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Formation of Turbulence Closure Model 2.5 (M-Y)

B. Rajkovic
University of Belgrade
Serbia & Montenegro

Statement of the problem

- Closure of the equations for the second moments (momentum & heat fluxes and pot. t. variance) leads to a complex system of 10 (11) differential equations.

$$\frac{\partial}{\partial t}\overline{u_i u_j}+\frac{\partial}{\partial x_k}\left(U_k\overline{u_i u_j}+\overline{u_i u_j u_k}-\nu\frac{\partial}{\partial x_k}\overline{u_i u_j}\right)+\frac{\partial}{\partial x_i}\overline{p u_j}+\frac{\partial}{\partial x_j}\overline{p u_i}+f_k\left(\varepsilon_{jkl}\overline{u_l u_i}+\varepsilon\right.$$

$$=-\overline{u_i u_k}\frac{\partial U_j}{\partial x_k}-\overline{u_j u_k}\frac{\partial U_i}{\partial x_k}-\beta\left(g_j\overline{u_i\theta}+g_i\overline{u_j\theta}\right)+\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

$$\frac{\partial}{\partial t}\overline{u_i\theta}+\frac{\partial}{\partial x_k}\left(U_k\overline{u_i\theta}+\overline{\theta u_i u_k}-\alpha\overline{u_i\frac{\partial\theta}{\partial x_k}}-\overline{\nu\theta\frac{\partial u_i}{\partial x_k}}\right)+\frac{\partial}{\partial x_i}\overline{p\theta}+f_k\epsilon_{ikl}\overline{u_l\theta}$$

$$=-\overline{u_i u_k}\frac{\partial\Theta}{\partial x_k}-\overline{\theta u_k}\frac{\partial U_i}{\partial x_k}-\beta g_i\overline{\theta^2}+\overline{p\frac{\partial\theta}{\partial x_i}}-(\alpha+\nu)\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial\theta}{\partial x_k}}$$

$$\frac{\partial}{\partial t}\overline{\theta^2}+\frac{\partial}{\partial x_k}\left(U_k\overline{\theta^2}+\overline{u_k\theta^2}-\alpha\overline{\frac{\partial\theta^2}{\partial x_k}}\right)=-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\alpha\overline{\frac{\partial\theta}{\partial x_k}\frac{\partial\theta}{\partial x_k}}.$$

group A:	**group C:**
$\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$	$\overline{u_i u_j u_k}$
$\overline{p\frac{\partial \theta}{\partial x_j}}$	$\overline{\theta u_i u_k}$
	$\overline{\theta^2 u_k}$
group B:	**group D:**
$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$	$\overline{p u_j}$
$(\alpha+\nu)\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial \theta}{\partial x_k}}$	$\overline{p\theta}$
$(2\alpha)\overline{\frac{\partial \theta}{\partial x_k}\frac{\partial \theta}{\partial x_k}}$	

Relations are linear

All constitutive tensors are *isotropic*.

- Based on the ordering in terms of deviation from isotropy MY have created three level hierarchy of equations (expanded to four for historical reasons).

- The fourth level is the original one with all equations.

- The first simplification, the third level, leads to two differential equations for *tke* and $\overline{\theta^2}$. It was, on the empirical grounds, reduced by Yamada to the single equation for *tke*. Being in the middle, between levels $2$and 3, it was termed 2.5 turbulence closure model.

- The second lavel has only algerbaric equations. Later, Galperin further simplifies 2.5 level by reducing the *tke* equation.

$$\frac{\partial}{\partial t}\left(\frac{q^2}{2}\right) - \frac{\partial}{\partial z}\left[lqS_q\frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right] = P_s + P_b - \epsilon$$

$$P_s = -\overline{wu}\frac{\partial U}{\partial z} - \overline{wv}\frac{\partial V}{\partial z} \quad ; \quad P_b = \beta g\overline{w\theta} \quad ; \quad \epsilon = \frac{q^3}{B_1 l}$$

$$\overline{wu} = -K_M\frac{\partial U}{\partial z} \quad ; \quad \overline{wv} = -K_M\frac{\partial V}{\partial z} \quad ; \quad \overline{w\theta} = -K_H\frac{\partial \Theta}{\partial z}$$

$$K_M = lqS_M \quad ; \quad K_H = lqS_H$$

$$G_M = \frac{l^2}{q^2}\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right] \quad ; \quad G_H = -\frac{l^2}{q^2}\beta g\frac{\partial \Theta}{\partial z}$$

$$S_M(6A_1A_2G_M) + S_H(1 - 3A_2B_2G_H - 12A_1A_2G_H) = A_2$$

$$S_M(1 + 6A_1^2G_M - 9A_1A_2G_H) - S_H(12A_1^2G_H + 9A_1A_2G_H) =$$

$$A_1(1 - 3C_1)$$

where A_1, A_2, B_1, B_2 and C_1 are constants derived from the **neutral** case.

- When applied to a simulation of a growing boundary layer, occasionally problems were encountered that values of shear and bouncy forcing which are possible in the nature would give physically unrealistic (large) mixing which means to large levels of *tke*. Mathematically the problem comes in trying to solve the algebraic equations for S_M and S_H. In the unstable case, the determinant of the system may approach zero.

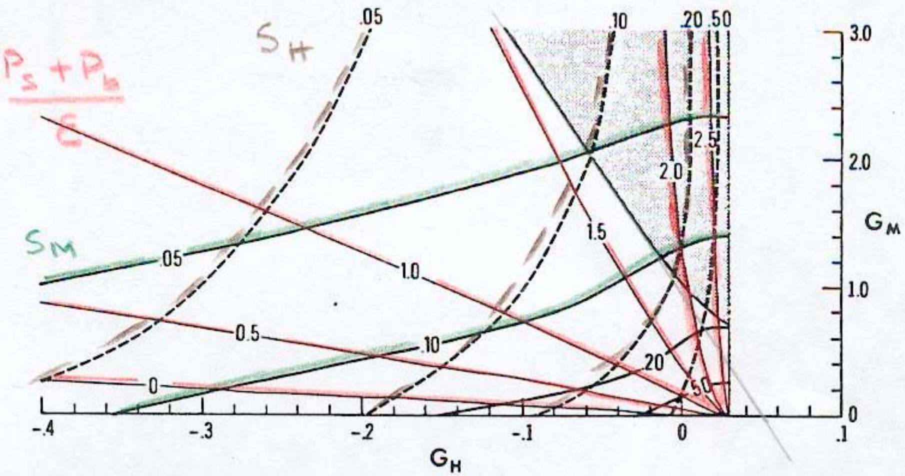


Fig. 3. The stability functions $S_M(G_H, G_M)$ and $S_H(G_H, G_M)$. The heavy solid lines are contours of S_M, whereas the dashed lines are contours of S_H. The lighter solid lines are contours of $(P_s + P_b)/\varepsilon$. One could also draw lines of constant $R_i = G_H/G_M$, which are radial lines on this diagram. The shaded portion is where $\langle w^2 \rangle/q^2 \leq 0.12$.

- This ZJ terms the realizability problem (physically acceptable forcing producing unrealistic response). There are more specific variants of this statement.

HISTORY

- A comprehensive analysis of the Level 2.5 model was performed by HL88. However, they examined the realizability of the model in the space of stability and shear parameters that were dependent on both the turbulence variables and the large-scale driving flow.
- Important further step was made by GBT94 who examined the dependence of the turbulent kinetic energy (TKE) production on the Ri_g number. The Ri_g number was computed from the large-scale variables alone, which provided a clearer insight into the relationship between the turbulence and the driving flow.

Joey Gerritie's ...analysis (GBT94)

The MY Level 2.5 turbulence closure model is governed by the equations (MY82):

$$\frac{\partial}{\partial t}\left(\frac{q^2}{2}\right) - \frac{\partial}{\partial z}\left[lqS_q\frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right] = P_s + P_b - \epsilon$$

or

$$\frac{\partial}{\partial t}\left(\frac{q^2}{2}\right) = -\frac{q^3}{B_1 l}\frac{\left(q^2 - r_1\right)\left(q^2 - r_2\right)}{\left(q^2 - p_1\right)\left(q^2 - p_2\right)}$$

which is of the form

$$\frac{\partial q}{\partial t} = Aq^2$$

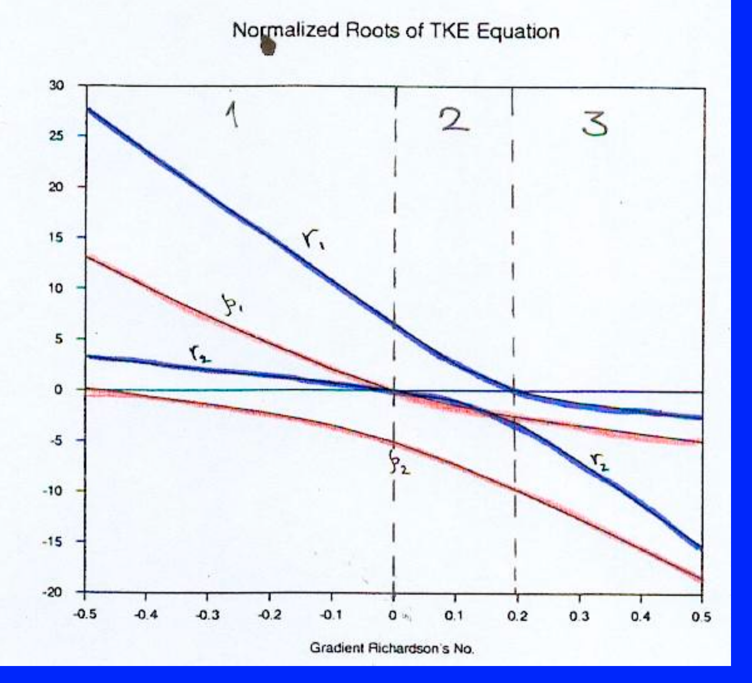
solution (with $A = const$) is

$$q(t) = \frac{q(t_0)}{1 - A\, q_0\, (t - t_0)}$$

- $A < 0 \Rightarrow \quad q \downarrow$

- $A > 0 \Rightarrow \quad q \uparrow$ if $t - t_0 < \frac{1}{Aq_0}$

In region $(1) r_1 > p_1 > r_2 > p_2$ so with $q > p_1 \quad q \Rightarrow r_1$ In region (2), where shear prod. is stronger then boyoncy destruction, with $q > 0 \quad q \Rightarrow r_1$ In region $(3) \quad q > 0 \quad q \Rightarrow 0$



Zaviša Janjić's analysis

- Although the Richardson number covers the whole range of stability and shear, it has a singularity for the case of vanishing wind shear. In order to avoid a special treatment of this singularity, a two-dimensional space will be used, with the stability and shear of the driving flow on the coordinate axes.

With the q $l > 0$, eqation for *tke* could be rewritten as

$$l\frac{\partial}{\partial t}\left(\frac{1}{q}\right) = -\left[S_M G_M + S_H G_H - \frac{1}{B_1}\right]$$

Instead of G_M and G_H, ZJ introduces new parameters

$$g_m = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \quad ; \quad g_h = \frac{\partial \Theta}{\partial z}$$

$$l\frac{\partial}{\partial t}\left(\frac{1}{q}\right) = \left[\frac{A\left(\frac{l}{q}\right)^4 + B\left(\frac{l}{q}\right)^2}{C\left(\frac{l}{q}\right)^4 + D\left(\frac{l}{q}\right)^2 + 1} - \frac{1}{B_1}\right]$$

- Roots P_1 and P_2 of the denominator are poles which can lead to instabilities.
- Both are real for very wide range of the forcing parameters g_m and g_h.

Again with g_m and g_h condition for instability can be rewritten in the form

$$C\left(\frac{l}{q}\right)^4 + D\left(\frac{l}{q}\right)^2 + 1 = 0.$$

- Solutions (roots), $(l/q)^2$, are positive in the relevant part of the g_hxg_m plane, except at the coordinate origin $g_h = g_m = 0$ where it vanishes.
- The requirement for the equilibrium has the form

$$S_M G_M + S_H G_H - \frac{1}{B_1} = 0.$$

which may be rewritten as

$$E\left(\frac{l}{q}\right)^4 + F\left(\frac{l}{q}\right)^2 + 1 = 0.$$

Comparison with the results of GBT94, form the equations for $(q/l)^2$. In this way we have, for the singularity condition

$$\left(\frac{l}{q}\right)^4 + D\left(\frac{l}{q}\right)^2 + C = 0.$$

and for the equilibrium condition

$$\left(\frac{l}{q}\right)^4 + F\left(\frac{l}{q}\right)^2 + E = 0.$$

This equation has two real zeros, S_1 and S_2 with $S_1 > S_2$ in the unstable region. Similar to JG, for the unstable region $S_1 > P_1 > S_2 > P_2$ so that condition

$$\frac{q}{l} > \sqrt{P_1} \qquad or \qquad \frac{l}{q} < \frac{1}{\sqrt{P_1}}$$

What about the STABLE situation ?

Consider the decaying turbulence case and the ratio

$$U = \frac{\overline{w^2}}{q^2} = \frac{\overline{w^2}}{q^2} = \frac{1}{3} - 2A_1 S_M G_M + 4A_1 S_H G_H.$$

Now consider the vanishing *equilibrium* case i.e.

$$\frac{q}{l} = s_1 = 0$$

which is on the line

$$g_m = R_{eq} g_h$$

This menas that free term in the equation for U must be zero. That is again biqudratic equation and has two roots, t_1 and t_2 with t_2 negative in this case. This value of $U = U_{min}$ i.e

$$U \geqq U_{min}$$

This condition transaltes into

$$\boxed{\frac{l}{q} < \frac{1}{\sqrt{t_1}}}$$

R E S I M E

For both, *stable* and *unstable* conditions we can write

$$0 < l < a \cdot q$$

where

$$a = \sqrt{\frac{1}{p_1}} \quad , \quad p_1 > 0 \quad ; \quad a = \sqrt{\frac{1}{t_1}} \quad , \quad p_1 < 0$$

- This implies that the master length scale should locally approach zero for vanishing turbulent kinetic energy. But also puts the limit for the growing turbulence and thus limits the growths of K_m and K_h !
- The explanation of the realizability problem proposed here is that in the case of growing turbulence the diagnostic method for calculating l overestimates the master length scale in the unstable and neutral ranges for a given level of *tke*, leading to a violation of this criterion.
- The proposed interpretation suggests that the non-singularity problem in the unstable range should be controlled by restricting the diagnostically computed master length scale l using. This condition is interpreted as the upper limit on l in the stable range as well.