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Non-Riemannian Vortex Acoustics

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Non-Riemannian geometry of nonlinear Schrödinger equation and Hasimoto soliton transformations

by

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Abstract

Schrödinger nonlinear (NLS) equation describing relativistic superfluids extended to non-Riemannian spacetime allows us to show that the there is a relation between the phase of the wave function and the Hasimoto soliton transformation, where to the Cartan vector torsion in the non-Riemannian geometry of NLS equation correponds the scalar Serret-Frenet torsion of the vortex filament curve in the Hasimoto soliton transformation. Unfortunatly the same analogy seems not to occur in the case of scalar Frenet curvature of the vortex filament and the Riemannian curvature since in the spacetime the modulus of the wave solution of the NLS equation does not coincide with the Riemann scalar curvature as demanded by the Hasimoto transfornation for the Frenet curvature. **PACS numbers:**

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I Introduction

In 1991 R. Ricca [1] has presented an algebraic-geometrical generalization of the Da Rios-Betchov dynamical intrinsic equations of curvature and torsion of an isolated vortex string moving in an unbounded perfect fluid. The filament in his case is embbeded in a manifold homeomorphic to an odd-dimensional Euclidean space, which possesses a Cartan torsion [2]. In his paper he suggests also how to take into account the fluid compressibility in his geometrical model. In this contribution we show that by extending the NLS equation to non-Riemannian spacetime endowed with curvature and torsion there appears a transformation similar to Hasimoto transformation where the phase of the wave function is proportional to the Serret-Frenet scalar torsion of the vortex filament. The issue of comparing Cartan to Serret-Frenet torsions in the framework of condensed matter systems [3] is a very interesting one, for example is possible to show that Cartan torsion loops [4] recently considered in the context of vortex acoustics in a rotational superfluid ^4He [5], since it does not possess Serret-Frenet torsion of the curve described by the torsion loop since it is a planar curve in the Euclidean two dimensional space E^2 , however it does possess Serret-Frenet line curvature since it is a loop in E^2. The paper is organized as follows: In section 2 to simplify matters we consider the case of NLS in relativistic non-Riemannian spacetime and investigate the role of torsion integral as the corresponding phase of wave field solution of the NLS equation. In section 3 conclusions and discussions are presented.

II Non-Riemannian geometry and soliton transformations

Hasimoto [6] has shown that there is a transformation

$$\psi = \kappa e^{i\theta} \tag{II.1}$$

$$\theta = \int \tau ds \tag{II.2}$$

which takes Da Rios-Betchov equation of a curve

$$\vec{X}_t = \vec{X}_s \times \vec{X}_{ss} \tag{II.3}$$

into the NLS equation

$$-i\psi_t = \psi_{ss} + \frac{1}{2}|\psi|^2\psi \tag{II.4}$$

as long as we take the Serret-Frenet equations

$$\vec{X}_s = \vec{l} \tag{II.5}$$

$$\vec{l}_s = \kappa\vec{n} \tag{II.6}$$

$$\vec{n}_s = -\kappa\vec{l} + \tau\vec{b} \tag{II.7}$$

$$\vec{b}_s = -\tau\vec{n} \tag{II.8}$$

are obeyed and $\vec{X}_t$ can be expressed in terms of the Frenet frame $(\vec{l}, \vec{n}, \vec{b})$ as

$$\vec{X}_t = u\vec{l} + v\vec{n} + \omega\vec{b} \tag{II.9}$$

Here τ represents the Serret-Frenet torsion of the vortex filament and κ its scalar curvature. By expressing the NLS equation in terms of the Dirac constant $\hbar$

$$i\hbar\partial_t\psi = -(\frac{\hbar^2}{2m})\nabla^2\psi + g|\psi|^2\psi \tag{II.10}$$

whose Riemannian extension has been given by Anandan [7] as

$$\Box\psi + (\frac{mc}{\hbar})^2\psi = -\frac{2mg}{\hbar}|\psi|^2\psi \tag{II.11}$$

where $\Box$ is the Riemannian Laplace-Beltrami wave operator. The extension to non-Riemannian NLS can be obtained by the relation between the non-Riemannian Laplace-Beltrami operator $\Box^{NR}$ and the Riemannian one as

$$\Box^{NR}\psi = \Box\psi - K^{\mu}{}_{\mu}{}^{\beta}\nabla_{\beta}\psi \tag{II.12}$$

where $(\beta, \mu = 0, 1, 2, 3)$ are space-time indices and $K_{\mu\gamma\beta}$ is the Cartan contortion tensor, while ∇_{μ} is the Riemannian covariant derivative. Substitution of the wave operator by the minimal coupling principle into the Riemannian NLS equation yields the non-Riemannian NLS equation

$$\Box\psi + (\frac{mc}{\hbar})^2\psi = K^{\mu}{}_{\mu}{}^{\beta}\nabla_{\beta}\psi - \frac{2mg}{\hbar}|\psi|^2\psi \tag{II.13}$$

where the amplitude α and the phase shift ϕ are real functions of space-time. In order we are able to solve this complex variables equation we should use a similar ansatz as used in general to solve the NLS equation , which is $\psi = \alpha e^{i\phi}$. Substitution of this ansatz into the non-Riemannian NLS equation we are left with two real PDEs

$$g_{\mu\nu}v^{\mu}v^{\nu} = 1 + f(\alpha) - (\frac{\hbar}{mc})^2 K^{\mu}{}_{\mu}{}^{\beta}\partial_{\beta}\psi \tag{II.14}$$

where here $g_{\mu\nu}$ is the Riemann metric and $f(\alpha)$ is the function given by

$$f(\alpha) := (\frac{\hbar}{mc})^2 \frac{\Box\alpha}{\alpha} + \frac{2g}{m^2c^2}\alpha^2 \tag{II.15}$$

where v_{μ} is the superfluid fluid flow

$$v_{\mu} = -\frac{\hbar}{mc}\partial_{\mu}\phi \tag{II.16}$$

The other real equation is the matter non-Riemannian current conservation

$$\nabla_{\mu}J^{\mu} = K^{\mu}{}_{\mu}{}^{\beta}J_{\beta} \tag{II.17}$$

where the current flow is given by $J^{\mu} = \alpha^2 v^{\mu}$. Now to simplify matters we consider the approximation given by the Minkowskian space-time background endowed with Cartan torsion underlying the NLS equation which reduces the previous non-Riemannian NLS equation to

$$[-(\frac{\hbar}{mc})^2(\nabla^2 - (\nabla\phi)^2) - (1 + (\frac{2g\alpha^2}{mc^2} - \vec{K}.\nabla))]\psi = K^{\mu}{}_{\mu}{}^{\beta}\nabla_{\beta}\psi - \frac{2mg}{\hbar}|\psi|^2\psi \tag{II.18}$$

where $\vec{K}$ is the Cartan torsion vector in E^3 manifold. The conservation equation in this framework becomes

$$\nabla^2\phi + [\vec{K} + \frac{\nabla\alpha^2}{\alpha^2}].\nabla\phi = 0 \tag{II.19}$$

where to simplify matters we consider the following approximation $\nabla\alpha^2 = 0$. By considering potential flows in superfluids we have $\vec{v} = \nabla\phi$ which yields

$$\partial_r v_r + (K + \frac{1}{r})v_r = 0 \tag{II.20}$$

whose solution is

$$v_r = \frac{e^{-Kr}}{r} \tag{II.21}$$

Note that in the region near the vortex filament the flow velocity can be approximated by

$$v_r = \frac{1}{r} - K \tag{II.22}$$

which means that in the near region to the vortex filament there is a constant contribution to the flow due to torsion which yields the following contribution to the phase

$$\phi = -\int K dr \tag{II.23}$$

which is closely analogous to the previous Hasimoto transformation for the soliton. For a more detailed and complete solution of the NLS equation the reader is refered to reference [?].

III Conclusions

We have consider the close analogy between the Hasimoto transformation that takes Da Rios-Betchov equation to the non-relativistic NLS equation through the use of Frenet scalar torsion and the transformation using the Cartan vector torsion in the realm of a non-Riemannian relativistic NLS equation which describes the superfluid dynamics of relativistic fluids.

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