The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR 1646 - 1

Conference on
Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and
Non-Commutative Geometry in Condensed Matter Physics and Field Theory
1 - 4 March 2005

MOMENTUM-SPACE TOPOLOGY OF FERMION ZERO MODES, CHERN-SIMONS TERM & QUANTUM PHASE TRANSITION

G. VOLOVIK
Helsinki University of Technology, Espoo, Finland
&
Landau Institute, Moscow, Russia

Momentum-space topology of fermion zero modes, Chern-Simons term & quantum phase transition

G. Volovik

Helsinki University of Technology & Landau Institute, Moscow

Trieste, March 2005

* **Momentum-space topology in 3+1**
  Fermi points, chiral fermions, chiral anomaly, Standard Model, Chern-Simons term, spinons & holons
* **Momentum-space topology in 2+1**
  Chern-Simons term, quantum statistics of skyrmions, QHE and spin QHE
* **Momentum-space topology of edge states**
  index theorem
* **Momentum-space topology in higher dimensions**
  Brane fermions
* **Quantum phase transitions dictated by momentum-space topology**
  BEC-BCS and neutrino oscillations
* **Conclusion**

Example of *p*-space topology

Compare these two Hamiltonians

Bogoliubov-Nambu
for quasiparticles
in ^{3}He-A & chiral superconductors

Weyl for **right**-handed
neutrino

$$H = +\, c\, \sigma \cdot \mathbf{p}$$

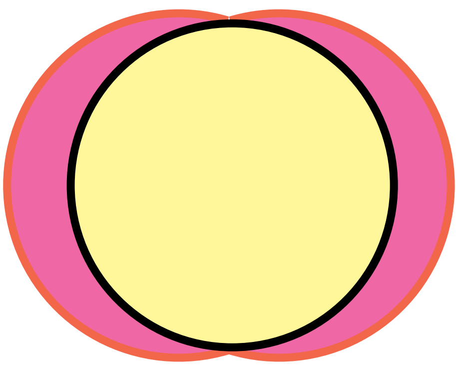
$$H = \begin{pmatrix} \frac{p^2 - p_F^2}{2m} & c_\perp (p_x + i p_y) \\ c_\perp (p_x - i p_y) & -\frac{p^2 - p_F^2}{2m} \end{pmatrix} \qquad H = \begin{pmatrix} c p_z & c\,(p_x + i p_y) \\ c(p_x - i p_y) & -\, c p_z \end{pmatrix}$$

What is common for them?

$$H(\mathbf{p}) = \tau \cdot \mathbf{g}(\mathbf{p}) \qquad E^2(\mathbf{p}) = \mathbf{g}^2(\mathbf{p})$$

1. $E(\mathbf{p}) = 0$ ***at points*** **(Fermi points)**

$\mathbf{p} = +\, p_F\, \mathbf{e}_z$ $N = +1$



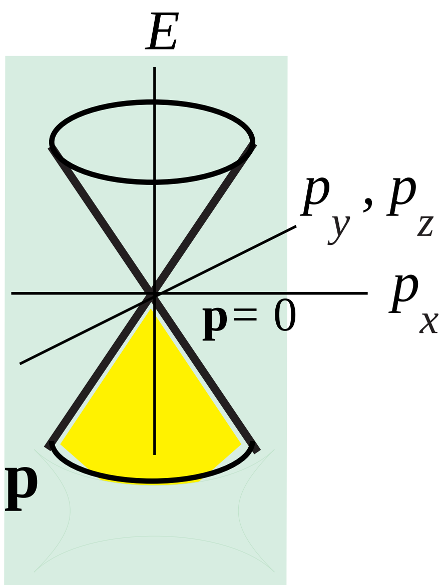
$\mathbf{p} = -\, p_F\, \mathbf{e}_z$ $N = -1$

$N = +1$
right-handed
meutrino

left-handed
meutrino

$H = -\, c\, \sigma \cdot \mathbf{p}$

$N = -1$



2. Fermi points are topologically stable
& described by topological invariant in momentum space

$$N = \frac{1}{8\pi}\, e_{ijk} \int dS^i\, \hat{\mathbf{g}} \cdot (\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

over 2D surface around Fermi point

3. Close to Fermi points (quasi)particles are
relativistic left or **right**-handed **chiral Weyl fermions**

$$L = e^\mu{}_a \sigma^a (p_\mu - e A_\mu)$$

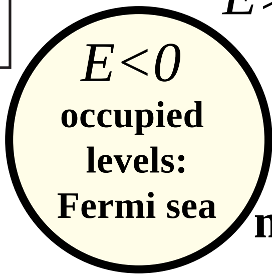
Universality classes of fermionic vacua from *p*-space topology

Systems with Fermi surface

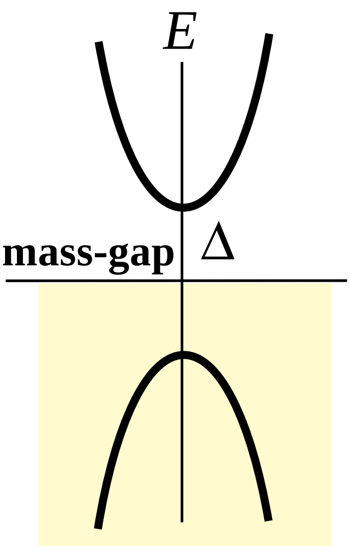
$$E(\mathbf{p}) = \frac{p^2 - p_F^2}{2m}$$

Fermi surface $E=0$

$E>0$



cod = 1
co-dimension
cod =
space dimension D
minus dimension of zeroes;
exist for D ≥ 1



Fully gapped Fermi systems

no zeroes

Superconductor

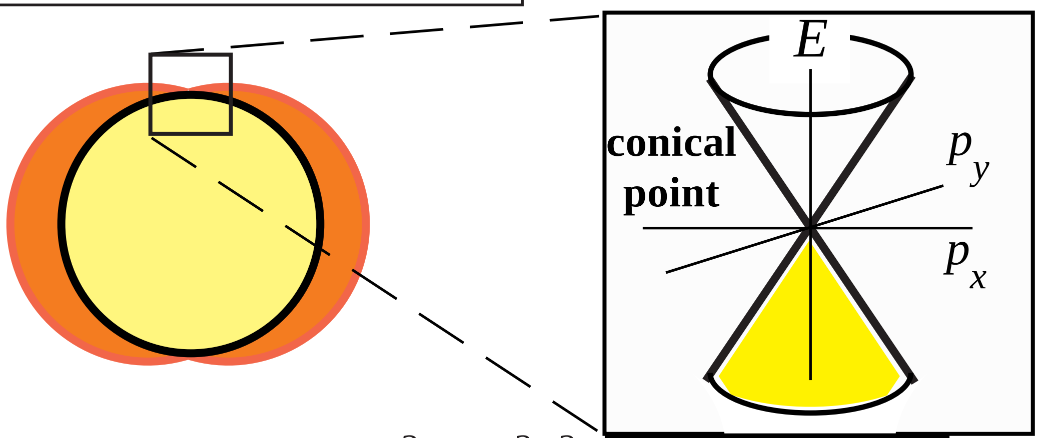
$$E^2(\mathbf{p}) = v_F^2 (p-p_F)^2 + \Delta^2$$

Dirac fermions

$$E^2 = p^2c^2 + M^2$$

***p*-space topology**
is non-trivial
in even
space dimensions
D=2,4,6...

Systems with Fermi points



$$E^2 = p^2c^2$$

$$H = c\,\sigma \cdot \mathbf{p}$$

^{3}He-A
&
Standard Model
fermions

cod = 3
D ≥ 3

Higher dimensional Fermi points

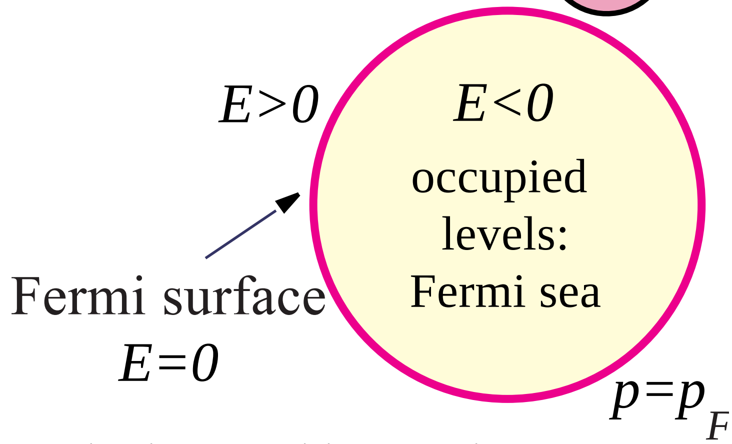
odd co-dimensions
cod = 5 , 7 , ...

D ≥ 5, 7 , ...

Fermi surface as vortex in 3+1 *p*-space

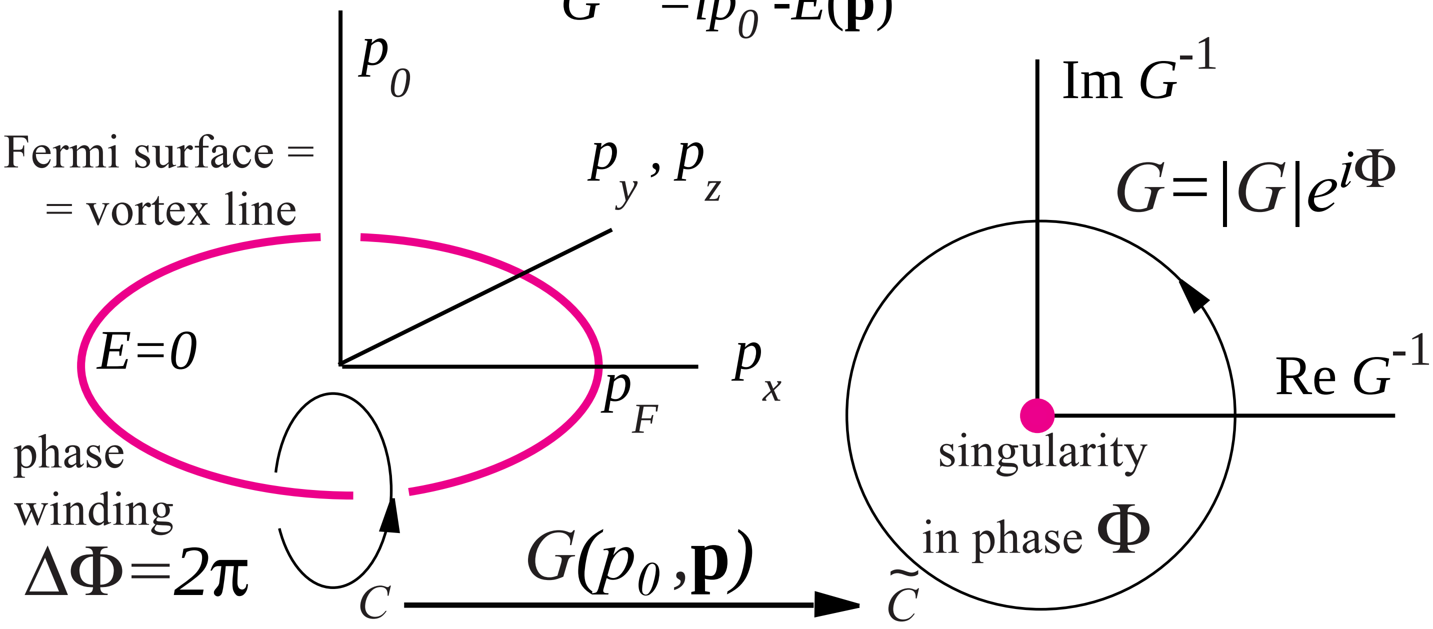
* Fermi gas $E = \frac{p^2}{2m} - \mu$

$$E(\mathbf{p}) = \frac{p^2 - p_F^2}{2m}$$



Fermi surface is robust to perturbations and interactions
as vortex -- topologically stable singularity of Green function:

$$G^{-1} = ip_0 - E(\mathbf{p})$$

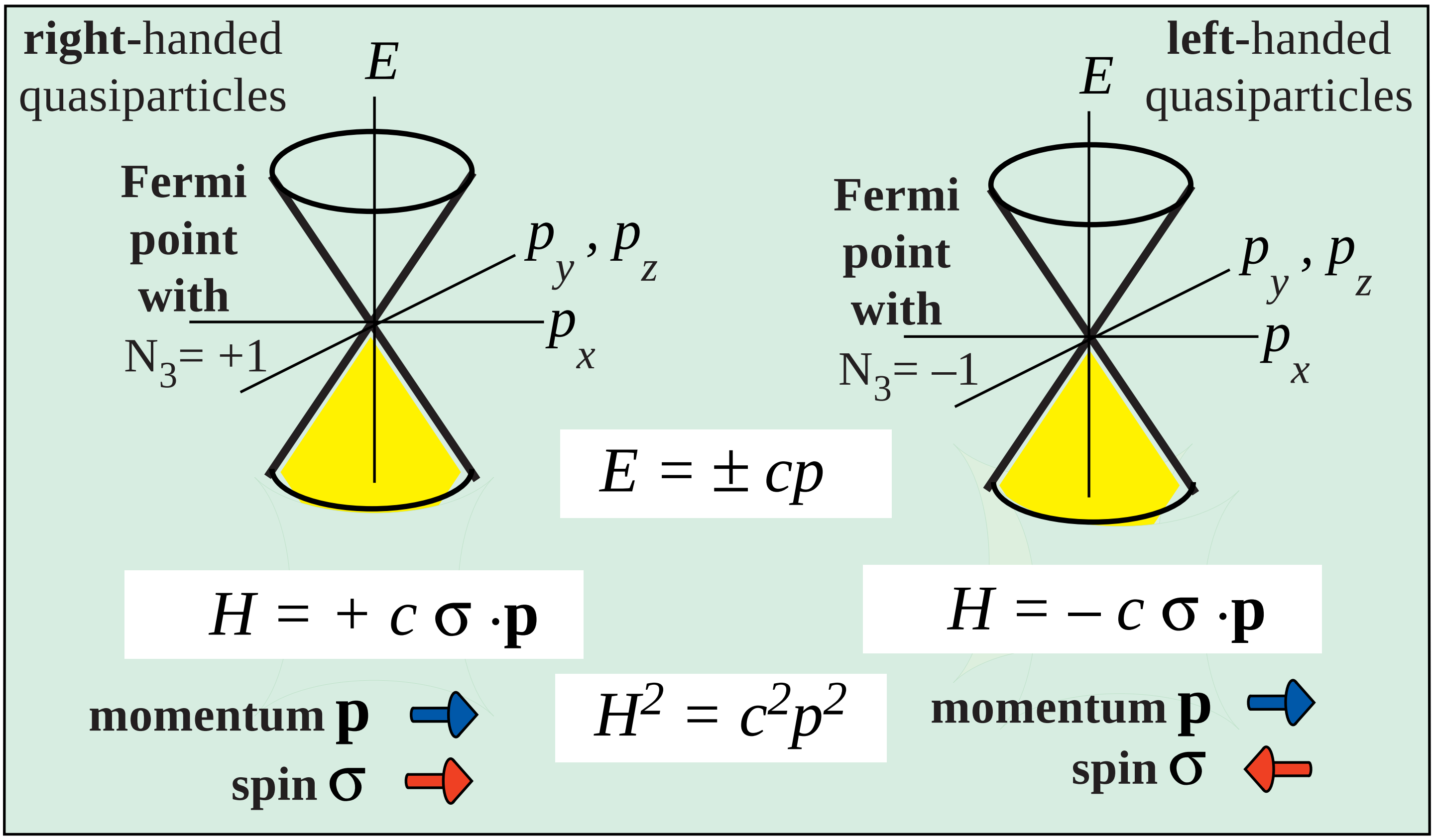


* Interacting Fermi system:
  general topological invariant

$$N_1 = \frac{1}{2\pi i}\, \mathbf{tr} \oint_{\text{around Fermi surface}} dp^\mu\, \mathbf{G}\, \partial_{p_\mu} \mathbf{G}^{-1}$$

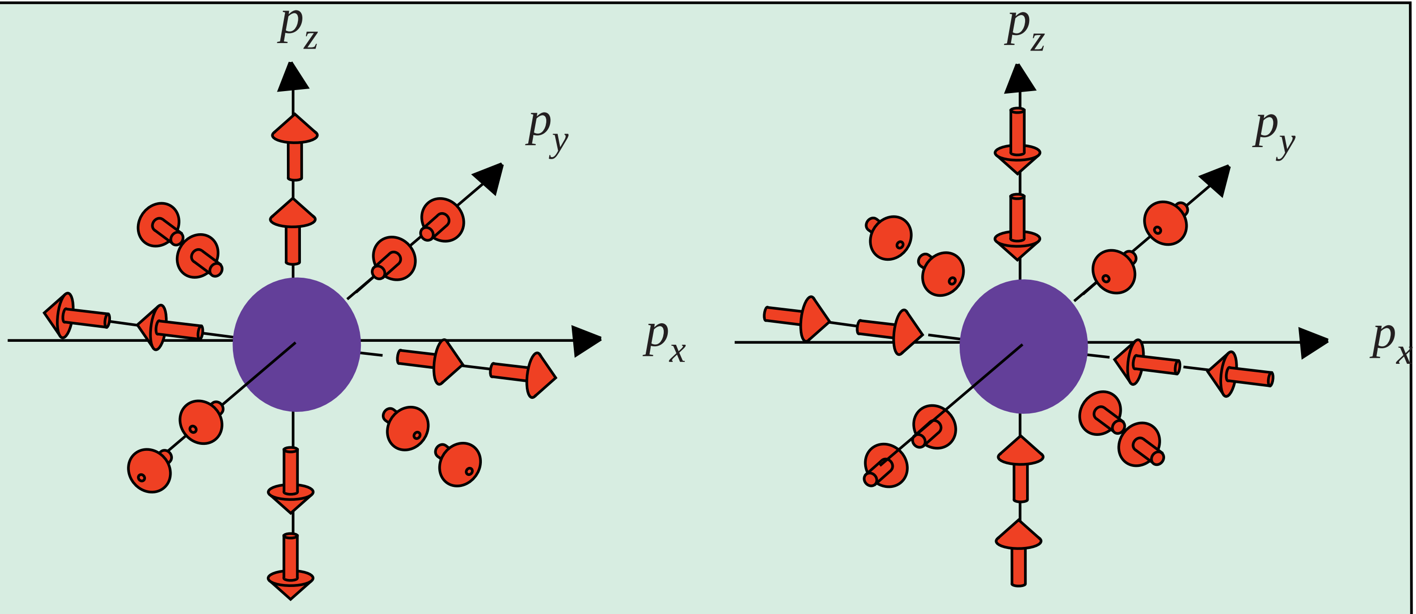
Chiral particles

Quasiparticles near **Fermi points** are **relativistic:**
left or **right**-handed **chiral Weyl fermions**



Chiral particles

Topological stability of Fermi point: hedgehog in momentum space



hedgehog with spines (spins) outward (N= +1)

hedgehog with spines (spins) inward (N= –1)

$$H = + c\, \sigma \cdot \mathbf{p}$$

$$\mathbf{g}(\mathbf{p}) = +c\mathbf{p}$$

$$H = - c\, \sigma \cdot \mathbf{p}$$

$$\mathbf{g}(\mathbf{p}) = -c\mathbf{p}$$

$$N = \frac{1}{8\pi} e_{ijk} \int dS^i\, \hat{\mathbf{g}} \cdot (\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

over 2D surface around Fermi point

Topological stability of Fermi point
(general case)

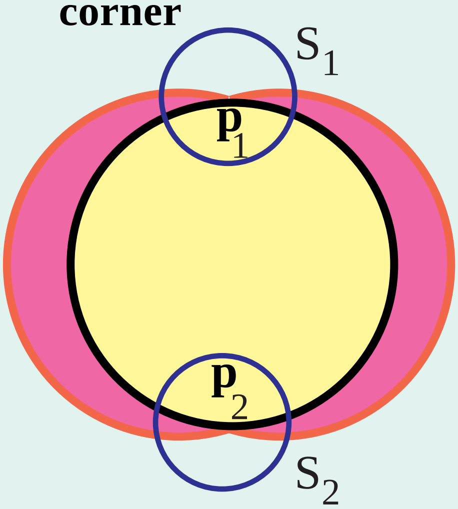
Topological invariant in 4D momentum space ($\mathbf{p}$, p_0)
in terms of fermionic propagator:
matrix Green's function $\mathbf{G}(\mathbf{p}, p_0)$

$$N = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma} \, \mathbf{tr} \int dS^{\gamma} \, \mathbf{G} \, \partial^{\mu} \mathbf{G}^{-1} \, \mathbf{G} \, \partial^{\nu} \mathbf{G}^{-1} \mathbf{G} \, \partial^{\lambda} \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

in low-energy corner

$$\mathbf{G}^{-1} = i\,p_0 + c\,\sigma \cdot (\mathbf{p} - \mathbf{p}_1)$$



$N = +1$ **right-handed particles**

$N = -1$ **left-handed particles**

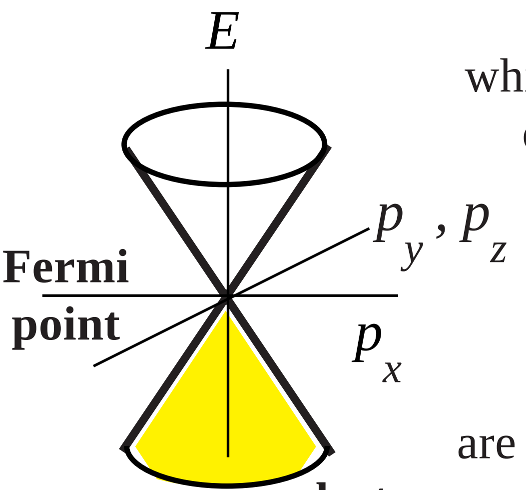
$$\mathbf{G}^{-1} = i\,p_0 - c\,\sigma \cdot (\mathbf{p} - \mathbf{p}_2)$$

top. invariant determines chirality in low-energy corner

Collective modes of fermionic vacua of Fermi-point Universality Class:

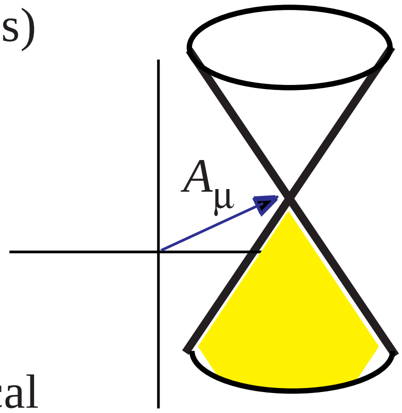
gauge fields & gravity

Vacuum low-energy dynamics cannot destroy the Fermi point.
Shifts **A** and slopes g^{ik} are propagating collective modes:

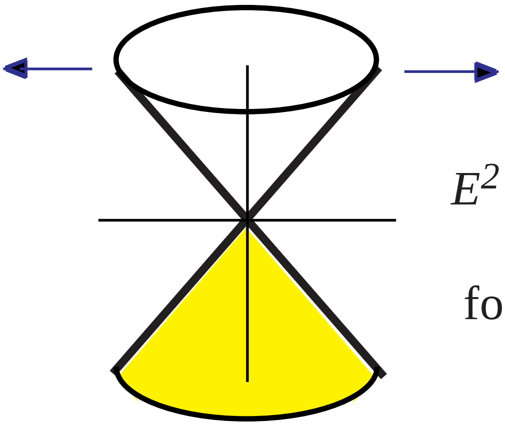


vacuum modes
which shift **position**(s)
of Fermi point(s)

$$p_\mu \rightarrow p_\mu - eA_\mu$$



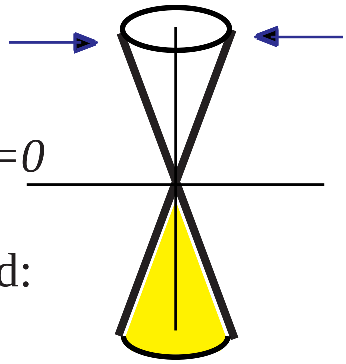
are effective dynamical
electromagnetic and other **gauge fields**



vacuum modes
where **slopes** vary

$$E^2 - c^2p^2 = 0 \ \rightarrow \ g^{\mu\nu} p_\mu p_\nu = 0$$

form effective metric field:
gravitational field



Quasiparticle near **Fermi point** is
left or **right** particle moving in effective
gravitational, **electromagnetic**, **weak** fields

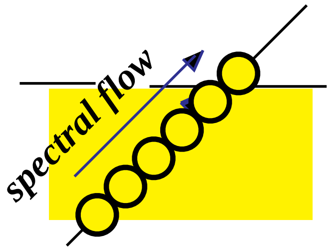
$$g^{\mu\nu}(p_\mu - eA_\mu - e\tau\cdot\mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau\cdot\mathbf{W}_\mu) = 0$$

chiral fermions,
gauge fields and **gravity**
appear
in low-energy corner
together with spin and
physical laws:
Lorentz and gauge invariance,
and **general covariance**

Chiral anomaly

nucleation of fermionic charge from *vacuum*

* **chiral particles*****: quarks & leptons in Standard Model***
& ***quasiparticles in*** ^{3}*He-A*
are created from* vacuum *one by one
by spectral flow



creation of momentum from
^{3}He-A vacuum

creation of baryonic charge from
Standard Model vacuum

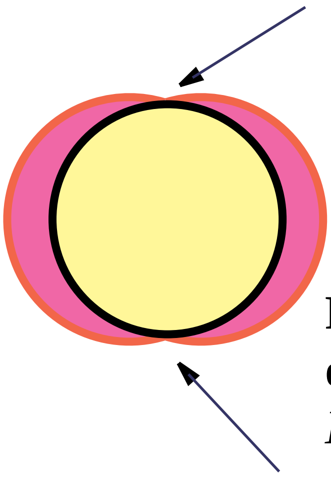
$$\dot{\mathbf{P}} = (1/4\pi^2)\,\mathbf{B}\cdot\mathbf{E}\sum_a \mathbf{P}_a N_a e_a^2$$

$$\dot{B} = (1/4\pi^2)\,\mathbf{B}_Y\cdot\mathbf{E}_Y \sum_a B_a N_a Y_a^2$$

$\mathbf{P}_a$ -- **momentum of Fermi point**
e_a -- **effective electric charge**

B_a -- **baryonic charge**
Y_a -- **hypercharge**

N_a -- **topological charge of Fermi point**



Fermi point
$\mathbf{P}_a$
$e_a = +1$
$N_a = +1$

$\mathbf{P}_a$
$e_a = -1$
$N_a = -1$
Fermi point

Chiral anomaly

General anomaly equation in terms of topological invariant protected by symmetry :

nucleation of fermionic charge Q **by gauge field** A^{Y}

$$\dot{Q} = (1/4\pi^2)\, \mathbf{B}_Y \bullet \mathbf{E}_Y\, N_{YYQ}$$

$$N_{YYQ} = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma} \mathbf{tr}\, YYQ \int dS^{\gamma}\, \mathbf{G}\, \partial^{\mu}\, \mathbf{G}^{-1}\, \mathbf{G}\, \partial^{\nu}\, \mathbf{G}^{-1} \mathbf{G}\, \partial^{\lambda}\, \mathbf{G}^{-1}$$

Y - charge interacting with gauge field A_{μ}^{Y}

^{3}He-A

$Q = \mathbf{P}_a$ **-- momentum of Fermi point**

$Y = e_a$ **-- effective electric charge**

Standard Model

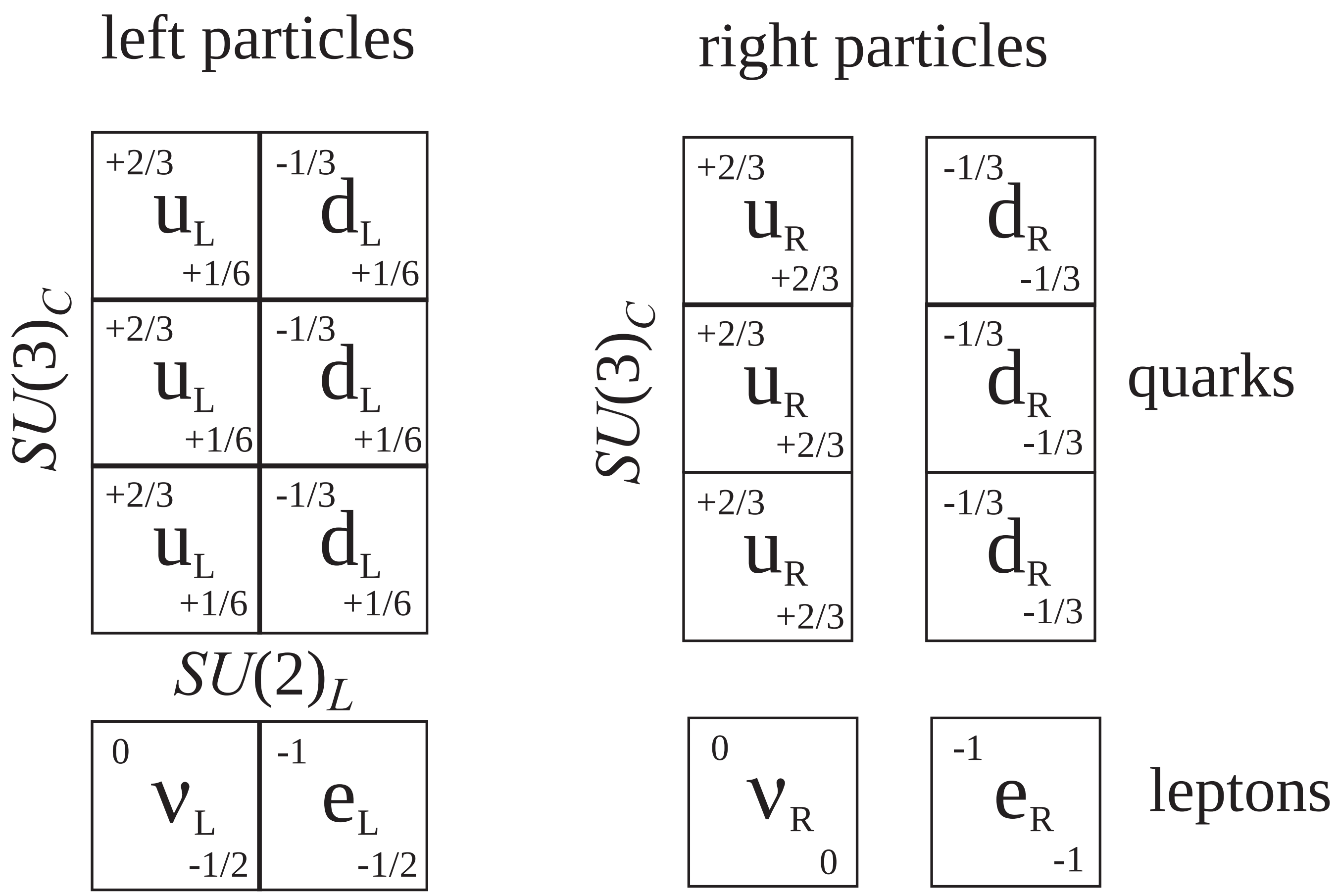
$Q = B_a$ **-- baryonic charge**

$Y = Y_a$ **-- hypercharge**

Chiral fermions in Standard Model

Fermi-point Universality Class

Family #1 of quarks and leptons

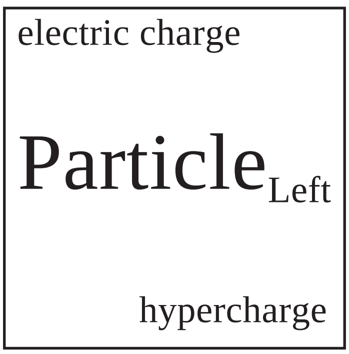


$$H = -c\,\boldsymbol{\sigma}\cdot\mathbf{p} \qquad\qquad H = +c\,\boldsymbol{\sigma}\cdot\mathbf{p}$$

$$N = -1 \qquad\qquad N = +1$$

$$N = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int dS^{\gamma}\, \mathbf{G}\, \partial^{\mu}\, \mathbf{G}^{-1}\, \mathbf{G}\, \partial^{\nu}\, \mathbf{G}^{-1} \mathbf{G}\, \partial^{\lambda}\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space



Pati-Salam unification of quarks & leptons
&
Terazawa spinon-holon model of fermions

lepton as 4-th color

$SU(4)_C$

$SU(2)_L$

+2/3 u_L	-1/3 d_L
+2/3 u_L	-1/3 d_L
+2/3 u_L	-1/3 d_L
0 ν_L	-1 e_L

$SU(2)_R$

+2/3 u_R	-1/3 d_R
+2/3 u_R	-1/3 d_R
+2/3 u_R	-1/3 d_R
0 ν_R	-1 e_R

↓

composite model of elementary particles

Holons $^Q C$ carry color charge

$SU(4)_C$

+1/6 C
+1/6 C
+1/6 C
-1/2 C

$\otimes$

Spinons $^Q W$ carry spin and isospin

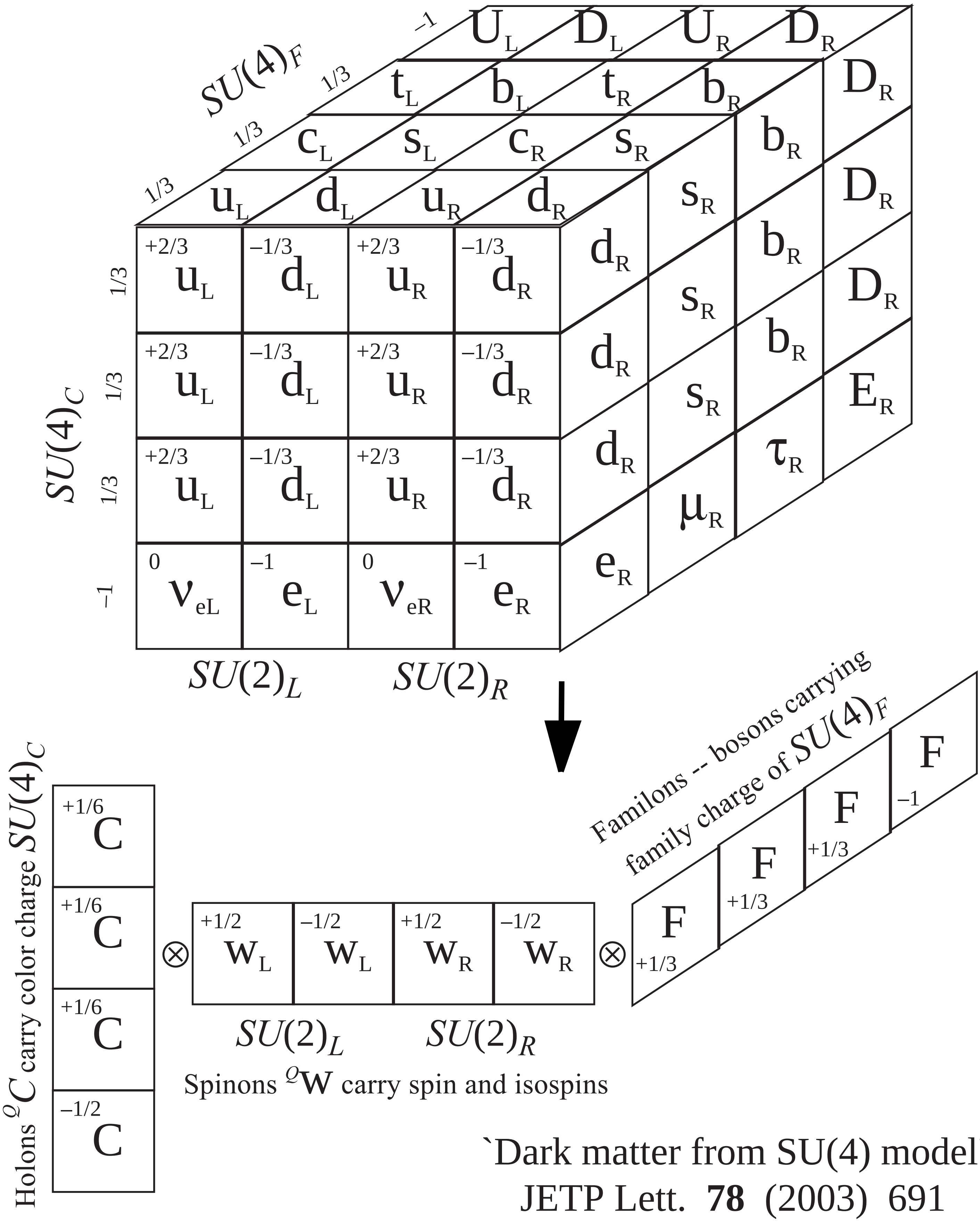
+1/2 W_L	-1/2 W_L	+1/2 W_R	-1/2 W_R

$SU(2)_L$ $SU(2)_R$

Q - electric charge

Periodic Table of elementary particles

Extended Pati-Salam model with 4 families



`Dark matter from SU(4) model'
JETP Lett. **78** (2003) 691

Momentum-space topological invariant in 2+1

general case:

$$N = \frac{1}{24\pi^2} \; e_{\mu\nu\lambda} \, \mathbf{tr} \int d^2p \, d\omega \, \mathbf{G} \, \partial^\mu \, \mathbf{G}^{-1} \, \mathbf{G} \, \partial^\nu \, \mathbf{G}^{-1} \mathbf{G} \, \partial^\lambda \, \mathbf{G}^{-1}$$

simple case:

$$\mathbf{G}^{-1} = ip_0 + \mathrm{H}(p_\mathrm{x}, p_\mathrm{y}) \quad , \quad \mathrm{H}(\mathbf{p}) = \boldsymbol{\tau} \cdot \mathbf{g} \, (p_\mathrm{x}, p_\mathrm{y})$$

$$\mathrm{H}(\mathbf{p}) = \tau_1 \mathrm{g}_1(p_\mathrm{x}, p_\mathrm{y}) + \tau_2 \mathrm{g}_2(p_\mathrm{x}, p_\mathrm{y}) + \tau_3 \mathrm{g}_3(p_\mathrm{x}, p_\mathrm{y})$$

$$N = \frac{1}{4\pi} \; e_{\mu\nu\lambda} \int d^2p \;\; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

Skyrmion in momentum space

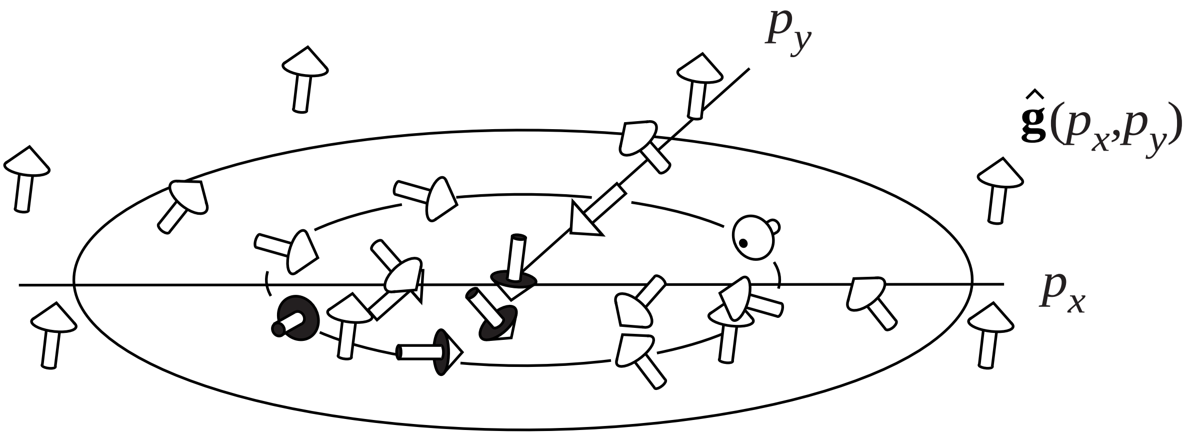
p-wave 2D superconductor & ^{3}He-A film :

$$H = \begin{pmatrix} \frac{p^2 - p_F^2}{2m} & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2 - p_F^2}{2m} \end{pmatrix}$$

$$g_1 = cp_x \qquad g_2 = cp_y$$

$$g_3 = \frac{p^2 - p_F^2}{2m}$$

$N = 1$ skyrmion in momentum space



$$N = \frac{1}{4\pi} e_{\mu\nu\lambda} \int d^2p \ \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}}) = 1$$

N=2 skyrmion in momentum space

$(d_{xx\text{-}yy} + id_{xy})$ -wave 2D superconductor

$$\mathrm{H}(\mathbf{p}) = \boldsymbol{\tau} \cdot \mathbf{g}\,(p_x, p_y)$$

$$g_3 = \frac{p^2 - p_F^2}{2m} \qquad g_1 = a\,(p_x^2 - p_y^2) \qquad g_2 = b p_x p_y$$

$$\mathrm{H} = \begin{pmatrix} \dfrac{p^2 - p_F^2}{2m} & a\,(p_x^2 - p_y^2) + i\,b\,p_x p_y \\ a\,(p_x^2 - p_y^2) - i\,b\,p_x p_y & -\dfrac{p^2 - p_F^2}{2m} \end{pmatrix}$$

$$N = \frac{1}{4\pi}\; e_{\mu\nu\lambda} \int d^2p\;\; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}}) = 2$$

Chern-Simons terms
&
momentum-space invariant
(interplay of *r*-space and *p*-space topologies)

general case of several gauge fields

$$S_{CS} = \frac{1}{16\pi} N_{IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ I} F_{\nu\lambda}^{\ J}$$

***r*-space invariant**

***p*-space invariant protected by symmetry**

$$N_{IJ} =$$

$$\frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr}\, Q_I\, Q_J \int d^2p\, d\omega\, \mathbf{G}\, \partial^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \partial^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \partial^\lambda\, \mathbf{G}^{-1}$$

Q_I - charge interacting with gauge field $A_\mu^{\ I}$

$Q=e$ for electromagnetic field A_μ

$Q=s_z$ for effective spin-rotation field $A_\mu^{\ z}$ $(A_0^{\ z} = \gamma H^{\ z})$

Intrinsic spin-current quantum Hall effect
&
momentum-space invariant

spin current $J_x^{\;z} = \delta S_{CS} / \delta A_x^{\;z}$

$$S_{CS} = \frac{1}{16\pi} N_{IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\;I} F_{\nu\lambda}^{\;J}$$

$$J_x^{\;z} = \frac{1}{4\pi} \left(\gamma N_{ss}\, dH^z/dy + N_{se} E_y\right)$$

2D singlet superconductor: $N_{ss} = N/4$

quantized spin Hall conductivity:

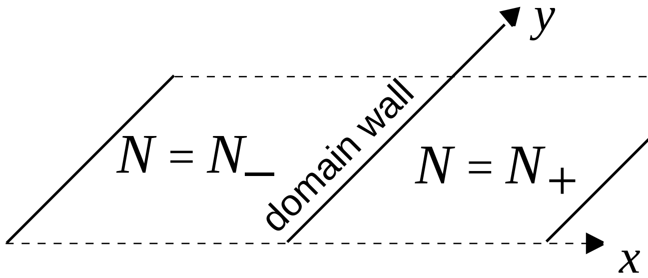
$$\sigma_{xy}^{\;s} = \frac{\gamma N}{16\pi}$$

s-wave: $N = 0$

$p_x + ip_y$: $N = 2$

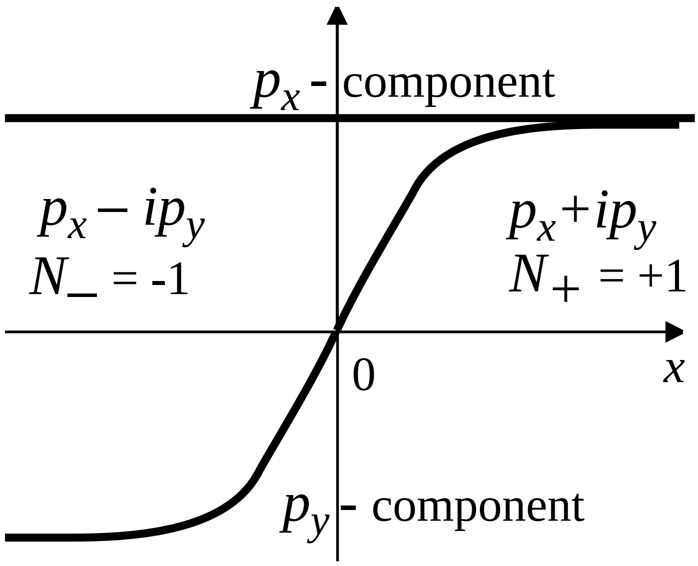
$d_{xx\text{-}yy} + id_{xy}$: $N = 4$

Edge states & momentum-space invariant

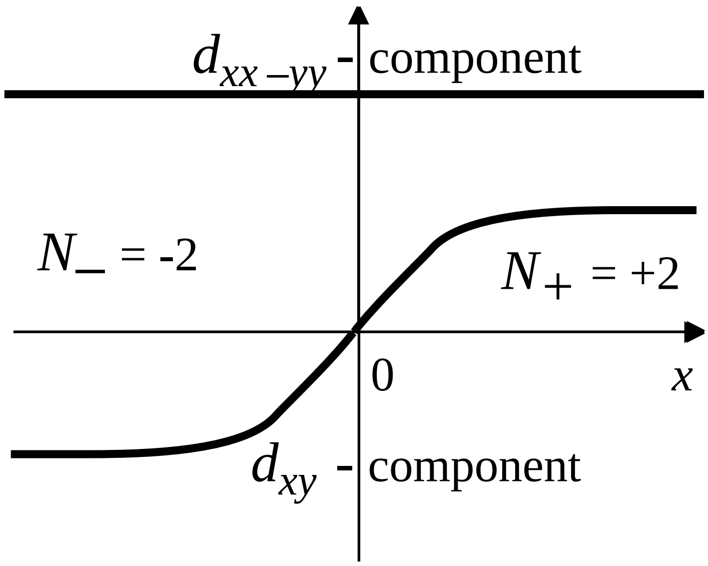


Index theorem: number of edge states $\nu = N_+ - N_-$

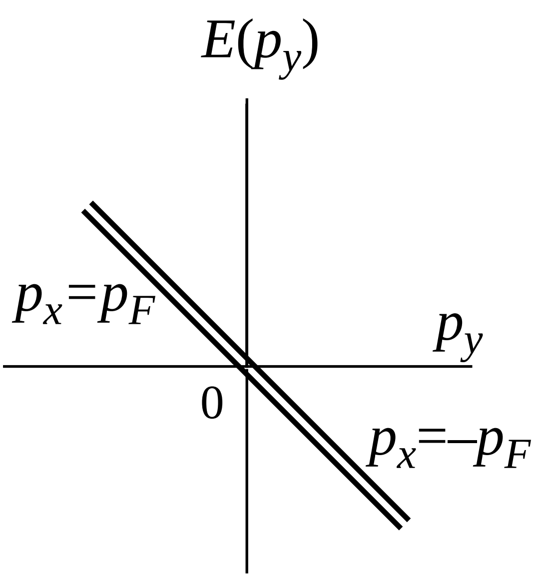
2D superconductors:

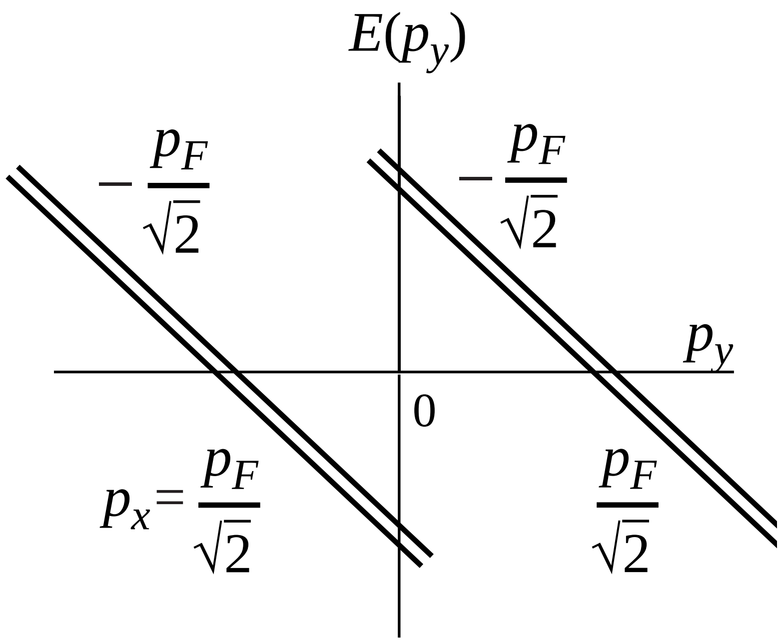


Number of edge states $\nu = 2$

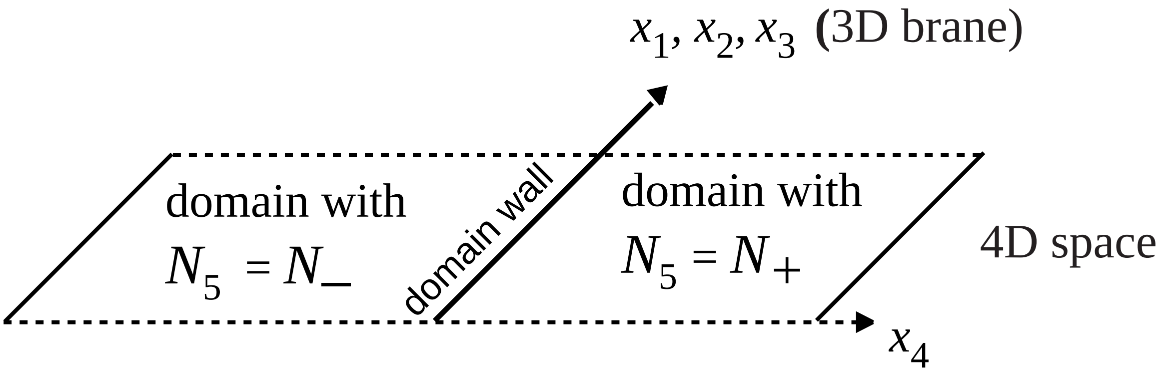


Number of edge states $\nu = 4$





Edge states -- fermion zero modes in higher dimensions & *p*-space invariant



Index theorem:

number of edge states living at the brane (chiral 3+1 fermions)

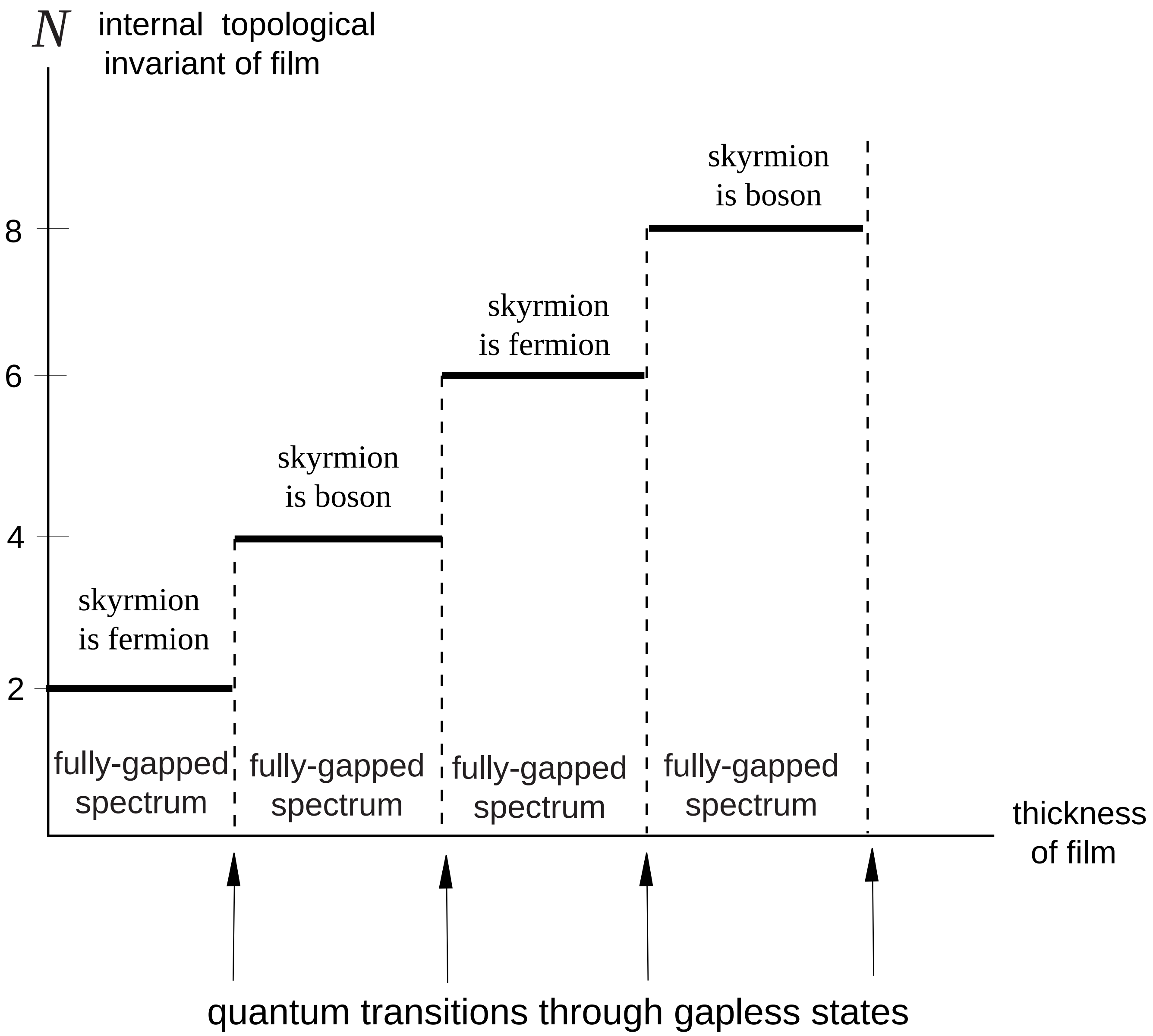
$$\nu = N_+ - N_-$$

$$N_5 = \mathbf{tr} \int d^4p\, d\omega \; (\mathbf{G}\,\partial\,\mathbf{G}^{-1}) \wedge (\mathbf{G}\,\partial\,\mathbf{G}^{-1}) \wedge (\mathbf{G}\,\partial\,\mathbf{G}^{-1}) \wedge (\mathbf{G}\,\partial\,\mathbf{G}^{-1}) \wedge (\mathbf{G}\,\partial\,\mathbf{G}^{-1})$$

Quantum phase transition at T=0
as abrupt change of *p*-space topology

example of ^{3}He-A film

$$N = \frac{1}{24\pi^2} e_{\mu\nu\lambda}\, \mathbf{tr} \int d^2p\, d\omega\, \mathbf{G}\, \partial^\mu \mathbf{G}^{-1}\, \mathbf{G}\, \partial^\nu \mathbf{G}^{-1} \mathbf{G}\, \partial^\lambda \mathbf{G}^{-1}$$

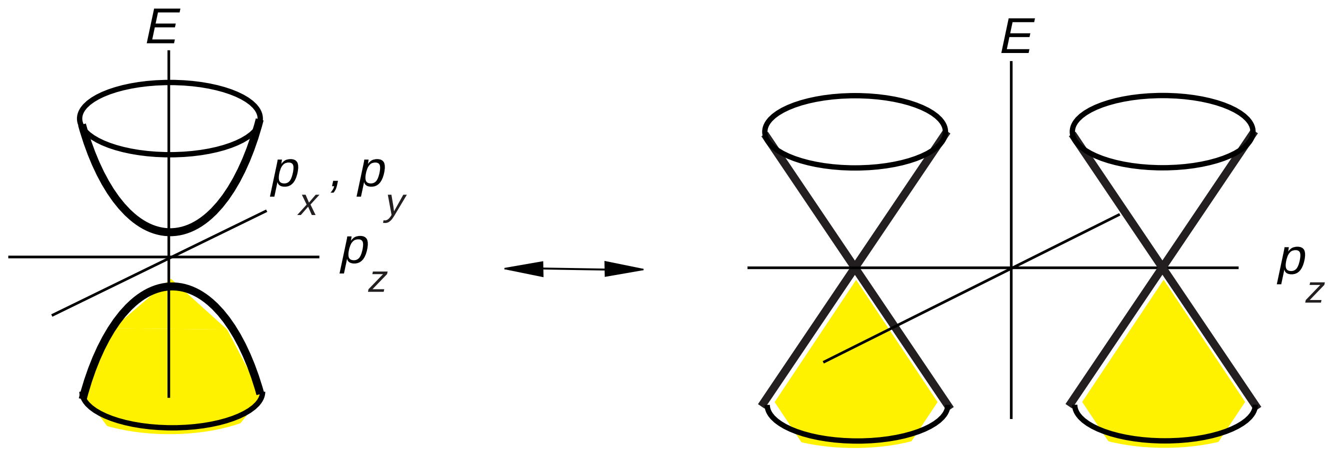


Quantum statistics of solitons (skyrmions)
abruptly changes at quantum phase transition

parameter of Chern-Simons action: $\theta = N\, \pi/2$

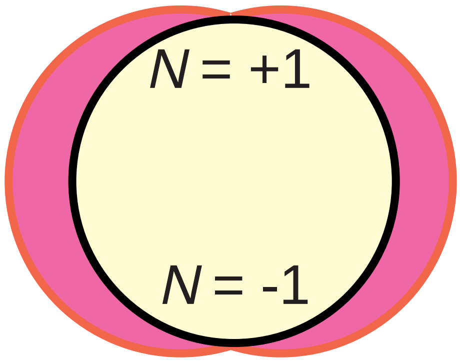
BEC-BCS quantum phase transition

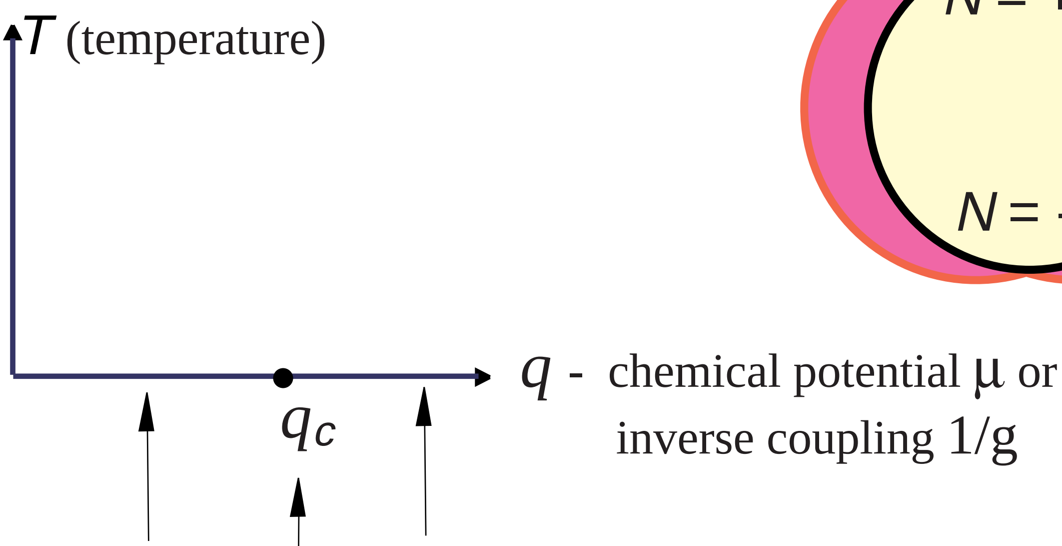
***p*-wave BEC** **p*-wave BCS**



fully gapped spectrum in strong coupling

2 Fermi points in weak coupling regime of BCS





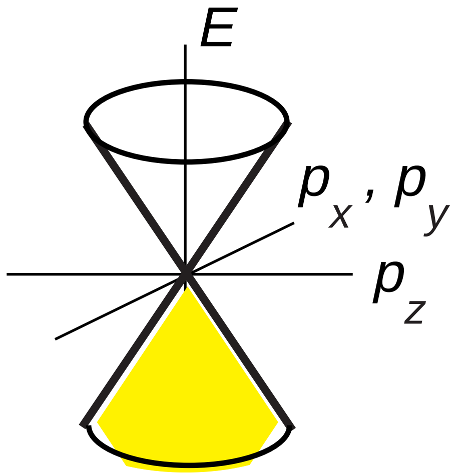
quantum phase transition at $q=q_c$, marginal Fermi point with $N=0$ appears

for $q>q_c$, marginal Fermi point has split into two Fermi points with $N=\pm 1$

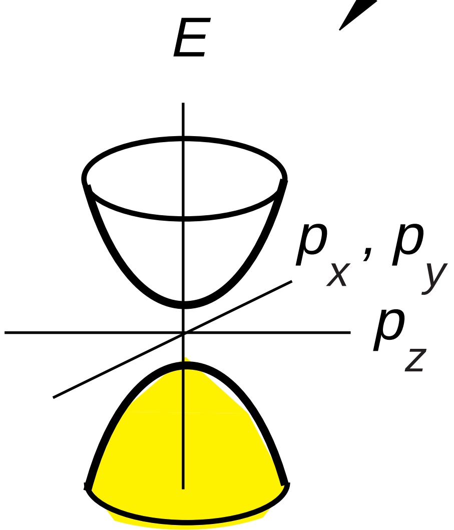
Quantum phase transition in Standard Model and BEC

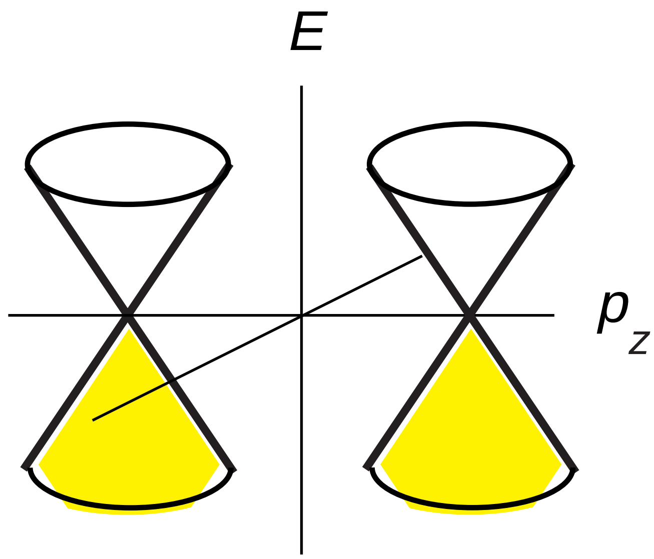
Two topological scenaria in Standard Model

Spectrum
of chiral (left & right)
Weyl fermions
in Standard Model



Marginal
Fermi point
N=+ 1 - 1 =0





Marginal Fermi point
disappears,
massive Dirac fermions are formed

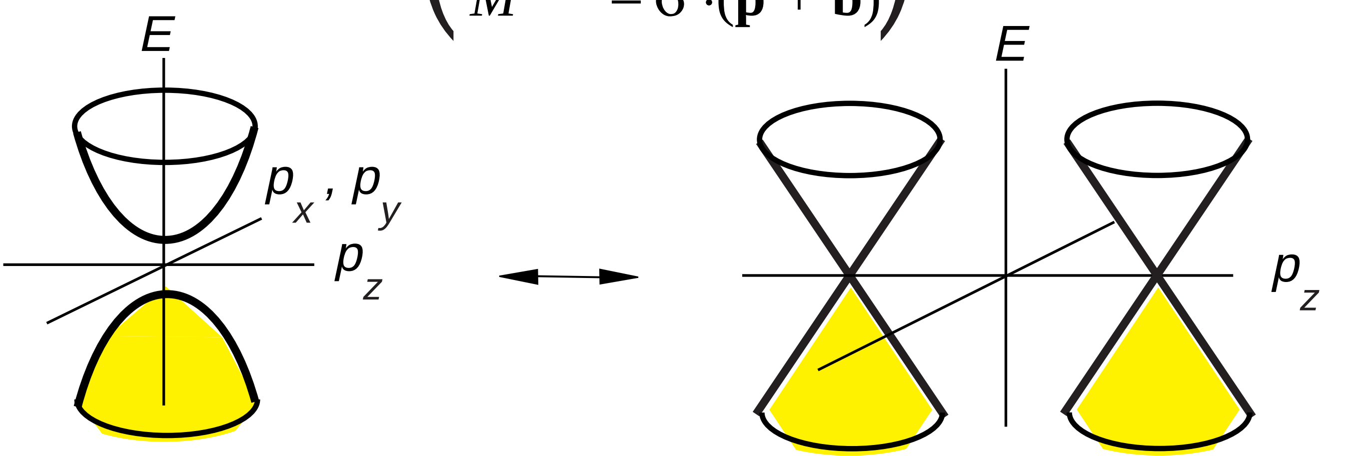
Marginal Fermi point
splits
into topologically protected
Fermi points with N=+1 and N=-1

Topological quantum phase transition in Standard Model

Hamiltonian for neutrino

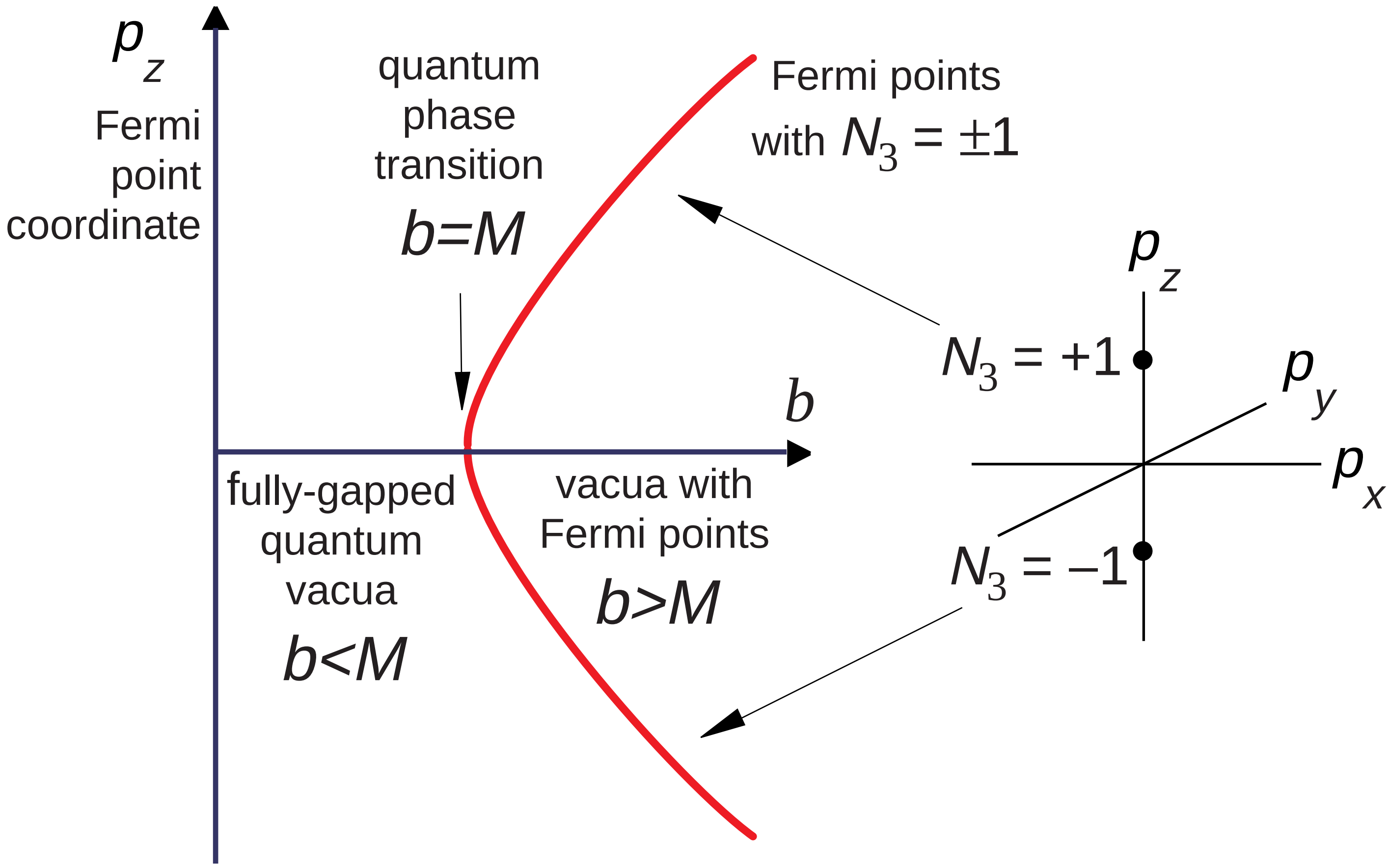
$$H = \begin{pmatrix} \sigma \cdot (\mathbf{p} - \mathbf{b}) & M \\ M & -\sigma \cdot (\mathbf{p} + \mathbf{b}) \end{pmatrix}$$

possible scenario for neutrino since its mass M is small



fully gapped at $b < M$

Fermi points $\mathbf{P}_a = \pm\, \mathbf{b}\, (1 - M^2/b^2)^{1/2}$ appear at $b = M$ & split at $b > M$



Chern-Simons term in 3+1 action from splitting of Fermi points & *p*-space topology

$$S_{CS} = e^{\mu\nu\lambda\gamma} k_\mu \int d^4x \; A_\nu(x) \, \partial_\lambda A_\gamma(x)$$

parameter k is determined by topology in momentum space

topological term in real space

$$\mathbf{k} = \mathbf{N}_{\mathbf{p}QQ} = (1/24\pi^2) \, \Sigma_a \; \mathbf{P}_a \, N_a \, Q_a^{\,2}$$

position of *a*-th Fermi point in p-space

U(1) charge of fermions living near Fermi point

topological charge of *a*-th Fermi point

Conclusion

The momentum-space topology determines
universality classes of quantum vacua

Vacuum of Standard Model belongs to Fermi-point universality class;
elementary particles in vacua of this universality class are
chiral fermions emerging near Fermi points;
gravity and gauge fields are low-energy collective modes,
either fundamental or emerging due to Fermi points

The momentum-space topology determines also:

quantization of Hall and spin-Hall conductivity;
prefactor in topological Chern-Simons & Wess-Zumino terms;
quantum statistics of topological objects;
chiral anomaly & vortex dynamics
spectrum of edge states & fermion zero modes on branes & strings;
quantum phase transitions;
existence of mass of Standard-Model fermions;
etc.

Quantum Fermi liquids

liquid ^{3}He

10^{23} atoms

complicated many-body system of strongly interacting strongly correllated atoms

normal ^{3}He, ^{3}He-A, ^{3}He-B, normal metals, semiconductors, superconductors, etc

Effective theory in low temperature limit

Fermionic quasiparticles + Bosonic collective modes = QFT

elementary particles of effective theory

Type of Quantum Field Theory depends on universality class

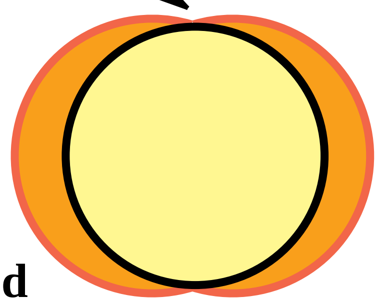
universality class of Fermi points

^{3}He-A, Standard Model

Effective theory in low temperature limit

Chiral fermions + Gauge fields & gravity = Relativistic QFT

left-handed fermions live here



right-handed fermions live here

emergent phenomena:

Gauge invariance
Lorentz invariance
General covariance (partly)
chiral fermions
gauge fields
gravity
spin