scale, etc.

B. DIRECT CALCULATION → LARGE NUMBERS OF FEYNMAN DIAGRAMS ($\gtrsim$ Millions) → DIFFICULT TASK

C. WHAT CAN TWISTORS DO? DONE?

1. A FORMULA FOR THE TREE LEVEL S-MATRIX IN QCD ($\mathcal{N}=4$ SYM)

2. ONE LOOP : MHV FOR ALL n
SOME AT NEXT-TO-MHV
8 gluons, NEXT-TO-NEXT-TO MHV
} $\mathcal{N}=4$ SYM

→ COULD BE USEFUL FOR ANOMALOUS DIMENSIONS.

MHV AMPLITUDES (MAXIMALLY HELICITY VIOLATING)

GLUONS : p_μ, $p^2=0$

$$\sigma^\mu = (\mathbb{1}, \sigma^i) \qquad (p_\mu \sigma^\mu)_{A\dot{A}} = \begin{bmatrix} p_0+p_3 & p_1 - i p_2 \\ p_1 + i p_2 & p_0 - p_3 \end{bmatrix}$$

$\det p = 0 \Rightarrow$

$$p_{A\dot{A}} = \pi_A \bar{\pi}_{\dot{A}}$$

$\bar{\pi}_{\dot{A}} = (\pi^A)^*$

$$(\pi_A, e^{i\theta}\pi_A) \longrightarrow \text{SAME } p$$

Physically identify π_A, $e^{i\theta}\pi_A$.

$$\pi = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - i p_2 \\ p_0 - p_3 \end{pmatrix}$$

EVERY GLUON MOMENTUM $\rightarrow$ SPINOR MOMENTUM π^A

LORENTZ TRANSFORMATION

$$\pi^A \rightarrow (g\pi)^A \qquad g \in SL(2,\mathbb{C})$$

LORENTZ-INVARIANT SCALAR PRODUCT

$$\langle 1\,2\rangle = \pi_1 \cdot \pi_2 = \epsilon_{AB}\,\pi_1^A\,\pi_2^B \qquad \begin{cases} \langle 12\rangle = 0 \\ \text{if } \pi_1^A = c\,\pi_2^A \end{cases}$$

SCATTERING AMPLITUDES CAN BE EXPRESSED IN TERMS OF THESE INVARIANTS.

GLUON HELICITY

$$\epsilon_\mu \rightarrow \qquad \epsilon_{A\dot{A}} = \begin{cases} \dfrac{\bar{\pi}_{\dot{A}}\,\lambda_A}{\pi\cdot\lambda} & +1 \text{ helicity} \\[2ex] \dfrac{\pi_A\,\bar{\lambda}_{\dot{A}}}{\bar{\pi}\cdot\bar{\lambda}} & -1 \text{ helicity} \end{cases}$$

RESULTS OBTAINED BY PARKE & TAYLOR IN 1986

n - EXTERNAL GLUONS, TREE LEVEL

$$\mathcal{A}(+ + + \cdots +) = 0$$

$$\mathcal{A}(- + + \cdots +) = 0$$

(MHV)

$$\mathcal{A}(1_-^{a_1}, 2_-^{a_2}, 3_+^{a_3} \cdots, n_+^{a_n}) = \mathrm{Tr}(t^{a_1} t^{a_2} \cdots t^{a_n})\ \delta^{(4)}(p_1 + p_2 + \cdots)\ \mathcal{M}$$

$$\mathcal{M} = \langle 1\, 2\rangle^4 \frac{1}{\langle 1 2\rangle\langle 2 3\rangle \cdots \langle n-1\, n\rangle\langle n 1\rangle}$$

$$\left[\text{FULL RESULT} = \mathcal{A} + \text{noncyclic permutations of } \frac{\mathrm{Tr}(t^{a_1} t^{a_2} \cdots t^{a_n})}{\langle 1 2\rangle \cdots \langle n 1\rangle} \right]$$

$$= \frac{1}{n}\left(\mathcal{A} + \text{all perm's}\right)$$

WE REWRITE THIS IN THREE STEPS.

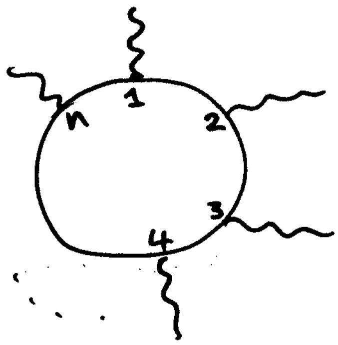
FIRST STEP

CHIRAL FERMION DETERMINANT IN 2 dim

$$\mathrm{Tr}\log D_{\bar z} = \mathrm{Tr}\log(\partial_{\bar z} + A_{\bar z})$$

$$= \mathrm{Tr}\log\left(1 + \frac{1}{\partial_{\bar z}} \cdot A_{\bar z}\right) + \text{CONST.}$$

$$= \sum \int \frac{(-1)^{n+1}}{n} \frac{\mathrm{tr}[A_{\bar z}(1)\, A_{\bar z}(2) \cdots A_{\bar z}(n)]}{(z_1-z_2)(z_2-z_3) \cdots (z_n-z_1)} \frac{d^2x_1}{\pi} \frac{d^2x_2}{\pi} \cdots$$



$$\left(\frac{1}{\partial_{\bar z}}\right)_{12} = \frac{1}{\pi(z_1-z_2)}$$

REGARD z's AS LOCAL COORDINATES ON $\mathbb{C}P^1$.

$$u^A = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad u^A \sim \lambda u^A, \quad \lambda \in \mathbb{C} - \{0\}$$

LOCAL COORDINATE (ON PATCH WITH $\alpha \neq 0$)

$z = \beta/\alpha$

$$z_1 - z_2 = \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} = \frac{\beta_1\alpha_2 - \beta_2\alpha_1}{\alpha_1\alpha_2}$$

$$= \frac{u_1 \cdot u_2}{\alpha_1\alpha_2}$$

IF $A_{\bar z}\, \alpha^2 = \mathcal{A}_{\bar z} = \bar{\mathcal{A}}$

$$\mathrm{Tr}\log D_{\bar z} = -\sum \frac{1}{n} \int \underbrace{\frac{\mathrm{tr}[\bar{\mathcal{A}}(1)\, \bar{\mathcal{A}}(2) \cdots \bar{\mathcal{A}}(n)]}{(u_1\cdot u_2)\,(u_2\cdot u_3) \cdots (u_n\cdot u_1)}}$$

IF $u^A = \pi^A$, this is denominator of amplitude

SECOND STEP

FROM LORENTZ TRANSFORMATION,

$$\text{helicity} = -\frac{1}{2}\, \pi^A \frac{\partial}{\partial \pi^A}$$

TO GET $\langle 12\rangle^4$, USE GRASSMAN PARAMETERS

$$\int d^2\theta\; \theta_A\, \theta_B\, \pi_1^A\, \pi_2^B = \epsilon_{AB}\, \pi_1^A\, \pi_2^B = \langle 12\rangle$$

$$\int d^8\theta\; \xi_1^1\, \xi_1^2\, \xi_1^3\, \xi_1^4\; \xi_2^1\, \xi_2^2\, \xi_2^3\, \xi_2^4 = \langle 12\rangle^4$$

$$\xi_1^\alpha = \theta_A^\alpha\, \pi_1^A \qquad \alpha = 1, 2, 3, 4.$$

$\rightarrow$ $\mathcal{N} = 4$ EXTENDED SUSY.

DEFINE A "WAVE FUNCTION" INVOLVING ξ's FOR $\mathcal{N} = 4$ SUPERMULTIPLET

$$\phi = e^{\eta_\alpha \xi^\alpha} = \underset{h=+1}{1} + \underset{h=+\frac{1}{2}}{\eta_\alpha \xi^\alpha} \mp \underset{h=0}{\frac{\eta_\alpha \eta_\beta}{2!} \xi^\alpha \xi^\beta} + \underset{h=-\frac{1}{2}}{\frac{1}{3!} \eta\cdot\xi\, \eta\cdot\xi\, \eta\cdot\xi}$$

$$+ \underset{h=-1}{\frac{1}{4!} \eta\cdot\xi\, \eta\cdot\xi\, \eta\cdot\xi\, \eta\cdot\xi}$$

$$\int d^8\theta \prod_i e^{\eta^i \xi^i} = \langle 1 2 \rangle^4$$

if we look at coefficient of $(\eta^1)^4 \; (\eta^2)^4$

CHOOSE $\mathcal{A} = t^a \phi \times e^{ip\cdot x}$

$$\int d^4x \int d^8\theta \; \mathrm{Tr} \log D_{\bar z} = -\sum \frac{1}{n} \frac{\mathrm{Tr}(t^{a_1} \cdots t^{a_n})}{(u_1\cdot u_2)\cdots(u_n \cdot u_1)} \int d^8\theta\, d^4x \prod_{i=1}^{n} e^{(\eta\cdot\xi)_i} \prod_i e^{ip_i\cdot x}$$

$u^A = \pi^A$

$$= -\sum \frac{1}{n} \mathcal{A}(- - + + \cdots +)$$

(NAIR 1988)

<u>THIRD STEP</u> (WITTEN 2003)

INTRODUCE $\omega_{\dot A} = x_{\dot A A}\, \pi^A$

$$\exp i\, p\cdot x = \exp\left(\tfrac{i}{2} \bar\pi^{\dot A} \pi^A x_{\dot A A}\right)$$

$$= \exp\left(\tfrac{i}{2} \bar\pi^{\dot A} \omega_{\dot A}\right)$$

REGARD $\omega_{\dot{A}}$ AS A FREE VARIABLE

$$\int d^2\omega \; \delta(\omega_{\dot{A}} - x_{\dot{A}A}\pi^A) \; e^{\frac{i}{2}\bar{\pi}^{\dot{A}}\omega_{\dot{A}}} = e^{ip\cdot x}$$

DO SAME FOR ξ, AND TO SET $u^A \hat{=} \pi^A$.

$$\underbrace{\int d^2u \, d^2\omega \, d^4\xi}_{d\mu} \; \delta(\omega_{\dot{A}} - x_{\dot{A}A}u^A) \; \delta(\xi^\alpha - \theta^\alpha_A u^A) \cdot$$
$$\times \; \delta(\pi^A - u^A) \times e^{\frac{i}{2}\bar{\pi}^{\dot{A}}\omega_{\dot{A}}} \, e^{(\eta\cdot\xi)}$$

$$= e^{ip\cdot x} \; e^{\eta\cdot(\theta\pi)}$$

NOW WRITE

$$\tilde{\mathcal{A}} = t^a \; \delta(\omega - x\,u) \; \delta(\pi - u) \; \delta(\xi - \theta u)$$

$$\int d^4x \, d^8\theta \;\; \mathrm{Tr} \log D_{\bar{z}} \Big|_{n\text{-th term}}$$

$$= \tilde{\mathcal{A}}(1, 2, \ldots)$$

$$\int d\mu_i \; e^{i\bar{\pi}_i\omega_i} e^{\eta_i\cdot\xi_i} \; \tilde{\mathcal{A}} = \text{YM MHV amplitude}$$

PROPERTIES OF $\tilde{\mathcal{A}}$

1. ENTIRELY HOLOMORPHIC IN $(\underbrace{\omega, \pi}_{Z}, \xi)$

2. INVARIANT UNDER $u \to \lambda u$

$$\left.\begin{array}{l} \omega \to \lambda\omega \\ \pi \to \lambda\pi \end{array}\right\} Z \to \lambda Z$$
$$\xi \to \lambda \xi$$

(Use d^2u integration) ON $\mathbb{CP}^{3/4}$ SUPERTWISTOR SPACE

3. SUPPORT ON

$$\omega_{\dot{A}} = \underset{\sim}{x_{\dot{A}A}}\, u^A$$

$$\pi^A = u^A$$

$$\xi^\alpha = \underset{\sim}{\theta^\alpha_A}\, u^A$$

A LINE IN $\mathbb{CP}^{3/4}$

$$\begin{array}{ccc} \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^{3/4} \\ u & & (Z, \xi) \end{array}$$

degree 1 curve

4. INTEGRATE OVER MODULI OF LINE $\sim$

($\pi^A = a^A{}_B u^B$ $\quad a^A{}_B$ CAN BE SET TO $\mathbb{1}$ BY $SL(2,\mathbb{C})$ INVARIANCE.)

GENERAL FORM $= \displaystyle\int \frac{d(\text{moduli})}{\text{vol}(SL(2,\mathbb{C}))}\, \text{Tr} \log D_{\bar{z}}$

GENERALIZATION

HIGHER DEGREE CURVES FOR NON-MHV

$$\underset{\substack{\uparrow\\ \text{degree of}\\ \text{curve}}}{d} = \underset{\substack{\uparrow\\ \text{number}\\ \text{of}\\ -1 \text{ gluons}}}{q} - 1 + \underset{\substack{\uparrow\\ \text{number}\\ \text{of loops}}}{\ell} \qquad \underset{\substack{\uparrow\\ \text{genus}}}{g} \leq \ell$$

JUSTIFICATION

OPEN STRINGS IN TOPOLOGICAL B-MODEL.

TARGET SPACE $\mathbb{CP}^{3|4}$ CALABI-YAU SUPER MANIFOLD

TOPOLOGICAL SECTOR

$$I = \frac{1}{2} \int_Y \Omega \wedge \mathrm{Tr}\left(\mathcal{A}\, \bar{\partial} \mathcal{A} + \tfrac{2}{3} \mathcal{A}^3\right)$$

$\mathcal{A} = \mathcal{A}(z, \bar{z}, \xi)$ $\qquad$ $Y =$ subspace of $\mathbb{CP}^{3|4}$ with $\bar{\xi} = 0$

$\rightarrow$

$$I = \int \mathrm{Tr}\left[G^{AB} F_{AB} + \tilde{\chi}^{\dot{A}}_{\alpha} D_{A\dot{A}} \chi^{\dot{A}}_{\alpha} + \ldots\right]$$

Add a term like $\int \frac{G^2 \epsilon}{2}$ (GENERATED BY INSTANTONS) D1 BRANES

INTEGRATE OUT G

$\rightarrow \mathcal{N} = 4$ SYM WITH $\epsilon \sim g^2_{YM}$

$G \rightarrow$ helicity $= -1$

$A \rightarrow$ helicity $= +1$

$r\ G$'s $\rightarrow$ $(r-1)\ \epsilon$'s $\Rightarrow$ instanton number $= r-1 = d$

$\Rightarrow r = d+1$

FORMULA FOR ALL TREE AMPLITUDES

$$\mathcal{A}(\pi_i, \bar{\pi}_i) = \int \left[d^{2d+2}a \; d^{2d+2}b \; d^{4d+4}\gamma \right] \int \frac{d\sigma_1 \cdots d\sigma_n}{[\mathrm{vol}\ GL(2)]}$$

$$\times \frac{1}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_n-\sigma_1)}$$

$$\times\ \delta\left[\frac{\pi_i^2}{\pi_i^1} - \frac{\pi^2(\sigma_i)}{\pi^1(\sigma_i)}\right] e^{i\,\bar{\pi}_i^{\dot{A}}\,\pi_i^1\,\frac{\omega_{\dot{A}}(\sigma_r)}{\pi^1(\sigma_r)}}$$

$$\times\ \mathrm{Tr}\left[\phi(1)\cdots\phi(n)\right]$$

degree d

$\mathbb{CP}^1 \rightarrow \mathbb{CP}^{3|4}$

$$\begin{cases} \pi^A(\sigma) = \sum_0^d a_k^A\,\sigma^k & \omega_{\dot{A}}(\sigma) = \sum_0^d b_{\dot{A},k}\,\sigma^k \\ \xi^\alpha(\sigma) = \sum_0^d \gamma_k^\alpha\,\sigma^k & \sigma = \frac{u_2}{u_1} \end{cases}$$

LANDAU LEVEL CONNECTION

(related to BERKOVITS' WORK)
2004

START WITH $\mathbb{C}P^1 = S^2$ $\qquad u^A \sim \lambda u^A$.

$$S = \int \bar{q}(\bar{\partial} + \bar{\mathcal{A}})q + \overline{(\bar{D}Q)}\,\bar{D}Q$$

$(q, \bar{q})$ = fermion fields

$Q = (Z, \xi^\alpha) = (\pi, \omega, \xi)$ $\qquad \mathbb{C}^{4|4}$ $\Big\}$ on $S^2 = \mathbb{C}P^1$

$\bar{D} = \bar{\partial} + \bar{A}$

$\uparrow$ GL(1) field $\Rightarrow$ reduces $\mathbb{C}^{4|4}$ to $\mathbb{C}P^{3|4}$

$$\bar{A} = \bar{A}_d + \underbrace{\delta\bar{A}}_{\bar{\partial}\theta}$$

$\bar{\partial}\theta \leftarrow$ removed by GL(1) symmetry.

$\uparrow$ configuration of monopole number d

$\rightarrow$ *Landau levels, mode expansion*

$$Z = \sum \underbrace{a_{A_1 \cdots A_d}\, u^{A_1} \cdots u^{A_d}}_{LLL} + \text{nonzero modes}$$

higher LL involve $\bar{u}$

$$\xi^\alpha = \sum \gamma^\alpha_{A_1 \cdots A_d}\, u^{A_1} \cdots u^{A_d} + (\text{nonzero modes})$$

IF $\mathcal{A}$ has no $\bar{Z}$, HIGHER LANDAU LEVELS DO NOT CONTRIBUTE TO CORRELATORS OF $\mathcal{A}$

WE CHOOSE

$$\bar{\mathcal{A}}^a_\pi = \partial_{\bar{z}} \Phi^a$$

$$\Phi^a_\pi = t^a \frac{\delta(\pi\cdot Z)\; Z\cdot A}{\pi\cdot A} \exp\left[i \frac{\bar{\pi}\cdot Z \;\pi\cdot A}{Z\cdot A} + \xi\cdot\eta \frac{\pi\cdot A}{Z\cdot A}\right.$$

$A = (0, 0, 1, 0)$

$Z = (\omega_1, \omega_2, \pi_1, \pi_2)$

$$\int e^{-S} \frac{[dZ\, d\bar{Z}\, d\xi]\,[d\hat{A}]}{[\mathrm{vol}\; GL(2)]}$$

$$\sum_d c_d \int \frac{d(\mathrm{moduli})}{[\mathrm{Vol}\; GL(2)]} \oint \frac{\mathrm{Tr}[\phi(1) \cdots \phi(n)]\; u_1 du_1 \; u_2 du_2}{(u_1\cdot u_2)(u_2\cdot u_3)\cdots(u_n\cdot u_1)}$$

$$= \sum_d c_d \mathcal{A}_d$$

REMARKS

1. FERMIONIC INTEGRATION $\rightarrow \left(\prod^{d+1}{}' \langle ij \rangle\right)^4$

$= (\text{Laughlin})^4$ for $d+1$ fermions

2. Large d limit ?

BESIDES CITATIONS GIVEN

PRINCETON: BRITTO, CACHAZO, FENG, SVRČEK, E.W. + ..

QUEEN MARY BRANDHUBER, SPENCE, TRAVAGLINI + ..

DURHAM KHOZE, GLOVER, GEORGIOU

UCLA-SLAC - SACLAY BERN, KOSOWER, DIXON, BENA, DEL DUCA + ...

SANTA BARBARA ROIBAN, SPRADLIN, VOLOVICH, ..

BERKOVITS, SIEGEL, ...