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Oscillator in Constant Magnetic Field on CP(N)

(Oscillator on Complex Projective Spaces and Quantum Hall Effect)

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Oscillator on Complex Projective Spaces and Quantum Hall Effect

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Conference on Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and Non-Commutative Geometry in Condensed Matter Physics and Field Theory

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OVERVIEW

- We suggest a model for the oscillator on complex projective spaces CP^N, which is superintegrable. We construct the quantum oscillator interacting with a constant magnetic field B on CP^N, as well as on the non-compact counterparts, i.e. the N-dimensional Lobachewski spaces L_N.
- We find the spectrum and the complete basis of wavefunctions. Surprisingly, the inclusion of B does not yield any qualitative change in the energy spectrum.
- For N>1 B does not break superintegrability of the system, whereas for N=1 it preserves exact solvability.
- We extend the results to cones constructed over CP^N and L_N and perform a Kustaanheimo-Stiefel transformation, in the N=2 case, to 3-dimensional Coulomb-like systems.

INTRODUCTION

- The harmonic oscillator plays a fundamental role in quantum mechanics. On the other hand, there are few articles related with the oscillator on curved spaces.
- The most known generalization of the Euclidean oscillator is the oscillator on curved spaces with constant curvature (sphere and hyperboloid) [Higgs (1979); Leemon (1979)] given by the potential

$$V_{Higgs} = \frac{\omega^2 r_0^2}{2} \frac{\mathbf{x}^2}{x_0^2}, \qquad \epsilon \mathbf{x}^2 + x_0^2 = r_0^2, \quad \epsilon = \pm 1. \tag{1}$$

- This system received much attention since its introduction [Barut, Inomata, Junker (1987) (1990); Bonatsos, Daskaloyannis, Kokkotas (1994); Grosche, Pogosyan, Sissakian (1995); Kalnins, Miller, Pogosyan (2002)] and is presently known under the name of “Higgs oscillator”.

INTRODUCTION

- Recently the generalization of the oscillator to Kaehler spaces has also been suggested, in terms of the potential [Bellucci, Nersessian PRD2003]

$$V_{osc} = \omega^2 g^{\bar{a}b} \partial_{\bar{a}} K \partial_b K. \qquad (2)$$

- Various properties of the systems with this potential were studied [Bellucci, Nersessian PRD2003; Nersessian, Yeranyan JPA2004; Bellucci, Nersessian, Yeranyan PRD2004; Bellucci, Nersessian 2004].
- It was shown [BN PRD2003] that on the complex projective spaces CP^N such a system inherits the whole set of rotational symmetries and part of the hidden symmetries of the 2N-dimensional flat oscillator.

INTRODUCTION

- Then [NY JPA2004], the classical solutions of the system on CP^2, L_2 (the noncompact counterpart of CP^2) and the related cones were presented, and the reduction to three dimensions was studied. Particulary, it was found that the oscillator on some cone related with (CP^2, L_2) yields, after Hamiltonian reduction, a Higgs oscillator on the three-dimensional sphere (two-sheet hyperboloid) in the presence of a Dirac monopole field.
- Thirdly [BNY PRD2004], we presented exact quantum mechanical solutions for the oscillator on CP^2, L_2 and related cones. We also reduced these quantum systems to 3d and performed their (Kustaanheimo-Stiefel) transformation to 3d Coulomb-like systems.
- The ``Kaehler oscillator'' is a distinguished system with respect to supersymmetrisation as well. Its preliminary studies were presented in [BN 2004], [BN PRD2003].

INTRODUCTION

- In this talk we present an exactly solvable model of the quantum oscillator on CP^N, L_N and related cones *in the presence of a constant magnetic field*.
- Our model can be useful in higher-d quantum Hall effect (QHE). This theory was formulated initially on the 4d sphere [Zhang, Hu (Science 2001)] and further included as a particular case, in the theory of QHE on complex projective spaces [Karabali, Nair (NPB2002); Bernevig, Hu, Toumbas, Zhang (PRL2003); Fabinger (JHEP2002); Bellucci, Casteill, Nersessian (PLB 2003); Karabali, Nair (NPB2004); Hasebe, Kimura (2003)].
- The latter version is based on quantum mechanics on CP^N in a constant magnetic field. Our basic observation is that the inclusion of the constant magnetic field does not break any existing symmetries of CP^N-oscillator and, consequently, its exact solvability is preserved.

INTRODUCTION

- Let us consider first the (classical) oscillator on $R^{2N}=C^N$. It is described by the symplectic structure and Hamiltonian

$$\Omega_0 = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a \qquad (3)$$

$$\mathcal{H} = \pi\bar{\pi} + \omega^2 z\bar{z}. \qquad (4)$$

- It has a symmetry group U(2N) given by the generators of SO(2N) rotations

$$J^+_{ab} = \bar{z}^a \pi_b, \quad J^-_{\bar{a}\bar{b}} = z^a \bar{\pi}_b, \quad J_{a\bar{b}} = iz^b \pi_a - i\bar{\pi}_b \bar{z}^a \qquad (5)$$

and the hidden symmetries

$$I^+_{ab} = \pi_a \pi_b + \omega^2 \bar{z}^a \bar{z}^b, \quad I^- = \bar{\pi}_a \bar{\pi}_b + \omega^2 z^a z^b, \qquad (6)$$

$$I_{a\bar{b}} = \pi_a \bar{\pi}_b + \omega^2 \bar{z}^a z^b. \qquad (7)$$

INTRODUCTION

- The oscillator on CP^N (for N>1) [Bellucci, Nersessian (2003)] and that on Lobacewski space L_N are defined by the same symplectic structure as above (3) with Hamiltonian

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{\omega^2 r_0^2}{2} z\bar{z}, \qquad g^{\bar{a}b} = \frac{2}{r_0^2}(1+\epsilon z\bar{z})(\delta^{ab} + \epsilon z^a \bar{z}^b), \qquad \epsilon = \pm 1. \tag{8}$$

- The choice ε =1 corresponds to CP^N and ε =-1 is associated to L_N. This system inherits only part of the rotational and hidden symmetries of the C^N-oscillator given, respectively, by the following constants of motion:

$$J_{a\bar{b}} = i(z^b\pi_a - \bar{\pi}_b\bar{z}^a), \quad I_{a\bar{b}} = 2\frac{J_a^+ J_b^-}{r_0^2} + \frac{\omega^2 r_0^2}{2}\bar{z}^a z^b\ , \tag{9}$$

- where J^+, J^- are translation generators. J defines U(N) rotations, while I is just a CP^N counterpart of (7).

INTRODUCTION

- In order to include a constant magnetic field, we have to leave the initial Hamiltonian unchanged and replace the initial symplectic structure (3) by the following one:

$$\Omega_B = \Omega + iBg_{a\bar{b}}dz^a \wedge d\bar{z}^b, \qquad (10)$$

- where g is a Kaehler metric of the configuration space. N.B. the inclusion of a constant magnetic field preserves only the symmetries of the C^N-oscillator generated by *J* and *I*. On the other hand, the inclusion of the magnetic field preserves all symmetries for the oscillator on CP^N.
- Hence we can be sure that the CP^N-oscillator preserves its classical and quantum exact solvability in the presence of a constant magnetic field.

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- Next, we formulate the Hamiltonian and quantum-mechanical systems describing the CP^N and L_N- oscillators in a constant magnetic field and present their wavefunctions and spectra. We will also extend these results to the cones and discuss some related topics. The classical Kaehler oscillator in a constant magnetic field is defined by the Hamiltonian

$$\mathcal{H} = g^{a\bar{b}}\pi_a\bar{\pi}_b + \omega^2 g^{\bar{a}b}K_{\bar{a}}K_b \quad , \tag{11}$$

and the Poisson brackets corresponding to the symplectic structure (10)

$$\{\pi_a, z^b\} = \delta_a^b, \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_a^b, \quad \{\pi_a, \bar{\pi}_b\} = iBg_{a\bar{b}}. \tag{12}$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- Its canonical quantization assumes the following choice of momenta operators:

$$\widehat{\pi}_a = -i\left(\hbar\partial_a + BK_a/2\right), \quad \widehat{\bar{\pi}}_a = -i\left(\hbar\partial_{\bar{a}} - BK_{\bar{a}}/2\right), \qquad (13)$$

where $\partial_a = \partial/\partial z^a$, $K_a = \partial K/\partial z^a$ (non Hermitean momenta operators: can be redefined). The quantum Hamiltonian looks similar to the classical one

$$\widehat{\mathcal{H}} = \frac{1}{2}g^{a\bar{b}}(\widehat{\pi}_a\widehat{\bar{\pi}}_b + \widehat{\bar{\pi}}_b\widehat{\pi}_a) + \omega^2 g^{\bar{a}b}K_{\bar{a}}K_b. \qquad (14)$$

- In the specific case of the complex projective space CP^N and its noncompact version, i.e. the Lobacewski space L_N, we have to choose

$$K = \frac{r_0^2}{2\epsilon}\log(1+\epsilon z\bar{z}), \qquad K_a = \frac{r_0^2}{2\epsilon}\frac{\bar{z}^a}{1+\epsilon z\bar{z}}, \quad K_{\bar{a}} = \frac{r_0^2}{2\epsilon}\frac{z^a}{1+\epsilon z\bar{z}}. \qquad (15)$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- The scalar curvature R is related with the parameter r_0 : $R=2\varepsilon N(N+1)/r_0^2$.
- These systems possess the u(N) rotational symmetry generators

$$\hat{J}_{a\bar{b}} = i\epsilon(z^b\hat{\pi}_a - \bar{z}^a\hat{\bar{\pi}}_b) + i\epsilon(z^c\hat{\pi}_c - \bar{z}^c\hat{\bar{\pi}}_c)\delta^{\bar{a}b} - \frac{Br_0^2}{2}\frac{\epsilon\bar{z}^a z^b - \delta^{\bar{a}b}}{1+\epsilon z\bar{z}}, \qquad (16)$$

and the hidden symmetry defined by the generators

$$\hat{I}_{a\bar{b}} = \frac{\hat{J}_a^+\hat{J}_b^- + \hat{J}_b^-\hat{J}_a^+}{r_0^2} + \frac{\omega^2 r_0^2}{2}\bar{z}^a z^b\,, \qquad (17)$$

where J_a are the translation generators

$$\hat{J}_a^+ = i\hat{\pi}_a + i\epsilon\bar{z}^a(\bar{z}\hat{\bar{\pi}}) - \frac{Br_0^2}{2}\frac{\bar{z}^a}{1+\epsilon z\bar{z}}, \quad \hat{J}_a^- = -i\hat{\bar{\pi}}_a - i\epsilon z^a(z\hat{\pi}) - \frac{Br_0^2}{2}\frac{z^a}{1+\epsilon z\bar{z}}. \qquad (18)$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- The Hamiltonian (14) can be rewritten as follows:

$$\mathcal{H} = \frac{\hbar^2}{2r_0^2}\left[-\frac{(1+\epsilon x^2)^{N+1}}{x^{2N-1}}\frac{\partial}{\partial x}\left(\frac{x^{2N-1}}{(1+\epsilon x^2)^{N-1}}\frac{\partial}{\partial x}\right) + \frac{4(1+\epsilon x^2)}{\epsilon x^2}\widehat{\mathbf{J}}^2 + \epsilon(1+\epsilon x^2)(2\widehat{J}_0 + \frac{\mu_B}{\epsilon})^2\right] + \\ + \frac{\omega^2 r_0^2 x^2}{2} - \epsilon\frac{\hbar^2\mu_B^2}{2r_0^2}, \qquad (19)$$

where

$$x = |z|, \quad \mu_B = \frac{Br_0^2}{2\hbar}, \quad 2\widehat{J}_0 = z^a\partial_a - \bar{z}^{\bar{a}}\partial_{\bar{a}}, \qquad (20)$$

- $\boldsymbol{J}^2$ is the quadratic Casimir of the SU(N) momentum operator for N>1, and $\boldsymbol{J}=J_0$ for N=1.
- In order to get the energy spectrum of the system, let us consider the spectral problem

$$\widehat{\mathcal{H}}\Psi = E\Psi, \qquad \widehat{J}_0\Psi = s\Psi, \qquad \widehat{\mathbf{J}}^2\Psi = j(j+N-1)\Psi. \qquad (21)$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- Go to 2N-dimensional spherical coordinates (x, Φ_i), where i=1,…, 2N-1, x is a dimensionless radial coordinate taking values in the interval $[0,\infty)$ for ε=+1, and in [0, 1] for ε=-1, and Φ_i are appropriate angular coordinates. There is a convenient algorithm for the expansion of "Cartesian" coordinates to spherical ones. In the new coordinates the system can be solved by choosing the wavefunction:

$$\Psi = \psi(x) D^j_s(\phi_i), \tag{22}$$

where D is the eigenfunction of the operators $\mathbf{J}^2, J_0$. It can be explicitly expressed via 2N-dimensional Wigner functions $D^j_s(\Phi_i)=\Sigma_{m_i} c_{m_i} D^j_{\{m-i\},s}(\Phi_i)$, where j, m_i denote total and azimuthal angular momenta and s is the eigenvalue of J_0

$$m_i, s = -j, -j+1, \ldots, j-1, j \quad j = 0, 1/2, 1, \ldots \tag{23}$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

The energy spectrum is degenerate (as *E* depends on *n*+*j*) and reads

$$\tilde{E} = (2n + 2j + N + \epsilon\delta)^2 \,, \qquad (30)$$

or, explicitly,

$$E_{n+j,\,s} = \frac{\epsilon\hbar^2}{2r_0^2}\left[(2n + 2j + N + \epsilon\delta)^2 - \left(\frac{\omega^2 r_0^4}{\hbar^2} + N^2 + {\mu_B}^2\right)\right] \qquad (31)$$

Here *n* is the radial quantum number with the following range of definition:

$$n = \begin{cases} 0, 1, \ldots, \infty & \text{for } \epsilon = 1 \\ 0, 1, \ldots, n^{max} = [(\delta - j_1 - 1)/2] & \text{for } \epsilon = -1. \end{cases} \qquad (28)$$

- The spectrum is infinite on CP^N and finite on L_N

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- The regular wavefunctions, which form an orthonormal basis of the Schroedinger equation above, read

$$\psi = \begin{cases} C \sin^{j_1-1}\theta \cos^{\delta}\theta \; {}_2F_1(-n, n+\delta+j_1+1; j_1+1; \sin^2\theta), \text{ for } \epsilon = 1 \\ C \sinh^{j_1-1}\theta \cosh^{-\delta+2n}\theta \; {}_2F_1(-n, -n+\delta, j_1+1, \tanh^2\theta), \text{ for } \epsilon = -1, \end{cases} \qquad (27)$$

- The normalization constants are defined by

$$\frac{r_0^{2N} n! \Gamma^2(j_1+1)}{\Gamma(n+j_1+1)} C^2 = \begin{cases} (2n+j_1+1+\delta)\Gamma(n+j_1+1+\delta)/\Gamma(n+1+\delta), & \text{for } \epsilon = 1, \\ (\delta-2n-j_1-1)\Gamma(\delta-n)/\Gamma(\delta-n-j_1), & \text{for } \epsilon = -1 \; . \end{cases} \qquad (29)$$

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- The magnetic flux μ_B is quantized for ε=1 and nonquantized for ε=-1.
- In the flat limit $r_0^2 \to \infty$ we get the correct formula for the 2N-dimensional oscillator energy spectrum

$$E_{\tilde{n},s} = \hbar\sqrt{\omega^2 + \frac{B_0^2}{4}}(\tilde{n} + N) + \hbar B_0 s, \quad \tilde{n} = 0, 1, 2, \ldots, \tag{32}$$

i.e. n=2n+2j becomes the ``principal'' quantum number.

- Thus we get the following wonderful result: the inclusion of a constant magnetic field does not change the degeneracy of the oscillator spectrum on CP^N and L_N.
- For N=1, i.e. on complex projective plane and Lobacewski plane, s=j, hence the spectrum is nondegenerate.

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- For N>1 the spectrum depends on *s* and *n*, i.e. it is degenerate in the orbital quantum number *j*. This degeneracy is due to the existence of a hidden symmetry.
- On the other hand, for N=1 the complex projective plane/ Lobacewski plane coincides with the sphere/two-sheeted hyperboloid, while on these spaces there exists an oscillator system (Higgs oscillator) possessing a hidden symmetry. However, the inclusion of the constant magnetic field not only breaks the hidden symmetry (and the degeneracy of the energy spectrum) of that system, but makes it impossible to get the exact solution of its Schroedinger equation. So, opposite to the Higgs oscillator case, the Kaehler oscillator on a two-dimensional sphere/hyperboloid behaves, with respect to the magnetic field, similarly to the planar one.

OSCILLATOR IN A CONSTANT MAGNETIC FIELD

- In free particle limit, i.e. for $\omega = 0$, the energy spectrum is described by the principal quantum number J, which plays the role of the weight of the SU(N+1) group (when ε =1), and the SU(N.1) group (when ε = -1).
- For example, when ε =1, the energy spectrum reads

$$E_{n+j,\,s} = \frac{2}{r_0^2} J(J+N) - \frac{\hbar^2}{2r_0^2}\mu_B^2, \qquad J = n + j + |s + \frac{\mu_B}{2}|, \quad j \geq |s|. \tag{33}$$

- Now, the ground state becomes degenerate: the lowest value of J is equal to $|\mu_B|/2$.
- Just this degeneracy plays a key role in the use of quantum mechanics on CP^N in the theory of the higher dimensional quantum Hall effect.

CONIC OSCILLATOR

- Our results can be extended to the family of *v*-parametric cones (over CP^N and L_N) defined by the Kaehler potential

$$K = \frac{r_0^2}{2\epsilon}\log(1+\epsilon(z\bar{z})^\nu), \quad \nu > 0; \quad \epsilon = \pm 1. \tag{34}$$

N.B.: these cones have a non-constant curvature R (for v≠1).

- The energy *E* is defined as before

$$\tilde{E} = \frac{2Er_0^2}{\epsilon\hbar^2} + \frac{\omega^2 r_0^4}{\hbar^2} + N^2 + {\mu_B}^2, \qquad \delta^2 = \frac{\omega^2 r_0^4}{\hbar^2} + \left(2s + \frac{\mu_B}{\epsilon}\right)^2, \qquad j_1 = 2j + N - 1. \tag{26}$$

while the parameters *δ* and j_1 look as follows:

$$\delta^2 = \frac{\omega^2 r_0^4}{\epsilon^2\hbar^2} + \left(2\frac{s}{\nu} + \frac{B_0 r_0^2}{2\epsilon\hbar}\right)^2 \qquad j_1^2 = \frac{(2j+N-1)^2}{\nu} + \frac{\nu-1}{\nu}\left((N-1)^2 - \frac{4s^2}{\nu}\right). \tag{37}$$

CONIC OSCILLATOR

- Thus, wavefunctions and energy spectrum of the conic oscillator are defined, respectively, by the expressions

$$\psi = \begin{cases} C \sin^{j_1 - 1}\theta \cos^{\delta}\theta \; {}_2F_1(-n, n+\delta+j_1+1; j_1+1; \sin^2\theta), \text{ for } \epsilon = 1 \\ C \sinh^{j_1-1}\theta \cosh^{-\delta+2n}\theta \; {}_2F_1(-n, -n+\delta, j_1+1, \tanh^2\theta), \text{ for } \epsilon = -1, \end{cases} \tag{27}$$

$$n = \begin{cases} 0, 1, \ldots, \infty & \text{for } \epsilon = 1 \\ 0, 1, \ldots, n^{max} = [(\delta - j_1 - 1)/2] & \text{for } \epsilon = -1. \end{cases} \tag{28}$$

$$\tilde{E} = (2n + 2j + N + \epsilon\delta)^2 \,, \tag{30}$$

- The normalization constants C_{cone} are of the form $C_{cone}=C/v^{N/2}$, where C are still defined by the expressions

$$\frac{r_0^{2N} n! \Gamma^2(j_1+1)}{\Gamma(n+j_1+1)} C^2 = \begin{cases} (2n+j_1+1+\delta)\Gamma(n+j_1+1+\delta)/\Gamma(n+1+\delta), & \text{for } \epsilon = 1, \\ (\delta - 2n - j_1 - 1)\Gamma(\delta - n)/\Gamma(\delta - n - j_1), & \text{for } \epsilon = -1 \;. \end{cases} \tag{29}$$

CONIC OSCILLATOR

- Explicitly (plugging expression of δ), the energy spectrum of the conic oscillator reads

$$E_{n,j,s} = \frac{\epsilon\hbar^2}{2r_0^2}\left[(2n+j_1+1)^2 + \frac{4s^2}{\nu^2} - N^2\right] + \frac{|\epsilon|\hbar^2}{r_0^2}\delta(2n+j_1+1) + \hbar B_0 \frac{s}{\nu}\,. \quad (38)$$

- Hence, the spectrum becomes non degenerate: because j_1 is a nonlinear function of j, contrary to the ν=1 case.

KS-TRANSFORMATION

- There is a well-known Kustaanheimo-Stiefel (KS 1965) transformation relating the 4d oscillator with the 3d Coulomb (and MIC-Kepler Zwanziger 1968, McIntosh-Cisneros 1970) system.
- It allows for a straightforward extension to the oscillator on the 4d sphere S^4 and two-sheet hyperboloids H_4 (Nersessian, Pogosyan PRA2001). The KS-transformation of the oscillator on S^4, H_4 yields the MIC-Kepler system on a 3d two-sheet hyperboloid.
- We generalize here the KS transformation to the oscillator on CP^2 and L_2 and related cones. We show that it results in a MIC-Kepler system on 3d cones over H_3 equipped with the metric (Bellucci, Nersessian, Yeranyan PRD2004):

$$ds^2_{\nu,R_0^2} = \frac{8\nu R_0^2 y^{2\sqrt{\nu}-2}(dy)^2}{(1-y^{2\sqrt{\nu}})^2}, \qquad R_0 = r_0^2. \tag{39}$$

KS-TRANSFORMATIONS

- The Hamiltonian of the system is given by the expression

$$\widehat{\mathcal{H}}_{MIC} = \frac{1}{\sqrt{g}}\widehat{\pi}_i\sqrt{g}g^{ij}\widehat{\pi}_j + s^2\hbar^2\frac{(1-y^{2\sqrt{\nu}})^2}{8\nu^2R_0^2y^{2\sqrt{\nu}}} - \frac{\gamma}{2R_0}\frac{1+y^{2\sqrt{\nu}}}{y^{\sqrt{\nu}}}, \quad (40)$$

$$\widehat{\boldsymbol{\pi}} = -\imath\hbar\frac{\partial}{\partial\mathbf{y}} - s\mathbf{A}(\mathbf{y}), \quad [\widehat{\pi}_i, \widehat{\pi}_j] = \hbar s\epsilon_{ijk}\frac{y^k}{y^3}. \quad (41)$$

- The coordinates of the initial and final systems are related as:

$$\mathbf{y} = \left(\sqrt{1+\epsilon(z\bar{z})^\nu} - 1\right)^{2/\nu}\frac{z\boldsymbol{\sigma}\bar{z}}{(z\bar{z})^2} \quad (42)$$

- The energy and coupling constant *γ* of this system are defined by the energy and frequency of the respective 4d oscillator. The quantum number *s*=±0,½,1,… becomes a fixed parameter (“monopole number”), and instead of (23):

$$j = |s|, |s|+1, \ldots; \qquad m = -j, -j+1, \ldots, j-1, j. \quad (43)$$

$$m_i, s = -j, -j+1, \ldots, j-1, j \quad j = 0, 1/2, 1, \ldots \quad (23)$$

KS-TRANSFORMATIONS

- It appears that applying the KS-transformation to the 4d oscillator in a constant magnetic field, we have to get the modification of the MIC-Kepler system on the 3d hyperboloid (and related cones), which nevertheless remains superintegrable (exactly solvable).
- Surprisingly, repeating the whole procedure, we find that the inclusion of a magnetic field in the initial system yields, in the resulting system, a redefinition of the coupling constant γ and the energy ε only

$$\gamma = \frac{E_{osc}}{2} + \frac{\epsilon\hbar^2}{r_0^2}\left(1 - \frac{s^2}{\nu^2}\right) - \frac{B_0}{2\nu}\hbar s, \quad -2\mathcal{E} = \omega^2 + \frac{\epsilon E_{osc}}{r_0^2} + \frac{\hbar^2}{r_0^4}\left(1 + 2\frac{s^2}{\nu^2}\right) + \frac{\epsilon B_0}{r_0^2 \nu}\hbar s. \qquad (44)$$

- We can now convert the energy spectrum of the oscillator in the energy spectrum of the MIC-Kepler system

$$\mathcal{E} = -\frac{2\left(\gamma - \epsilon\hbar^2(2n + j_1 + 1)^2/(4R_0)\right)^2}{\hbar^2(2n + j_1 + 1)^2} - \frac{\epsilon\gamma}{R_0} + \frac{\hbar^2}{2R_0^2}, \qquad (45)$$

where j_1 is still defined by the expression (37).

SUMMARY AND CONCLUSIONS

- We have shown that the inclusion of a constant magnetic field preserves the hidden symmetries of the oscillator on the complex projective space CP^N, in its noncompact version, i.e. the Lobacewski space L_N.
- We constructed the complete basis of wavefunctions of these systems and their spectra and found that the inclusion of a constant magnetic field does not change the qualitative quantum properties of the systems. In particular, the inclusion of the magnetic field does not change the degeneracy of the energy spectra.
- These results are extended to the oscillators on cones related with CP^N and L_N.
- In some sense, we have shown that the oscillators on CP^N and L_N (with and without constant magnetic field) are more similar to the oscillator on C^N in the presence of a constant magnetic field, than to the one in its absence.

SUMMARY AND CONCLUSIONS

- Another observation concerns the reduction of the 4d oscillator and the 3d Coulomb-like system (KS transformation): we found, to our surprise, that the oscillators with and without (constant) magnetic fields result in the equivalent Coulomb-like systems.
- The oscillator on CP^3 does not allow for a reduction to S^4 contrary to the Landau problem on CP^3, which results in the particle on S^4 in the presence of an instanton, relating the configurations of two different QHE theories (Zhang-Hu with Karabali-Nair).
- Notice, that for N=1 our system remains exactly solvable in the presence of magnetic field, though it has no hidden symmetries.
- In contrast to our model, the well-known Higgs oscillator on CP^N and L_N looses its exact solvability property in the presence of constant magnetic field, while in its absence it has hidden symmetries.

SUMMARY AND CONCLUSIONS

- So, one can suppose that the considered system would preserve the exact solvability, also in the noncommutative case.
- Such a modification is interesting due to the interesting rotational properties of noncommutative quantum mechanics in a constant magnetic field, observed first for the planar case [Bellucci, Nersessian, Sochichiu 2001] and later on extended to the two-dimensional sphere and hyperboloid [Bellucci, Nersessian 2002].
- While the noncommutative planar oscillator with constant magnetic field remains superintegrable [Nair, Polychronakos 2001], on the noncommutative spheres and hyperboloids only the particle systems without potential terms do [Iengo, Ramachandran 2002, Karabali, Nair, Polychronakos 2002].

Recent related publications of the group

Mechanics on CP(N):

1) S. Bellucci, A. Nersessian, A. Yeranyan, Phys. Rev. D70 (2004) 085013.
2) S. Bellucci, A. Nersessian, A. Yeranyan, D70 (2004) 045006.
3) S. Bellucci, A. Nersessian, "Supersymmetric Kaehler oscillator in a constant magnetic field“, hep-th/0401232, Proc. SQS03, Dubna, Russia, 2003, p. 379.
4) S. Bellucci, A. Nersessian, Phys. Rev. D67 (2003) 065013.
5) S. Bellucci, P.-Y. Casteill, A. Nersessian, Phys. Lett. B574 (2003) 121.

Supersymmetric mechanics:

1) S. Bellucci, S. Krivonos, A. Sutulin, Phys. Lett. B605 (2005) 406.
2) S. Bellucci, S. Krivonos, A. Shcherbakov, "Two-dimensional supersymmetric mechanics in superspace", hep-th/0502245, to appear in Phys. Lett. B.
3) S. Bellucci, S. Krivonos, A. Nersessian, A. Shcherbakov, "2k-dimensional N=8 supersymmetric quantum mechanics“, hep-th/0410073.
4) S. Bellucci, S. Krivonos, A. Nersessian, Phys. Lett. B605 (2005) 181.
5) S. Bellucci, A. Nersessian, "Hamiltonian Model of N=4 Susy Mechanics", Proc. SQS01, Dubna, Russia, 2001, p.395.
6) S. Bellucci, A. Nersessian, Phys. Rev. D64 (2001) 021702(R).
7) S. Bellucci, A. Nersessian, Nucl. Phys. Proc. Suppl. 102 (2001) 227.

Recent related publications of the group

Noncommutative mechanics:

1) S. Bellucci, A. Yeranyan, hep-th/0412305, to appear in Phys. Lett. B.
2) S. Bellucci, Phys. Rev. D67 (2003) 105014.
3) S. Bellucci, A. Nersessian, Phys. Lett. B542 (2002) 295.
4) S. Bellucci, A. Nersessian, C. Sochichiu, Phys. Lett. B522 (2001) 345.

Spin chains and matrix models:

1) S. Bellucci, P.-Y. Casteill, A. Marrani, C. Sochichiu, Phys. Lett. B607 (2005) 180.
2) S. Bellucci, C. Sochichiu, "On matrix models for anomalous dimensions of super Yang-Mills theory", hep-th/0410010, submitted to Nucl. Phys. B.
3) S. Bellucci, P.-Y. Casteill, J.F. Morales, C. Sochichiu, Nucl. Phys. B707 (2005) 303.
4) S. Bellucci, P.-Y. Casteill, J.F. Morales, C. Sochichiu, Nucl. Phys. B699 (2004) 151.
5) S. Bellucci, P.-Y. Casteill, J.F. Morales, C. Sochichiu, hep-th/0408102.
6) S. Bellucci, C. Sochichiu, Phys. Lett. B571 (2003) 92.
7) S. Bellucci, C. Sochichiu, Phys. Lett. B564 (2003) 115.
8) S. Bellucci, C. Sochichiu, "On the dynamics of BMN operators of finite size and the model of string bits", hep-th/0404143.