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1) Workshop **on Design and Use of Regional Weather Prediction Models,** *April 11 - 19*

2) Conference on **Current Efforts Toward Advancing the Skill of Regional Weather Prediction. Challenges and Outlook,** *April 20 - 22*

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Eta Model Dynamics

Gravity Inertial Terms and Nonlinear Advection

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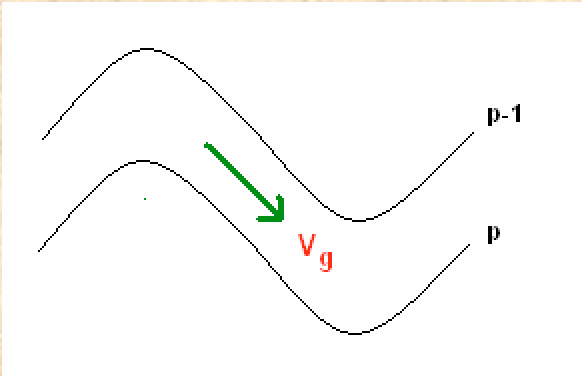
Key Issues of Atmospheric Dynamics

1) Process of geostrophic adjustment
2) Slowly changing quasi-geostrophic component of atmospheric flow

- This lecture will address the treatment of these two dynamical process in the Eta model
- More precisely, the lecture will explain how Eta model approximate the terms in the governing equations that are responsible for these two processes, that is:
  - Gravity-inertial (adjustment) terms
  - Nonlinear momentum advection terms

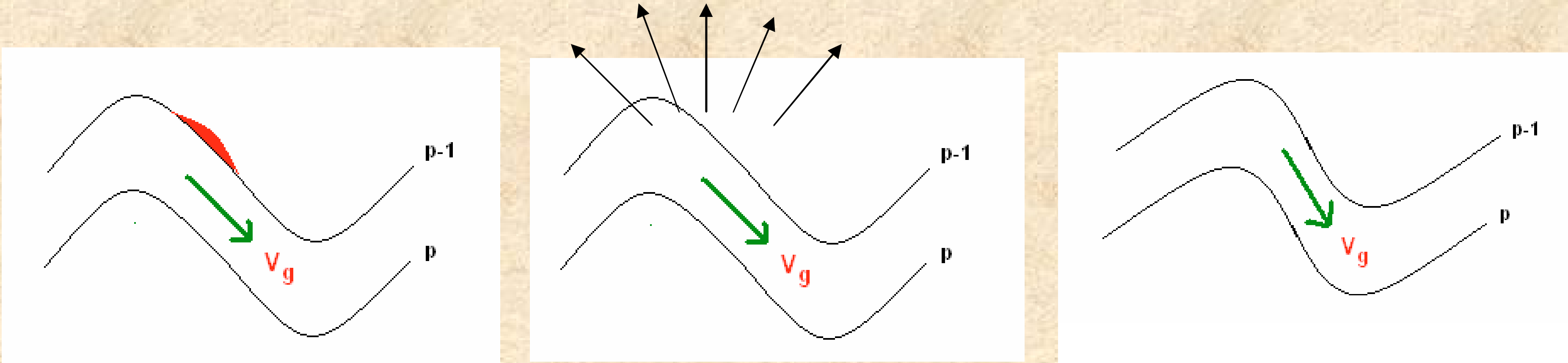
How does extratropial atmosphere work?

- There is a slowly varying (a quasi-geostrophic) balance between mass and wind fields



- Any forcing (such as, latent heat release during precipitation, orographic lifting, etc.) pushes the atmosphere out of balance

- Atmosphere reacts by exciting the gravity-inertial waves, until a new balance, slightly different than previous, is established



- In addition, the quasi-geostrophic flow is slowly changing by itself – for which the nonlinear advection is largely responsible

- The process of slipping off the balance and geostrophic readjustment is in the atmosphere happening simultaneously
- Similarly, the geostrophic adjustment and quasi-geostophic motion are also happening simultaneously
- It is just convenient for theoretical deliberations to separate them and consider them separately

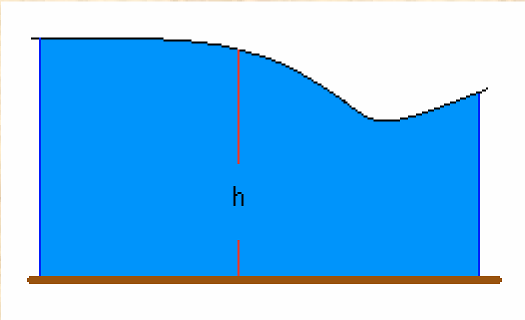
System of shallow water equations

- Shallow water model is a simple model that contains many important solutions as the full 3D atmosphere, and is therefore often used for theoretical studies

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - g\frac{\partial h}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} = -\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - g\frac{\partial h}{\partial y} - fu$$

$$\frac{\partial h}{\partial t} = -\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) - h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$



$$u, v, h = f(x, y, t)$$

Geostrophic Adjustment Terms

- Linearized system

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial h}{\partial y} - fu$$

$$\frac{\partial h}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

- Basic state is assumed to be that of zero motions and with a constant height h=H
- We will further assume that Coriolis parameter f=const
- Let us look for solutions of this system in the form of

$$\begin{bmatrix} u \\ v \\ h \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} e^{i(kx+ly-\nu t)}$$

- By replacing these solutions in the system, it becomes:

$$-i\nu\hat{u} = -gik\hat{h} + f\hat{v}$$

$$-i\nu\hat{v} = -gil\hat{h} - f\hat{u}$$

$$-i\nu\hat{h} = -H(ik\hat{u} + il\hat{v})$$

- … which could be rewritten using matrix notation as

$$(-i\nu)\begin{bmatrix}\hat{u}\\ \hat{v}\\ \hat{h}\end{bmatrix} = \begin{bmatrix}0 & f & -gik\\ -f & 0 & -gil\\ -Hik & -Hil & 0\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{v}\\ \hat{h}\end{bmatrix}$$

- By a closer inspection, we see that this is a classical formulation of an eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

- … with matrix **A**, vector **x** and eigenvalue λ defined respectively as

$$\mathbf{A} = \begin{bmatrix} 0 & f & -ik \\ -f & 0 & -il \\ -Hik & -Hil & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix} \qquad \lambda = (-i\nu)$$

- We can rewrite the eigenvalue equation above as

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{I}\mathbf{x}$$

Identity matrix

- This matrix equation may be rewritten as

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

- …or in the explicit form

$$\begin{bmatrix} i\nu & f & -gik \\ -f & i\nu & -gil \\ -Hik & -Hil & i\nu \end{bmatrix}\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{bmatrix}=0$$

- For a nontrivial solution is required that the matrix is singular, that is, that its determinant is equal to zero:

$$(i\nu)\left[(i\nu)^2-(-gil)(-Hil)\right]-f\left[-f(i\nu)-(-gil)(-Hik)\right]$$
$$+(-gik)\left[f(-Hil)-(i\nu)(-Hik)\right]=0$$

$$\nu\left[-\nu^2 + f^2 + gH(k^2 + l^2)\right] = 0$$

- We get two solutions

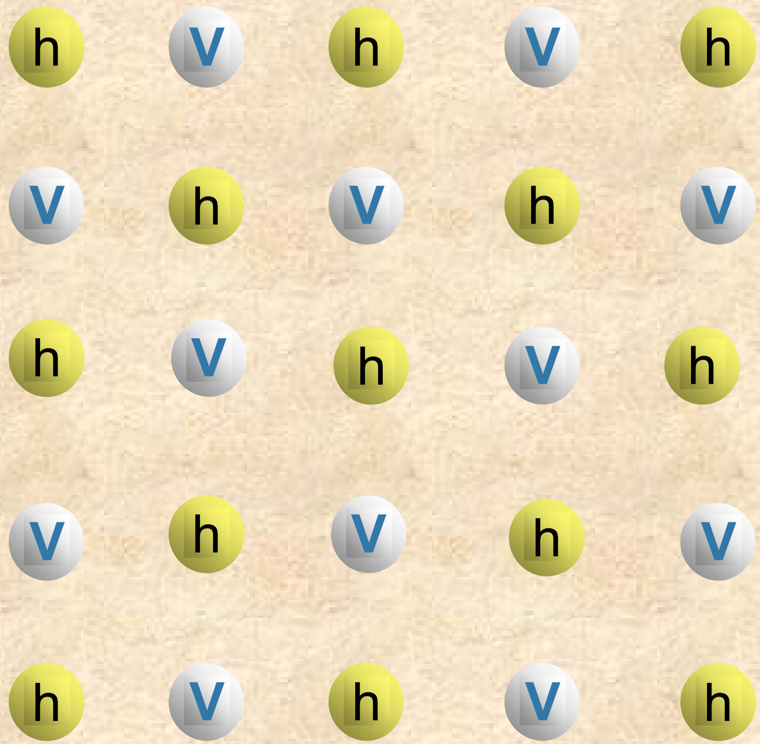
$$\nu = 0$$ Geostrophic mode

$$\nu = \pm\sqrt{f^2 + gH(k^2 + l^2)}$$ Gravity-inertial waves

- For g=0 we get pure inertial oscilation
- For f=0 we get pure gravity waves

- How does the Eta model approximate these terms?

E-grid (Eta)

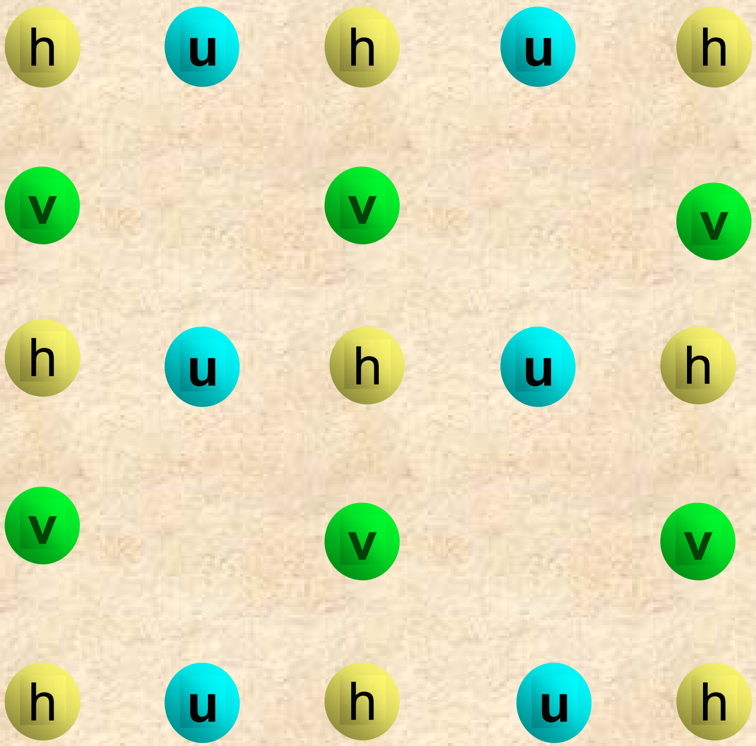


$$\frac{\partial u}{\partial t} = -g\delta_x h + fv$$

$$\frac{\partial v}{\partial t} = -g\delta_y h - fu$$

$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right)$$

C-grid



$$\frac{\partial u}{\partial t} = -g\delta_x h + f\overline{v}^{xy}$$

$$\frac{\partial v}{\partial t} = -g\delta_y h - f\overline{u}^{xy}$$

$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right)$$

What is the consequence of averaging on the C-grid ?

- Assuming that distance between two points carrying the same variable is d

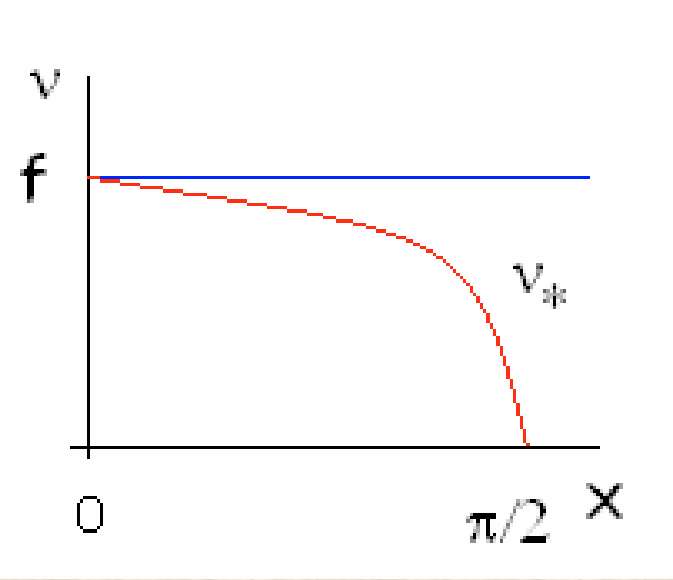
$$\overline{A}^{xy} = \hat{A}\left(e^{i(kd/2+ld/2)} + e^{i(kd/2-ld/2)} + e^{i(-kd/2+ld/2)} + e^{i(-kd/2-ld/2)}\right)$$

$$= \hat{A}\cos(kd/2)\cos(ld/2) \equiv \hat{A}\cos X \cos Y$$

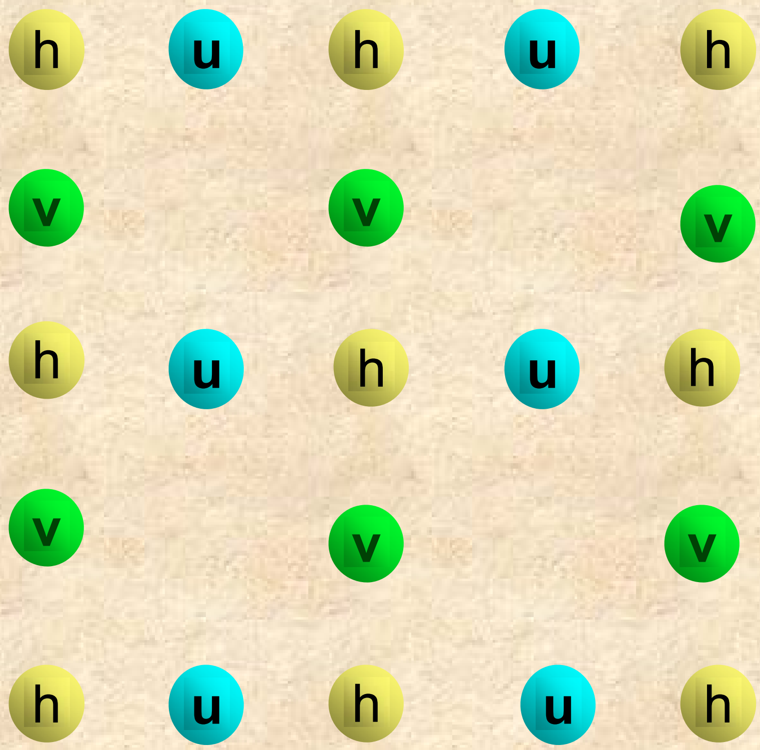
- Note that for the shortest resolvable wave
  k=2π/2d=π/d ⟹ X=π/2
  l=2π/2d=π/d ⟹ Y=π/2

- Thus, for the pure inertial wave on the C-grid, we get the following solution for numerical frequency

$$\left(\frac{\nu_*}{f}\right) = \cos X \cos Y$$



Gravity waves on the C-grid



$$\frac{\partial u}{\partial t} = -gv\delta_x h$$

$$\frac{\partial v}{\partial t} = -g\delta_y h$$

$$\frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right)$$

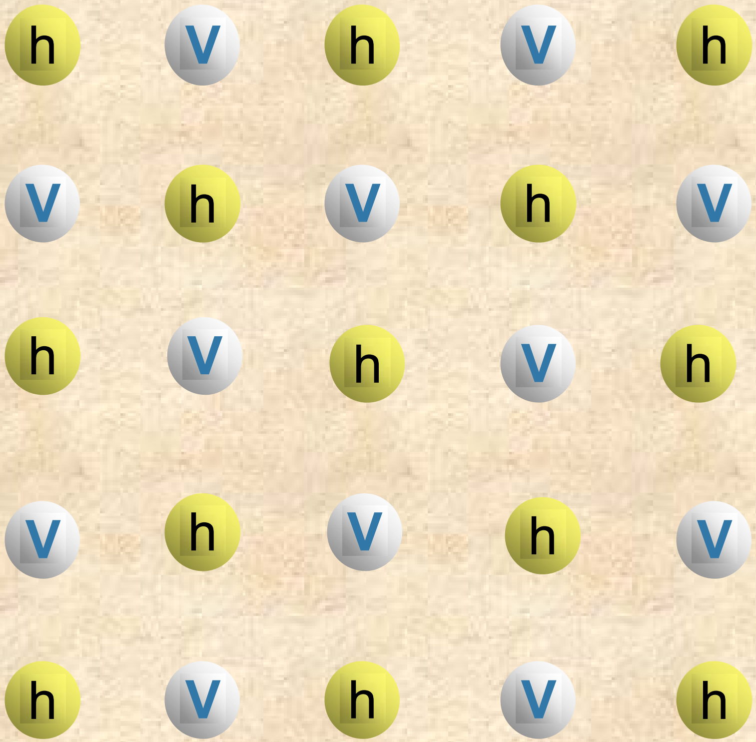
Rossby Radius of Deformation

- A spatial scale that describes properties of geostrophic adjustment

$$\lambda = \frac{\sqrt{gH}}{f}$$

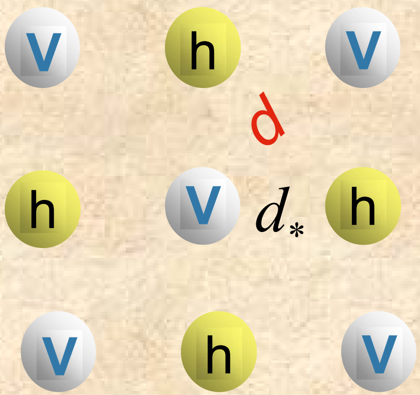
- In the general case, numerical frequency will depend on (λ/d)
- Because frequency of inertial waves tends to be erroneously presented for the short waves, it can be shown that in general case C-grid has difficulties to describe geostrophic adjustment for the small static stability when (λ/d) is small

E-grid



- There is no averaging of Coriolis term, and numerical frequency of inertial waves is exact
- The problem represents simulation of gravity waves!
- The diagonal rows 'does not feel' perturbation – which creates a two-grid-interval noise

- d_* represents here a distance between two successive points in x or y direction



Note that the shortest wave that grid may register is here with the wavelength in normal direction of

$$L = 2d_*$$

Corresponding wavenumber is

$$k = \frac{2\pi}{L} = \frac{2\pi}{2d_*} = \frac{\pi}{d_*}$$

- Thus, for 1D case (l=0), the frequency of the shortest wave is equal to zero

$$\nu_*^{\ 2} = gH\frac{\sin^2 kd_* + \sin^2 ld_*}{d_*^2} = gH\frac{\sin^2 \dfrac{\pi}{d_*}d_* + \sin^2 0d_*}{d_*^2} = 0$$

- … which describes stationary two-grid-interval noise
- This is a potentially serious drawback in high-resolution simulations at the scales that effect of Coriolis force may be neglected
- … as well as in simulations in tropical region

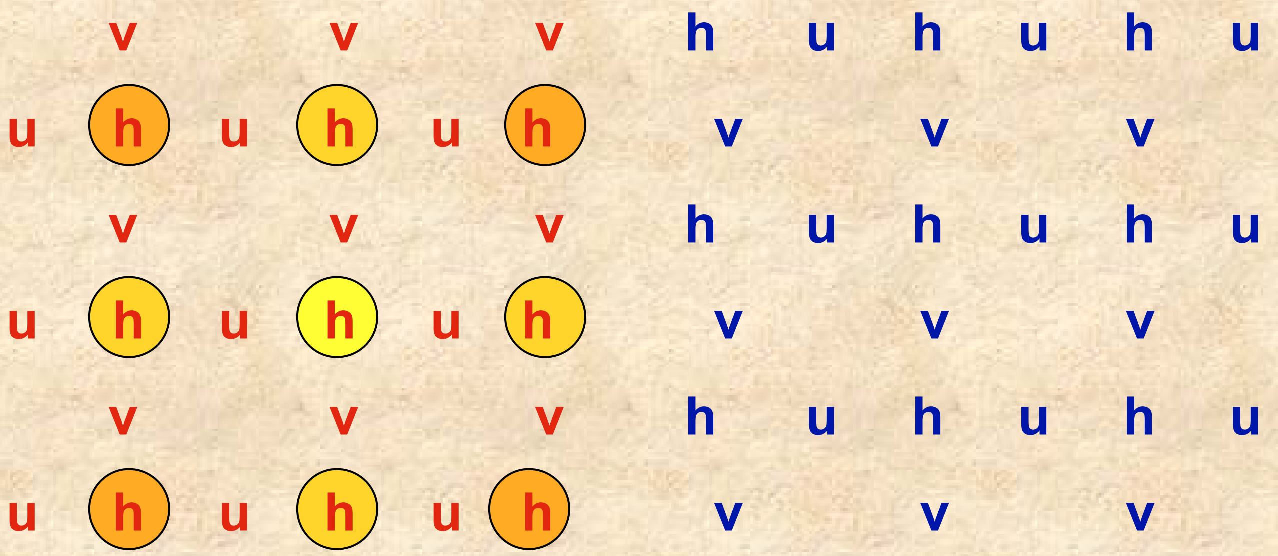
- A more formal description of this phenomena may be observed by replacing wave solution in equations for propagation of gravity waves

$$\frac{\partial u}{\partial t} = -g\delta_x h, \quad \frac{\partial v}{\partial t} = -g\delta_y h, \quad \frac{\partial h}{\partial t} = -H\left(\delta_x u + \delta_y v\right)$$

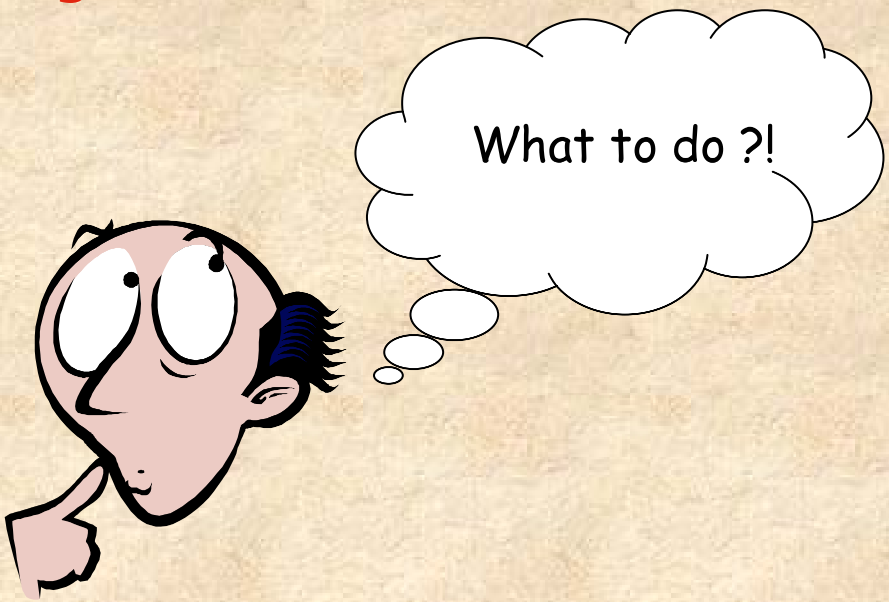
- …we get the following frequency relation for the gravity waves

$$\nu_*{}^2 = gH \frac{\sin^2 kd_* + \sin^2 ld_*}{d_*^2}$$

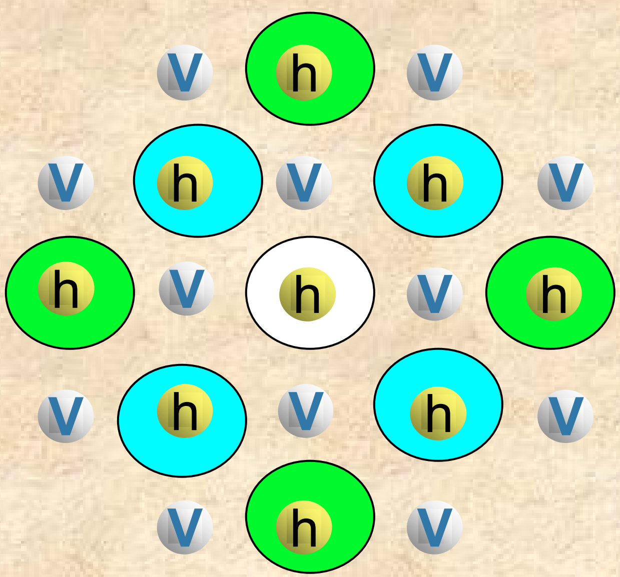
- This two-grid-interval problem may be better understood by a closer inspection of E-grid structure



- Thus, we see that E-grid is made of 2 elementary C-grids
- Two-grid-interval noise is essentially a consequence of separation of solutions on the elementary C-grids



- The nature of the problem may also be observed by considering the implied wave equation



$$\frac{\partial^2 h}{\partial t^2} = gH\left(\delta_{xx}h + \delta_{yy}h\right) = gH\nabla_{+}^{2}h$$

Can we somehow implement Jacobian made of the closest points?

$$\frac{\partial^2 h}{\partial t^2} = gH\left(\delta_{x'x'}h + \delta_{y'y'}h\right) = gH\nabla_{\times}^{2}h$$

Solution

- Mesinger (1973, 1974) introduced a solution
- In the context of an economical forward-backward time-differencing scheme

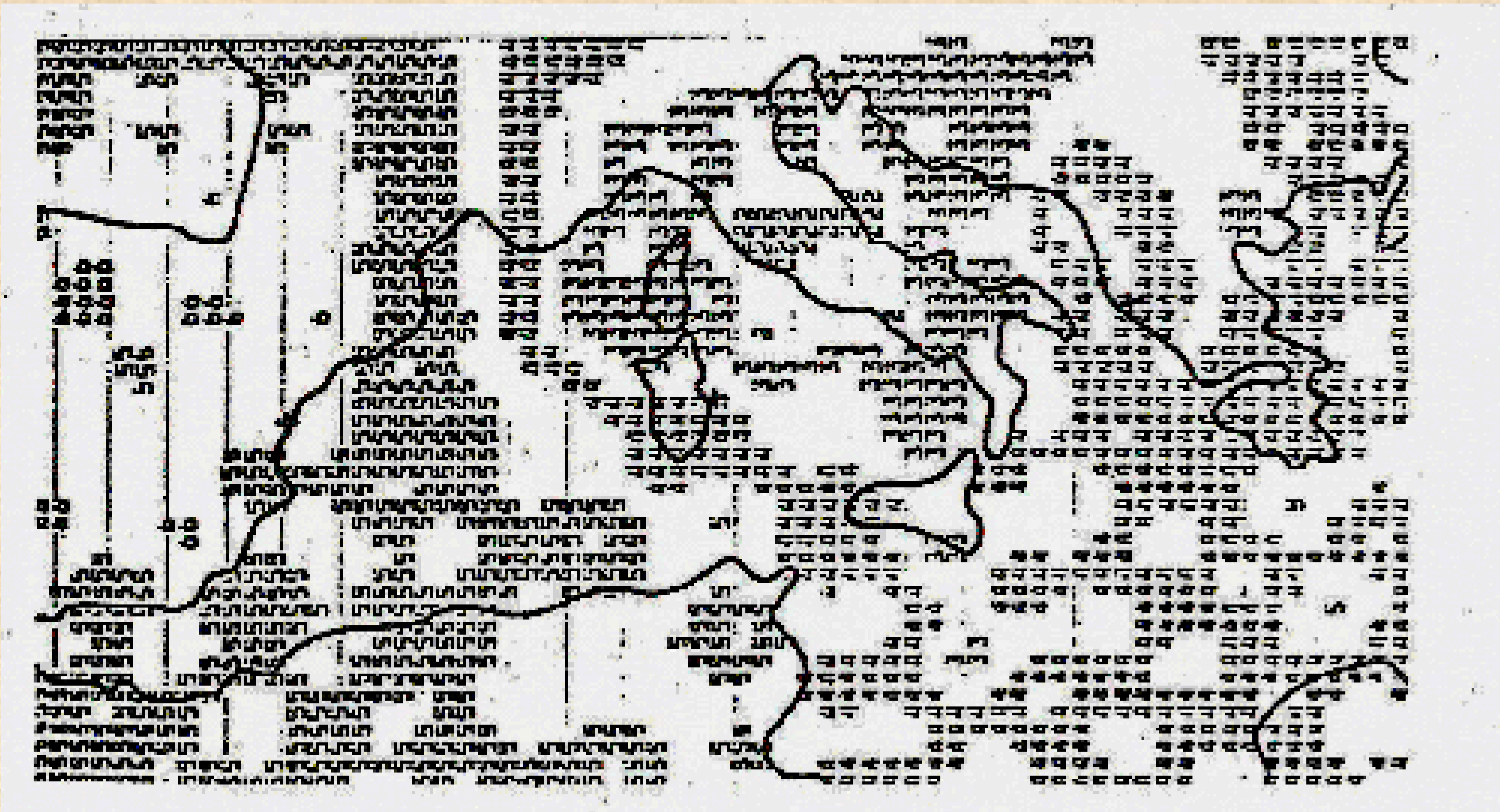
$$u^{\tau+1} = u^{\tau} - g(\Delta t)\delta_x h^{\tau}$$

$$v^{\tau+1} = v^{\tau} - g(\Delta t)\delta_y h^{\tau}$$

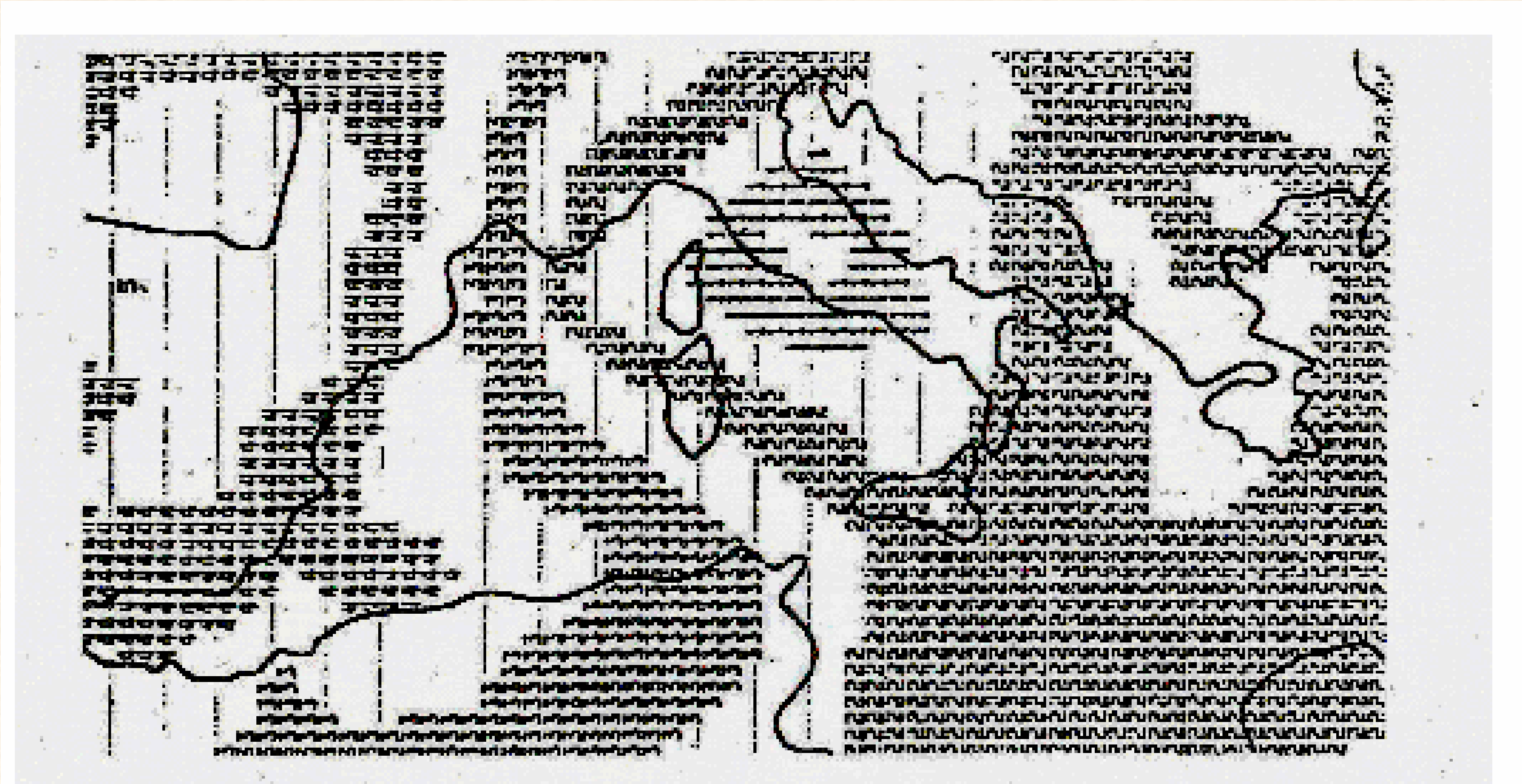
$$h^{\tau+1} = h^{\tau} - H(\Delta t)\left(\delta_x u + \delta_y v\right)^{\tau+1} + wgH(\Delta t)^2\left(\nabla_{\times}^2 - \nabla_{+}^2\right)h^{\tau}$$

- Here, w is a weight factor which is bounded by 0.25 in order to provide numerical stability

- This technique, known as a modification of the continuity equation, has been expanded to various time-differencing schemes (Janjic 1974, 1979)
- One clear advantage, in comparison to, for example, inclusion of diffusion in the momentum equation, is that it does not affect geostrophic part of the flow, but just gravity-inertial terms
- It has been generalized to A-grid
- There are alternative methods nowadays (Janjic 2000, Rancic 2005) that require application of elliptic solvers or solution of tridiagonal systems



Without modification (August 1975)



With modification (August 1975)

Nonlinear Advection Scheme

- Eta model is a model based on application of Arakawa conserving principles
- One of the key features of the model is application of Arakawa type momentum advection scheme formulated on the E-grid
- Janjic (1977) formulated an Arakawa type advection scheme in a vector invariant form that conserves both enstrophy ($\frac{1}{2}\zeta^2$) and kinetic energy of rotational flow $\frac{1}{2}\left(u^2+v^2\right)$ defined using E-grid wind components

Janjic (1977) scheme

- For a pure nondivergent flow is defined as

$$\frac{\partial u}{\partial t} = \frac{2}{3}\tilde{\zeta}^{xy} v + \frac{1}{3}\overline{\zeta \overline{v}^{y}}^{x}$$

$$\frac{\partial v}{\partial t} = -\frac{2}{3}\tilde{\zeta}^{xy} u - \frac{1}{3}\overline{\zeta \overline{u}^{x}}^{y}$$

- Here, vorticity is defined using E-grid wind components

$$\zeta = \delta_x v - \delta_y u$$

Brief review of Arakawa Jacobian

- By expanding vorticity written in terms of stream function in a closed domain using Fourier series, we get

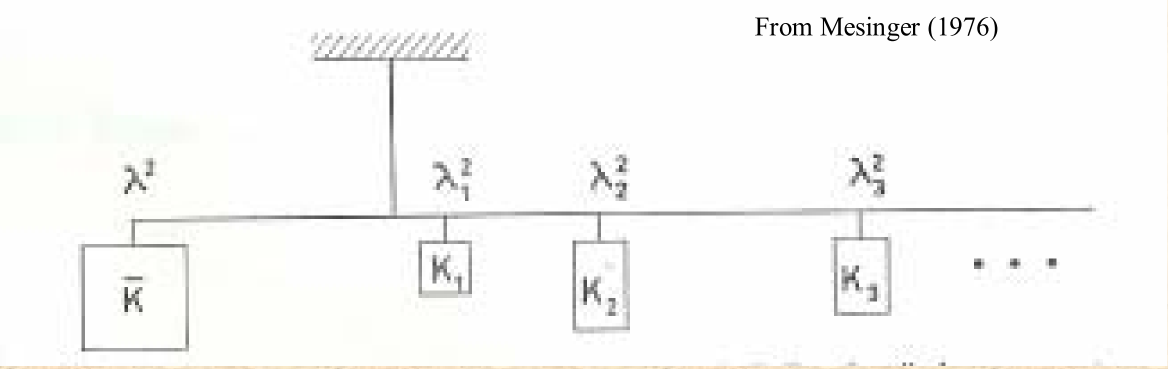
$$\zeta = \nabla^2 \psi_p = \lambda_p^2 \psi_p$$

- … where λ_p are eigenvalues of this Laplacian
- By applying the same development for the mean kinetic energy and mean enstrophy, we get

$$\overline{K} = \tfrac{1}{2}\overline{\left(u^2 + v^2\right)} = \tfrac{1}{2}\sum_p \lambda_p^2 \overline{\psi_p^2} = \sum_p K_p$$

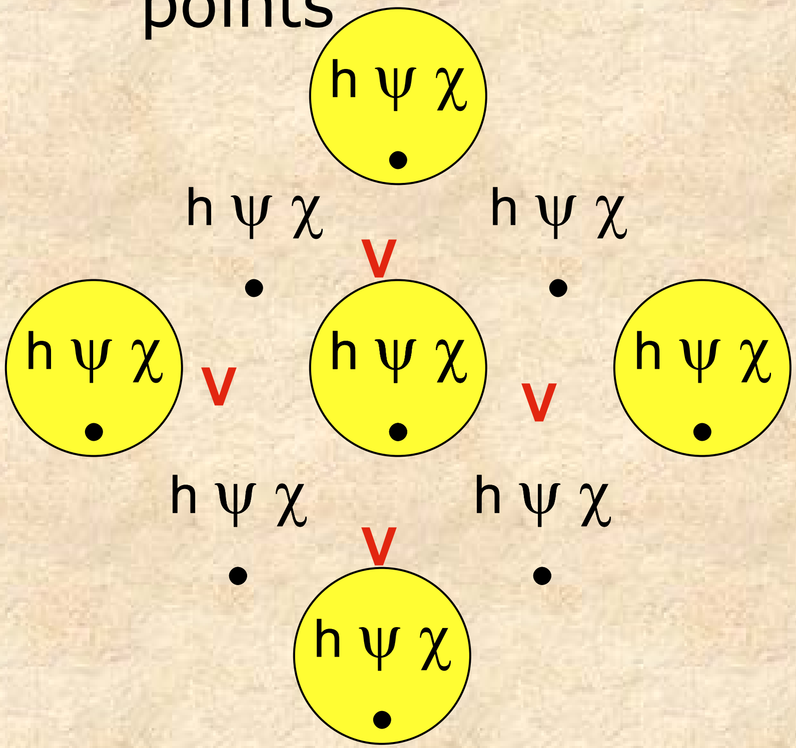
$$\tfrac{1}{2}\overline{\zeta^2} = \tfrac{1}{2}\sum_p \lambda_p^4 \overline{\psi_p^2} = \boxed{\sum_p \lambda_p^2 K_p \equiv \lambda^2 \overline{K}}$$

- Famous Charney (1966) mechanical analogy



- In other words, the nature of two-dimensional turbulence is such that energy cannot arbitrary cascade between different scales – but only in such a way as to preserve initial energy balance
- Shorter the waves, smaller amount of energy

- E-grid implies definition of stream function and velocity potential at the position of h scalar points



$$u = -\delta_y \psi$$

$$v = \delta_x \psi$$

$$\zeta = \delta_x v - \delta_y u =$$

$$= \delta_x(\delta_x \psi) + \delta_y(\delta_y \psi)$$

$$= \nabla_+^2 \psi$$

- Eigenvalues of this discrete + Laplacian are given by

$$\lambda_{mn}^2 = \frac{2}{d^2}\left(\sin^2 k_m \frac{d}{\sqrt{2}} + \sin^2 l_n \frac{d}{\sqrt{2}}\right)$$

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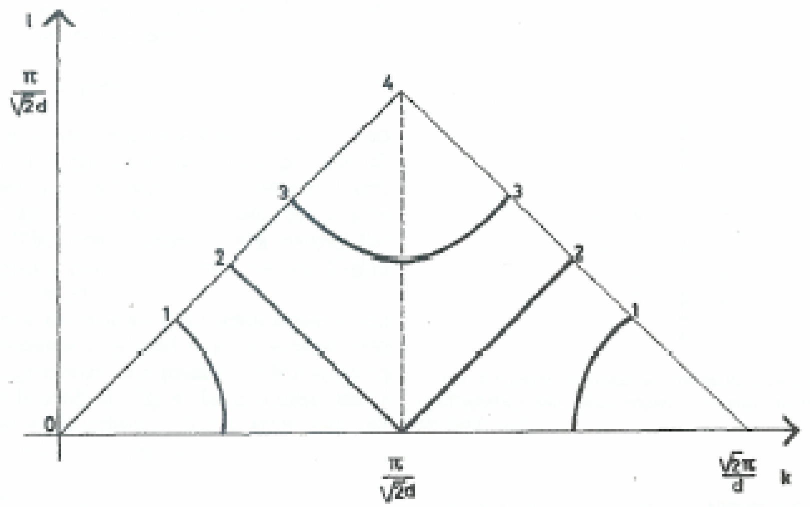


FIG. 5. Eigenvalues of the finite difference Laplacian ∇_*^2. The values are nondimensionalized by multiplication by d^2.

From Janjuc (1984)

This eigenvalues are not monotonically increasing functions!

- The consequence is that Charney mechanical analogy does not work any more – though formally both energy and vorticity of rotational flow are conserved!

From Janjic (1984)

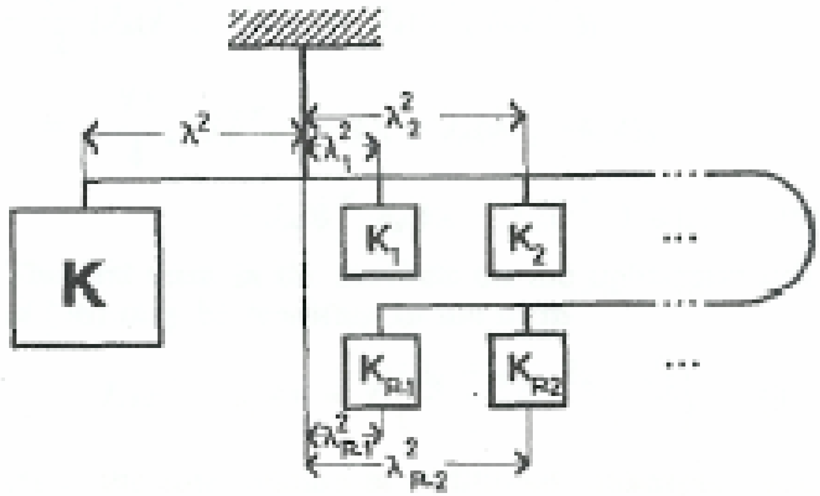


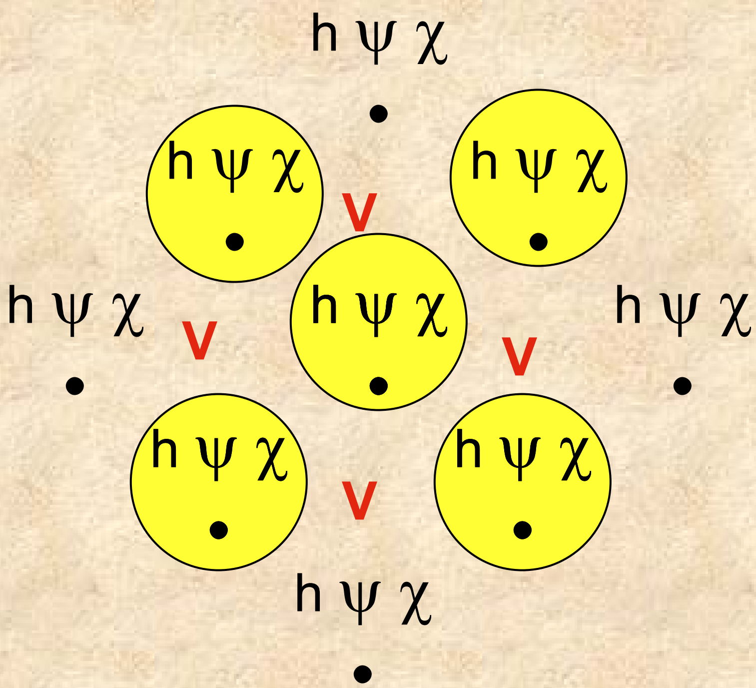
FIG. 6. Schematic representation of the limitations imposed on the nonlinear energy cascade by the energy and enstrophy conservation on the E grid.

The short and long waves may directly exchange energy

- Janjic (1984) asked the question:

Can we on the E-grid, using E-grid material, somehow conserve the other, x-Jacobian of the stream function - that is, the vorticity as defined on the Arakawa C-grid ????)

C-grid Jacobian



From Janjic (1984)

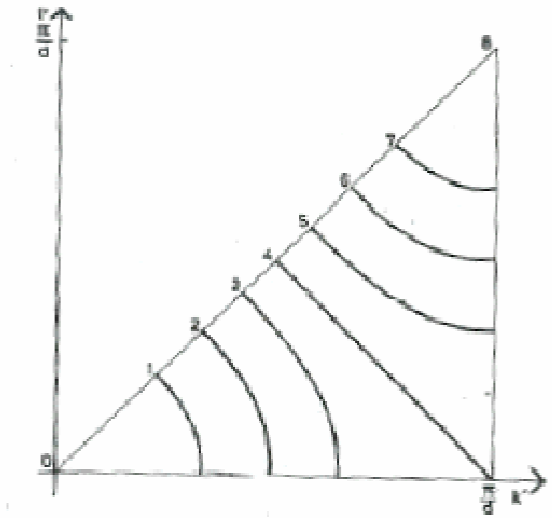


FIG. 3. Eigenvalues of the finite difference Laplacian ∇_+^2. The values are nondimensionalized by multiplication by d^2.

$$\lambda_{mn}^2 = \frac{4}{d^2}\left(\sin^2 k'_m \frac{d}{2} + \sin^2 l'_n \frac{d}{2}\right)$$

- It turned out that this is really possible, and by using properties of Arakawa Jacobian, Janjic was able to construct a new scheme that conserves C-grid vorticity/enstrophy and E-grid energy
- This scheme was ever since an integral part of the Eta model dynamics
- Eigenvalues of the new scheme are defined as

$$\Lambda_{mn}^2 = \frac{8}{d^2} \frac{\left(1 - \cos k_m \frac{d}{\sqrt{2}} \cos l_n \frac{d}{\sqrt{2}}\right)^2}{\sin^2 k_m \frac{d}{\sqrt{2}} + \sin^2 l_n \frac{d}{\sqrt{2}}}$$

- Not only that monotonicity is preserved, but they actually tends to infinity for the shortest resolvable wave – allowing virtually zero energy to be associated with it

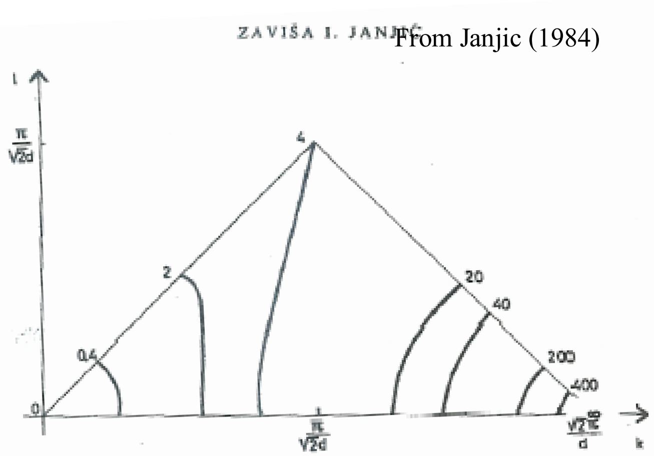


FIG. 9. The analogs of the eigenvalues of a finite difference Laplacian for a scheme which conserves E-grid energy and C-grid enstrophy. The values are nondimensionalized by multiplication by d^2.

This is a unique feature of the Eta model – which makes it ideal for long term ‘climate type’ simulations

The scheme for the shallow water case

$$\frac{\partial u}{\partial t} = -\frac{1}{\overline{h}^{x}}\left[\frac{1}{3}\left(\overline{\overline{U}^{x}\delta_x u}^{x} + \overline{\overline{V}^{x}\delta_y u}^{y}\right) + \frac{2}{3}\left(\overline{\overline{U'}^{x}\delta_{x'} u}^{x'} + \overline{\overline{V'}^{x}\delta_{y'} u}^{y'}\right)\right]$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\overline{h}^{y}}\left[\frac{1}{3}\left(\overline{\overline{U}^{y}\delta_x v}^{x} + \overline{\overline{V}^{y}\delta_y v}^{y}\right) + \frac{2}{3}\left(\overline{\overline{U'}^{y}\delta_{x'} v}^{x'} + \overline{\overline{V'}^{y}\delta_{y'} v}^{y'}\right)\right]$$

$$\frac{\partial h}{\partial t} = -\left[\frac{1}{3}\left(\delta_x U + \delta_y V\right) + \frac{2}{3}\left(\delta_{x'} U' + \delta_{y'} V'\right)\right]$$

- x and y are normal directions on E-grid
- x′ and y′ are diagonal directions on E-grid
- Normal and diagonal fluxes are respectively defined as

$$U = \overline{h}^{x} u$$

$$V = \overline{h}^{y} v$$

$$U' = \overline{h}^{x'} \frac{\sqrt{2}}{2} \overline{(u+v)}^{y'}$$

$$V' = \overline{h}^{y'} \frac{\sqrt{2}}{2} \overline{(-u+v)}^{x'}$$

Flat Square Earth Experiment

1244 From Janjic (1984) MONTHLY WE

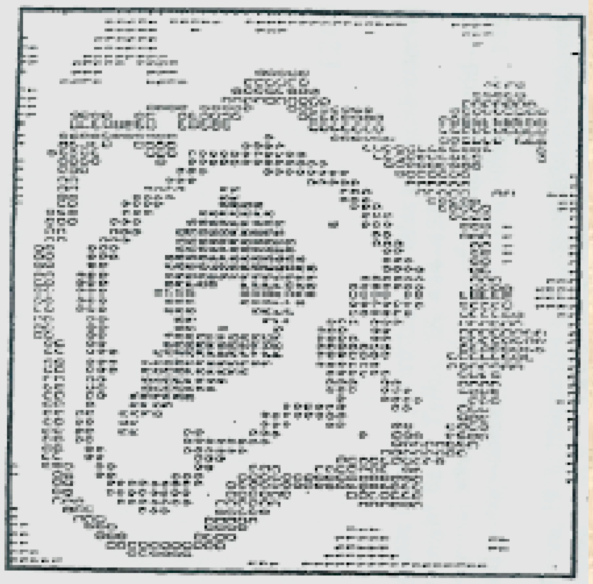


FIG. 13. Height field after 10 000 time steps in the control experiment. The shading interval is 160 m.

Old Scheme

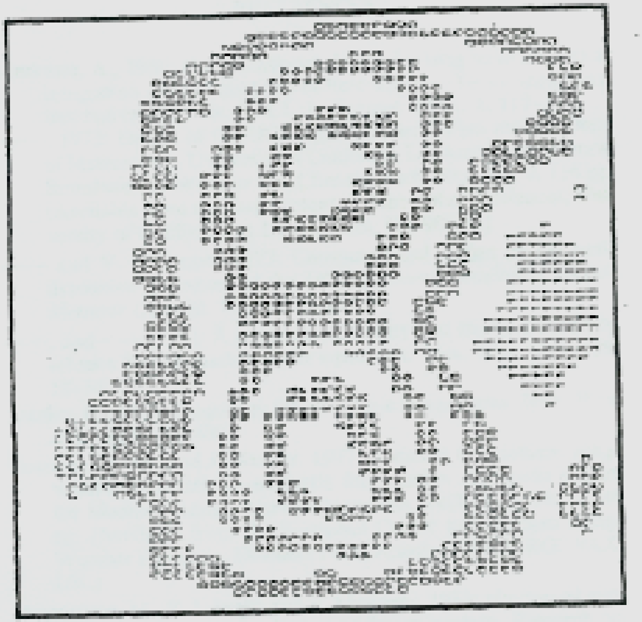


FIG. 12. Height field after 10 000 time steps in the main experiment. The shading interval is 160 m.

New Scheme

- A forth-order version of this scheme has been developed (Rancic, 1988)
- A new version of the scheme customized for application in the global version of Eta model on quasi-uniform grids has been recently developed (Rancic 2005)