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Knotted Vortex Rings from the Eikonal Equations

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Knotted vortex rings from the eikonal equation

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- Equation

$$\partial_\mu u \partial^\mu u = 0$$

- Topology

$$\vec{n} = \frac{1}{1+|u|^2}(u+u^*, -i(u-u^*), |u|^2-1)$$

  - Topological charge $\rightarrow$ Hopf index $Q_H \in \pi_3(S^2) \rightarrow$ knots

$$\text{if} \quad \lim_{|\vec{x}|\rightarrow\infty} \vec{n} = \vec{n}_0 \quad \text{then} \quad \vec{n} : R^3 \cup \{\infty\} \rightarrow S^2$$

- Topological charge $\longrightarrow Q \in \pi_2(S^2) \longrightarrow$ hedgehogs

$$\text{if} \quad \lim_{|\vec{x}|\to\infty} \vec{n} \neq \vec{n}_0 \quad \text{then} \quad \vec{n} : R^3_\infty \simeq S^2 \longrightarrow S^2$$

- Topological charge $\longrightarrow Q \in \pi_2(S^2) \longrightarrow$ braided strings

- Toroidal coordinates

$$x = \frac{\tilde{a}}{q} \sinh \eta \cos \phi$$

$$y = \frac{\tilde{a}}{q} \sinh \eta \sin \phi$$

$$z = \frac{\tilde{a}}{q} \sin \xi$$

where $q = \cosh \eta - \cos \xi$ and $\tilde{a} > 0$

then

$$\frac{q^2}{\tilde{a}^2} \left[(\partial_\eta u)^2 + (\partial_\xi u)^2 + \frac{1}{\sinh^2 \eta} (\partial_\phi u)^2 \right] = 0$$

- Ansatz

$$u = f(\eta)e^{i(m\xi+n\phi)}$$

where $m, \; n \in \mathcal{N}$

$$f' = \pm\sqrt{\left(m^2 + \frac{n^2}{\sinh^2\eta}\right)f}$$

- Basic solutions $\rightarrow$ unknots

$$f^{\pm} = A \sinh^{\pm|n|} \eta \frac{\left(|m| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta}\right)^{\pm|m|}}{\left(|n| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta}\right)^{\pm|n|}}$$

thus

$$u = A \sinh^{\pm|n|} \eta \frac{\left(|m| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta}\right)^{\pm|m|}}{\left(|n| \cosh \eta + \sqrt{n^2 + m^2 \sinh^2 \eta}\right)^{\pm|n|}} e^{i(m\xi + n\phi)}$$

- Hopf index

$$Q_H = -nm$$

- Position of the core of knots

$$\vec{n}_0 = -\vec{n}^{\infty} \quad \text{where} \quad \vec{n}^{\infty} = \lim_{\vec{x} \to \infty} \vec{n} = \lim_{\eta \to 0} \vec{n}$$

here

$$\lim_{\eta \to 0} f^{(-)}(\eta) = \infty \quad \Rightarrow \quad \vec{n}^{\infty} = (0, 0, 1) \quad \Rightarrow \quad \vec{n}_0 = (0, 0, -1)$$

- Symmetry

$$u \to F(u),$$

where F is any (anti)holomorphic function

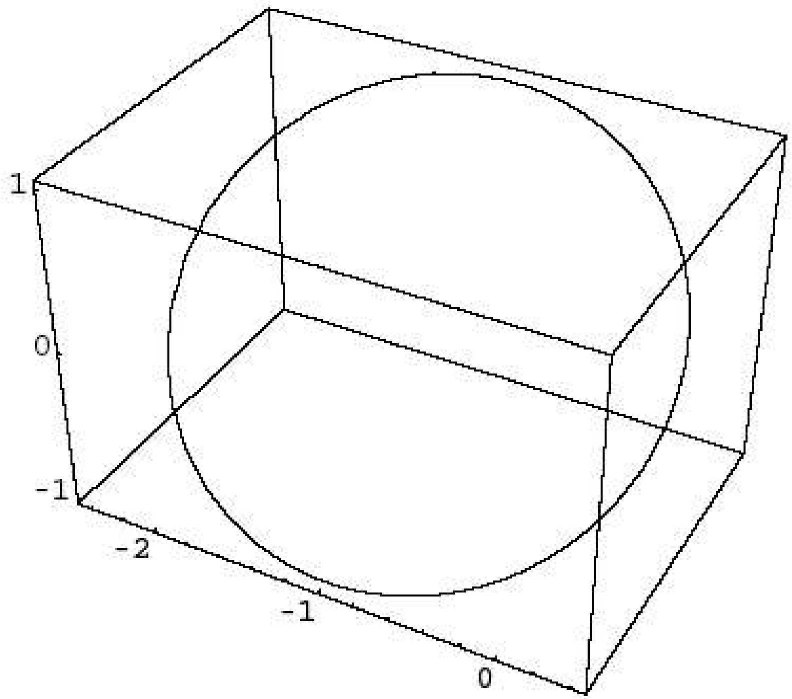
- Knotted solutions

$$u \to u + c_0, \quad m, n \text{ rel. prime numbers}$$

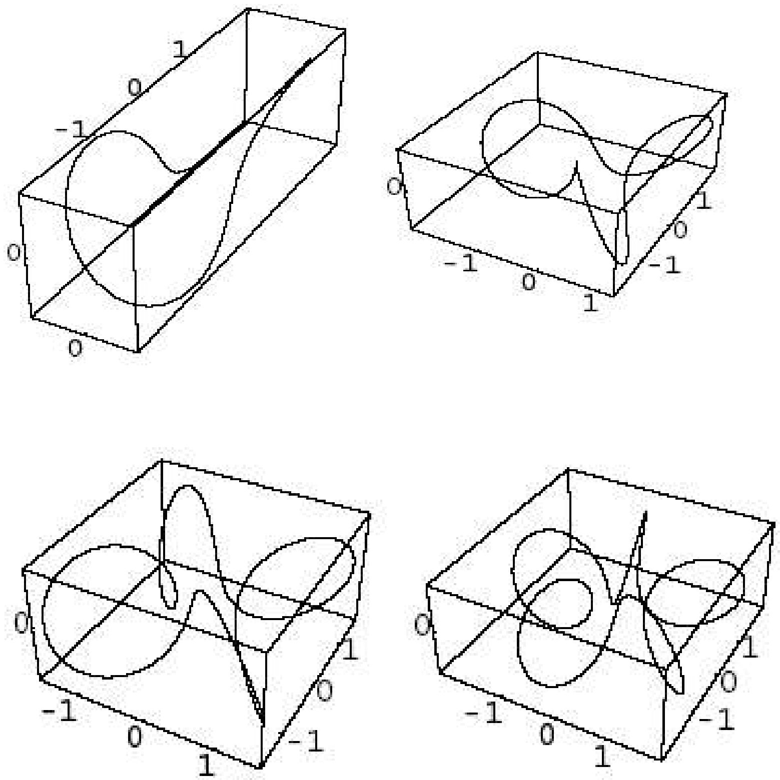
Position of the knot

$$f(\eta_0) = |c_0|, \quad m\xi + n\phi = \pi - \alpha_0$$

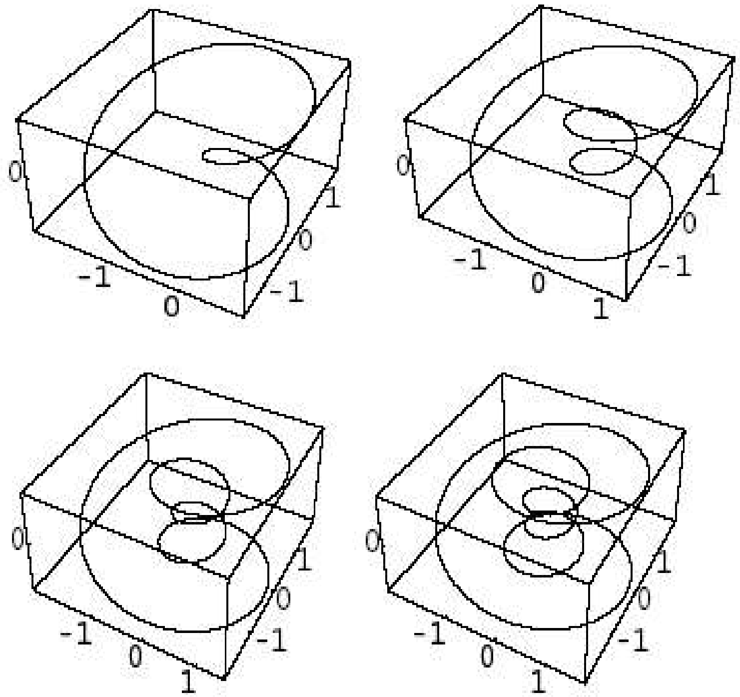
unknot $(m, n) = (1, 1)$



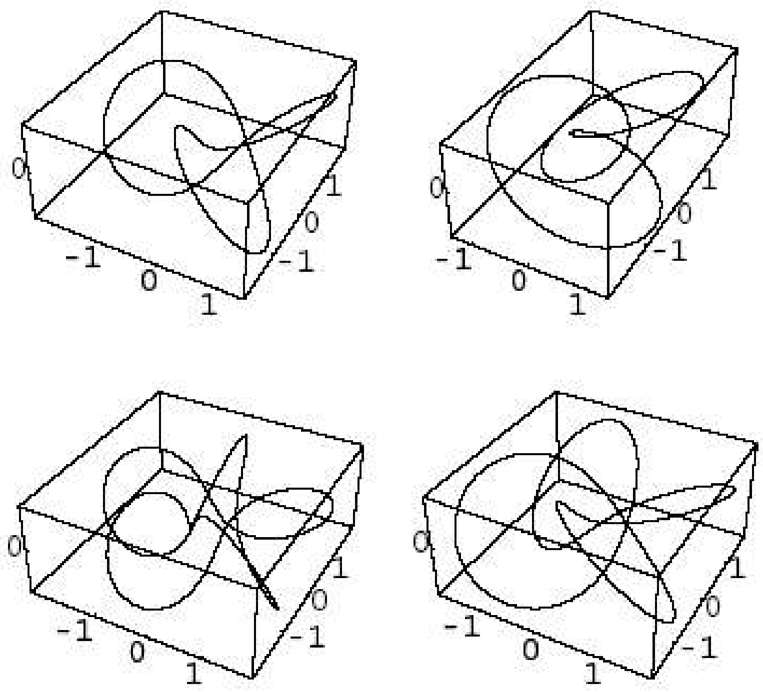
knots $(1,2), (1,3), (1,4)$ and $(1,5)$



knots $(2,1), (3,1), (4,1)$ and $(5,1)$



knots $(2,3), (3,2), (2,5)$ and $(3,4)$



◦ Linked knotted solutions

$$u \to u^N + c_0, \quad N \in \mathcal{N}$$

Position of the knots

$$f(\eta_0) = |c_0|, \quad m\xi + n\phi = \frac{1}{N}(\pi - \alpha_0 + 2\pi k), \quad k = 0, 1 \ldots N-1$$

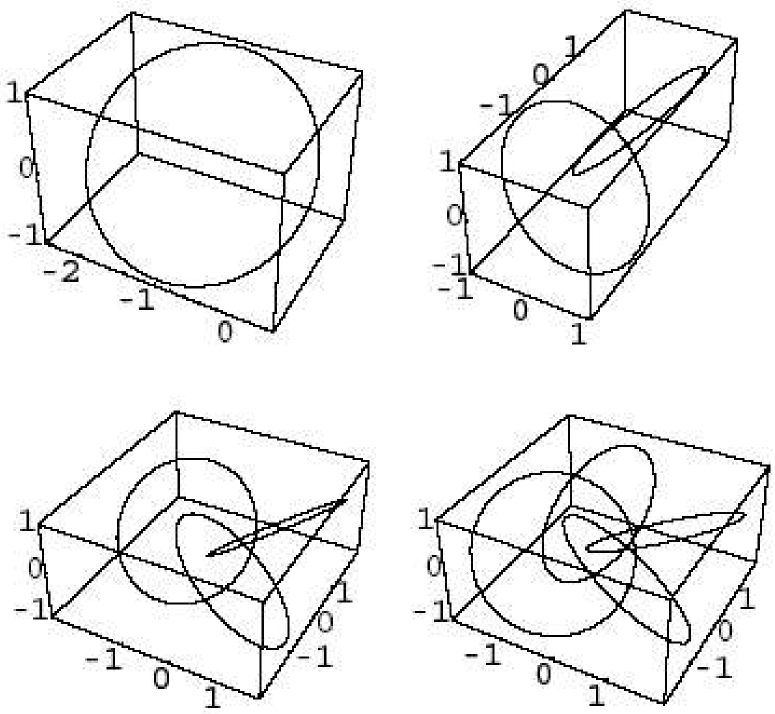
N elementary knots $\quad Q_e = -mn$

total topological charge $\quad Q_H = -N^2 mn$

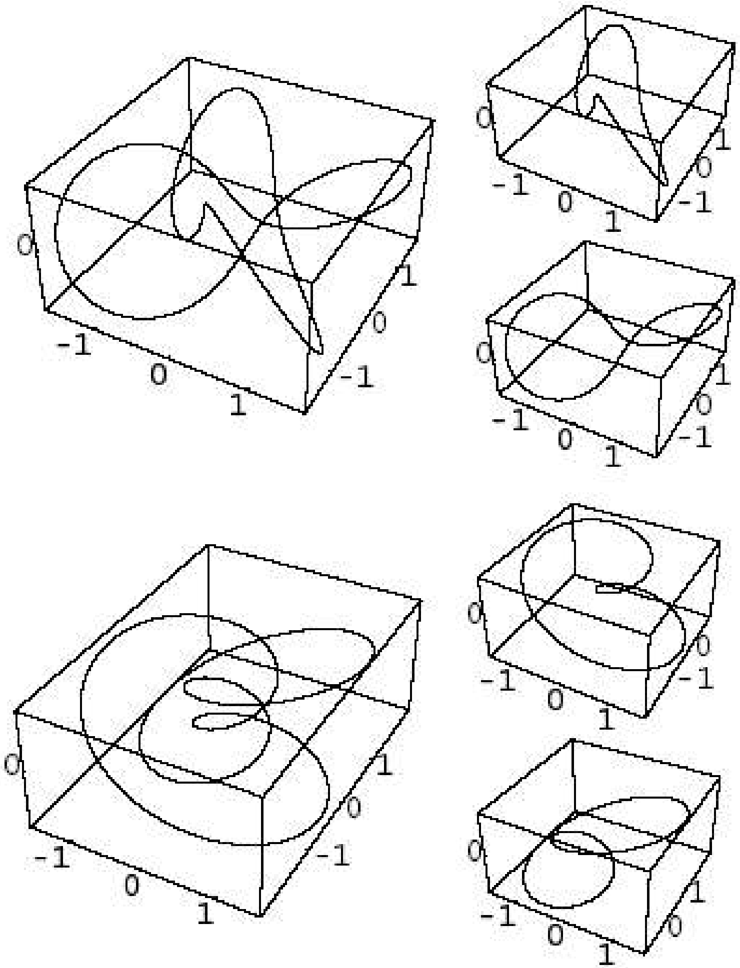
thus

$$Q_H = N \cdot Q_e - L, \quad L \ \text{linking number}$$

knots $(1,1), (2,2), (3,3)$ and $(4,4)$



knots $(2, 4)$ and $(4, 2)$



- Other knotted solutions

$$u \longrightarrow u^{N_1} + u^{N_2}, \quad N_1 > N_2$$

Position of the knots

$$\eta_0 = \infty$$

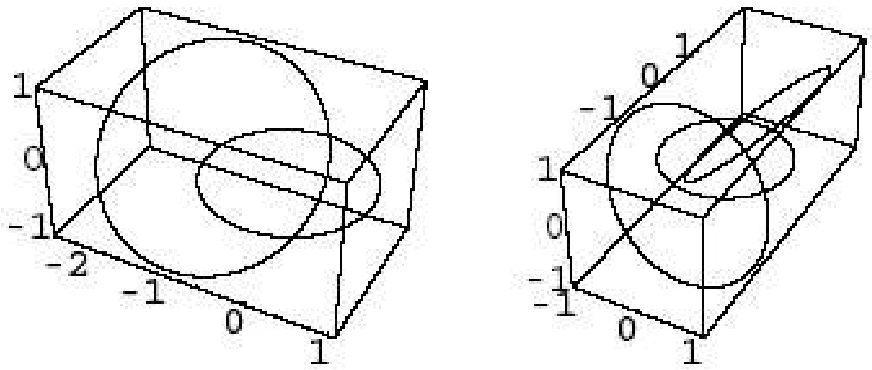
$$\frac{f_1(\eta_0)}{f_2(\eta_0)} = 1, \qquad m\xi + n\phi = \frac{1}{N_1 - N_2}(\pi + 2k\pi), \quad k = 0, 1 ... (N_1 - N_2) - 1$$

unique central knot $\qquad Q_c = -(N_1 - N_2 + 1)mn$

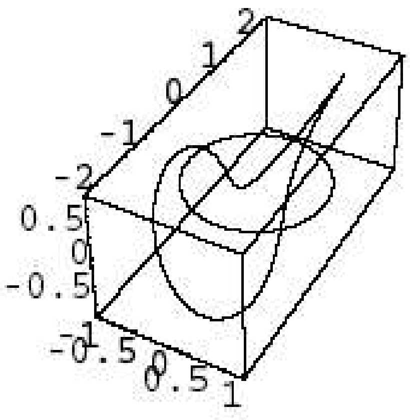
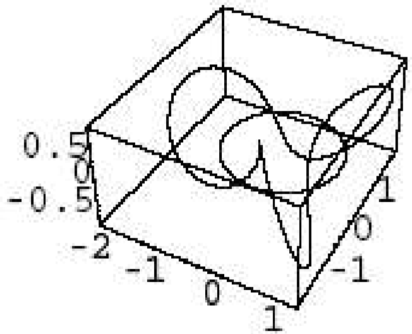
$N_1 - N_2$ satellite knots $\qquad Q_c = -mn$

knots $(1, 1)$

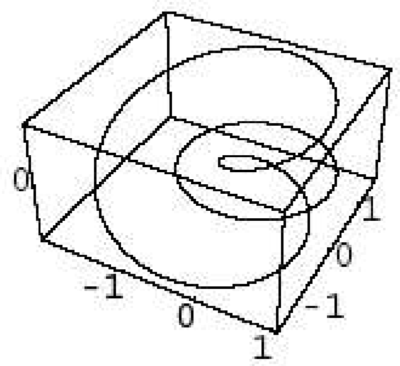
$N_1 = 2, N_2 = 1$ and $N_1 = 3, N_2 = 1$

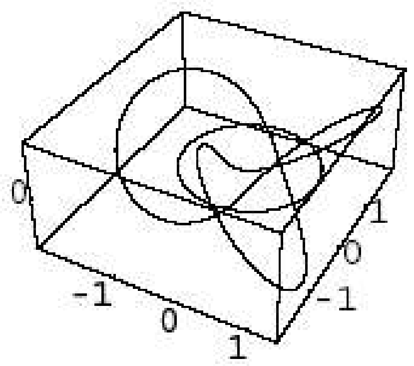


knots $(1, 2)$, $N_1 = 2, N_2 = 1$
knots $(1, 3)$, $N_1 = 2, N_2 = 1$

knots $(2,1)$, $N_1 = 2, N_2 = 1$
knots $(2,3)$, $N_1 = 2, N_2 = 1$





- Nicole Model

$$L = \frac{1}{2}(\partial_\mu \vec{n} \partial^\mu \vec{n})^{\frac{3}{2}} \quad \rightarrow \quad L = \frac{1}{(1+|u|^2)^3}(\partial_\mu u \partial^\mu u^*)^{\frac{3}{2}}$$

Equation of motion

$$\frac{1}{2}\partial_\mu \left[\frac{1}{1+|u|^2}(\partial_\nu u \partial^\nu u^*)^{\frac{1}{2}} \partial^\mu u \right] - \frac{u^*}{(1+|u|^2)^2}(\partial_\nu u \partial^\nu u^*)^{\frac{1}{2}}(\partial_\mu u)^2 = 0$$

solution

$$u = \frac{1}{\sinh \eta} e^{i(\xi+\phi)} \quad \text{with} \quad Q_H = -1$$

- Nicole-type Models

$$L=\frac{1}{2}\sigma(\vec{n})(\partial_\mu \vec{n}\partial^\mu \vec{n})^{\frac{3}{2}}, \quad \sigma(\vec{n})=\left(\frac{1+n^3}{1-n^3}\right)^{\frac{3}{2}\left(\frac{1}{m}-1\right)}\left[\frac{1+\frac{1+n^3}{1-n^3}}{1+\left(\frac{1+n^3}{1-n^3}\right)^{\frac{1}{m}}}\right]^3$$

$$\rightarrow \quad L=\left(\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}}\right)^3(\partial_\mu u\partial^\mu u^*)^{\frac{3}{2}}$$

Equations of motion (integrable subsystem)

$$\partial_\mu\left[\frac{|u|^{\frac{1}{m}-1}}{1+|u|^{\frac{2}{m}}}(\partial_\nu u\partial^\nu u^*)^{\frac{1}{2}}\partial^\mu u\right]=0 \quad \text{and} \quad \partial_\mu u\partial^\mu u=0$$

solution

$$u = \frac{1}{\sinh^m \eta} e^{im(\xi+\phi)} \quad \text{with} \quad Q_H = -m^2$$

- Energy-charge dependence

$$E = \sqrt{2}(2\pi)^2 |Q_H|^{\frac{3}{2}}$$

splitting of solitons (?)

- Faddeev-Niemi action

$$S = \int d^4x \, \frac{1}{2} m^2 (\partial_\mu \vec{n})^2 - \frac{1}{4e^2} \left[\vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \right]^2$$

Cho-Faddeev-Niemi decomposition

$$A^a_\mu = \epsilon^{abc} n^b \partial_\mu n^c + X^a_\mu$$

◦ Glueballs as knotted solitons

$$M \sim m_0 |Q_H|^{\frac{3}{4}}, \quad m_0 \simeq 1500 Mev$$

The Eikonal Anzatz can describe

- Qualitative features
    - really knotted (multi-knotted) configuration
- Quantitative features
    - topology - well defined Hopf index
    - energy (in each topological sector) is bounded the below
    - energy - approx. 20% accuracy

Q_H	type of the knot (m, n)	E_{min}	E_{num}	accuracy
1	(1,1)	304.3	252.0	20.0%
2	(1,2) (2,1)	467.9 602.7	417.5	12.0 %
3	(1,3) (3,1)	658.1 997.3	578.5	13.8 %
4	(1,4) (2,2) (4,1)	855.5 914.3 1466.4	743.0	15.0%
5	(1,5) (5,1)	1056.1 2000.0	905.0	16.8 %

- Equation

$$(\partial_\nu u)^2(\partial_\nu u^*)^2 + \alpha(\partial_\nu u \partial^\nu u^*)^2 = 0$$

- Generalized eikonal knots

$$u(\eta,\xi,\phi) = A \sinh^{\pm a|k|}\eta \frac{\left(|m|\cosh\eta + \sqrt{k^2 + m^2\sinh^2\eta}\right)^{\pm a|m|}}{\left(|k|\cosh\eta + \sqrt{k^2 + m^2\sinh^2\eta}\right)^{\pm a|k|}} e^{i(m\xi + n\phi)}$$

where

$$a^2(\alpha) = \frac{(1-\alpha)}{(1+\alpha)} \pm \sqrt{\frac{(1-\alpha)^2}{(1+\alpha)^2} - 1}$$

- ◦ general solutions $u \rightarrow F(u)$
- ◦ $Q_H = -mn$
- ◦ identical geometrical structure

- New integrable subsystems

$$L = G(|u|) \left(K_\mu^{(3)} \partial^\mu u^* \right)^{\frac{1}{2}}$$

where

$$K_\mu^{(3)} = \alpha(\partial_\nu u \partial^\nu u^*)^2 \partial_\mu u + \beta(\partial_\nu u)^2 (\partial_\nu u \partial^\nu u^*) \partial_\mu u^* + \gamma(\partial_\nu u)^2 (\partial_\nu u^*)^2 \partial_\mu u$$

- for

$$G_{m,a}(|u|) = \left(\frac{|u|^{\frac{1-am}{am}}}{1+|u|^{\frac{2}{am}}} \right)^3$$

solutions

$$u = \frac{1}{\sinh^{am} \eta} e^{im(\xi+\phi)}$$

- energy-charge relation

$$E = 2\pi^2 \sqrt{\lambda_+[(1+\beta)\lambda_-^2 + \alpha\lambda_+]} a^3 |Q_H|^{3/2}$$

Q_H	x	E_{min}	E_{num}	accuracy
1	1.170	296.0	252.0	17.5%
2	0.954	467.0	417.5	11.8 %
3	0.885	651.0	578.5	12.5 %
4	0.859	840.9	743.0	13.0%
5	0.859	1034.1	905.0	14.4 %

- knotted and multi-knotted (linked) structures with an arbitrary value of the Hopf index
  - analytical description
  - (new) integrable dynamical subsystems
  - energy-charge relation
- approximation of the Faddeev-Niemi hopfions

$\rightsquigarrow$ relation with the Faddeev-Niemi hopfions

$\rightsquigarrow$ the energy-charge formula - interaction of the knots