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Conference on

VORTEX RINGS AND FILAMENTS IN CLASSICAL AND QUANTUM SYSTEMS

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Vortex Rings With and Without Swirl

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VORTEX RINGS AND FILAMENTS ...

ICTP, 6-8 JUNE 2005

VORTEX RINGS WITH + WITHOUT SWIRL

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- 1. Review of magnetic relaxation technique for determination of steady vortex rings

  [H.K.M. Fluid Dynamics Research 3 (1988) 22-30]

  Steady State is a Problem of Existence — Saffman

- 2. Relaxation to steady solutions of the MHD equations

  [HKM 2000
  Vortex + Magnetodynamics – a Topological Perspective
  In: Mathematical Physics 2000 Ed. Fokas et al. ICP]

- 3. A "Hill's magnetic vortex"
  Hattori Y. & M. 2005 (submitted to JFM)
  The magnetohydrodynamic evolution of toroidal magnetic eddies.

GENERALISED VORTEX RINGS WITH AND WITHOUT SWIRL

SOME ELEMENTARY CONSIDERATIONS:

Def: VORTEX RING - A (STEADY) AXISYMMETRIC SOL^{N} OF THE EULER EQNS., WITH $\underset{\sim}{\omega} \neq 0$ in a BOUNDED DOMAIN D_c (the vortex core), and $\underset{\sim}{u} \sim \underset{\sim}{U}$ (cst.) as $|\underset{\sim}{x}| \to \infty$ ($-\underset{\sim}{U}$ = propagation velocity).

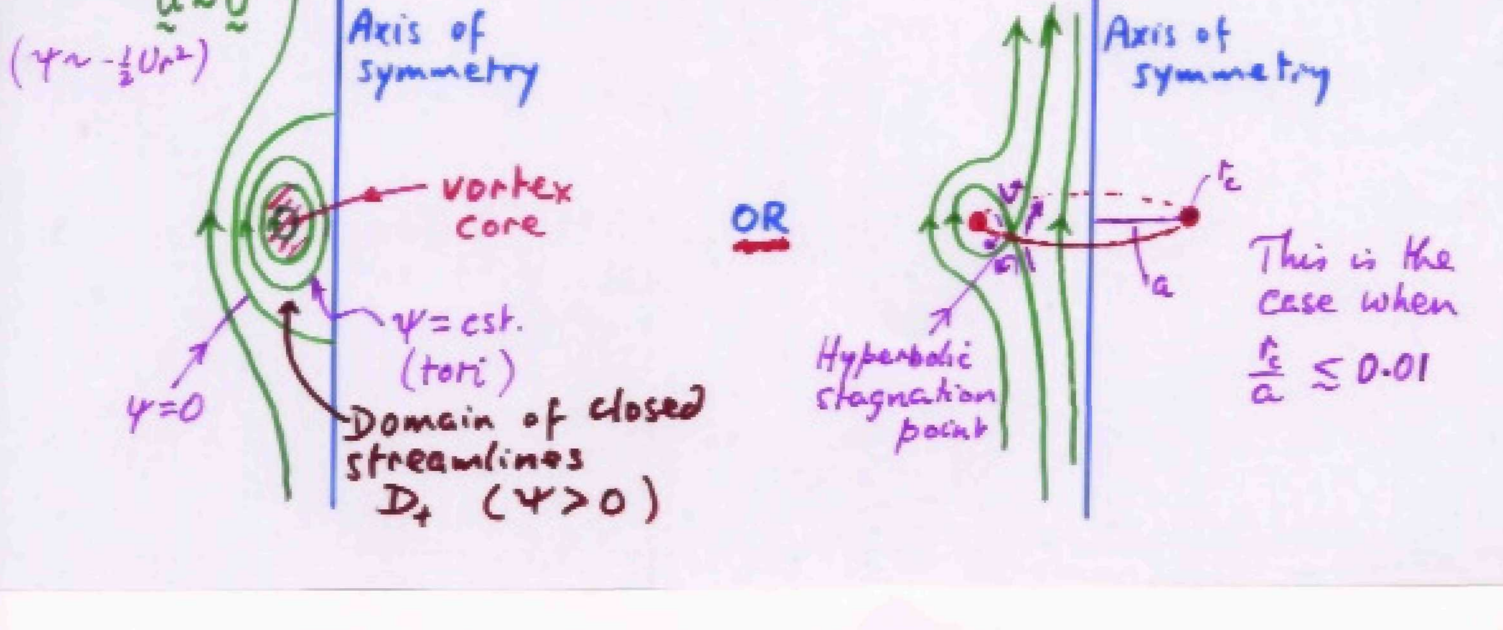
VORTEX RING WITHOUT SWIRL

With cylindrical polar coords (r, φ, z),

$$\underset{\sim}{u} = \left(\frac{1}{r}\frac{\partial \psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial \psi}{\partial r} \right) \qquad \psi = \psi(r, z)$$

(no swirl) (STOKES)

and we have essentially two possible streamline topologies:



In vortex core

$$\underline{\underline{\omega}} = (0, \omega_\varphi, 0) = (0, \tfrac{1}{r} D^2 \psi, 0)$$

Stokes op!

and the steady Euler eqn.

$$\nabla \times (\underline{\underline{u}} \times \underline{\underline{\omega}}) = 0$$

$$\Rightarrow \quad \frac{\omega_\varphi}{r} = F(\psi) \quad \text{for some } F.$$

Solutions are known for the case $F(\psi) = \text{cst.}$

e.g. HILL'S VORTEX:

$$\psi = \psi_H(r, z) = \begin{cases} 4\psi_m (r/a)^2 \left[1 - \frac{r^2 + z^2}{a^2}\right] & (r^2 + z^2 < a^2) \\ -\frac{1}{2} U r^2 \left[1 - \frac{a^3}{(r^2+z^2)^{3/2}}\right] & (r^2 + z^2 > a^2) \end{cases}$$

with $U = \frac{16}{3} \frac{\psi_m}{a^2}$

$$\text{giving} \quad F_H(\psi) = \begin{cases} -40\, \psi_m / a^4 & (0 < \psi < \psi_m) \\ 0 & (\psi < 0) \end{cases}$$

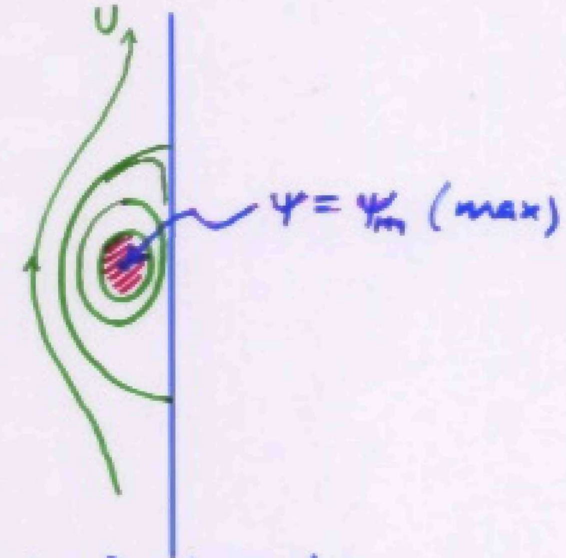
This is limiting member of a family of sol^{ns} with $F(\psi) = \text{cst.}$ $(\psi > 0)$ [Fraenkel 1970, 1972, Norbury 1973].

VERY LITTLE IS KNOWN CONCERNING SITUATION WHEN $F(\psi)$ IS GENERAL NONLINEAR FUNCTION - EVEN REGARDING THE QUESTION OF EXISTENCE OF SOLUTIONS.

WE NEED AN EXISTENCE PROOF WHICH YIELDS A NATURAL COMPUTATIONAL PROCEDURE FOR FINDING SOLUTIONS.

SIGNATURE OF VORTEX $V(\Psi)$

Instead of $F(\psi)$, we may characterise the ring structure by the function



$V(\Psi)$ = volume of toroid $\psi \geq \Psi$
$(0 \leq \Psi \leq \psi_m)$

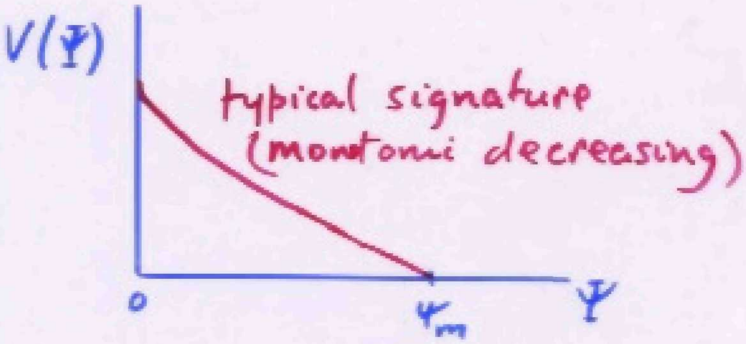
$V(0) = V_0$ = vol. of region of closed streamlines
$V(\psi_m) = 0$

e.g. for Hill's vortex

$$V = V_H(\Psi) = \frac{3V_0}{4\sqrt{2}}\left(1 - \frac{\Psi}{\psi_m}\right)\int_{-1}^{1}\left\{\frac{1-x^2}{1+x(1-\Psi/\psi_m)^{1/2}}\right\}^{1/2} dx$$

elliptic integral

$$\left[\, V_H(\Psi) \sim V_0\left[1 - \left(\frac{3}{16}\ln\frac{4\psi_m}{\Psi} + 1\right)\frac{\Psi}{\psi_m} + \cdots\right] \text{ as } \psi \to 0 \,\right]$$



WE MAY SHOW THAT FOR ANY REASONABLE $V(\Psi)$, (and any U!), THERE EXISTS A CORRESPONDING VORTEX RING (Moffatt JFM 1986, <u>173</u>, 289)

(AND THE PROOF INDICATES A COMPUTATIONAL PROCEDURE TO FIND IT)

THE PROOF IS BASED ON A WELL-KNOWN ANALOGY WITH MAGNETOSTATICS AND THE NATURAL PROCEDURE OF MAGNETIC RELAXATION TO MAGNETOSTATIC EQUILIBRIUM

STEADY EULER EQUATIONS

Ⓐ $$\boxed{\begin{array}{ll} \underset{\sim}{u} \times \underset{\sim}{\omega} = \nabla h & h = p/\rho + \tfrac{1}{2}u^2 \\ \nabla \cdot \underset{\sim}{u} = 0, & \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \end{array}}$$

EQUATIONS OF MAGNETOSTATIC EQUILIBRIUM (in perfectly conducting fluid)

Ⓑ $$\boxed{\begin{array}{ll} \underset{\sim}{j} \times \underset{\sim}{B} = \nabla p & \\ \nabla \cdot \underset{\sim}{B} = 0, & \underset{\sim}{j} = \nabla \times \underset{\sim}{B} \end{array}}$$

$\underset{\sim}{B}$ = mag. field
$\underset{\sim}{j}$ = current density
$\underset{\sim}{j} \times \underset{\sim}{B}$ = Lorentz force.

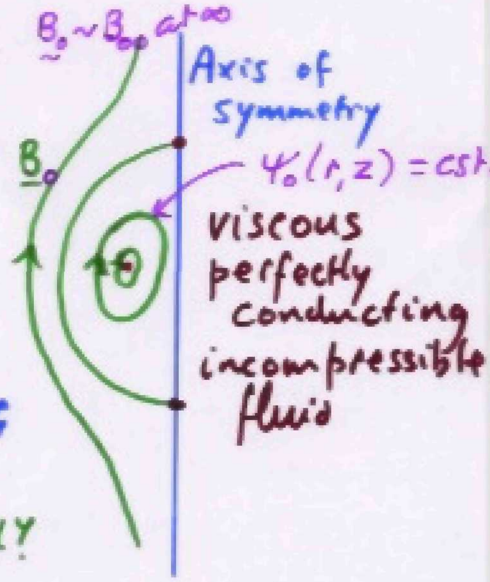
EXACT ANALOGY: $\underset{\sim}{u} \leftrightarrow \underset{\sim}{B}$, $\underset{\sim}{\omega} \leftrightarrow \underset{\sim}{j}$, $h \leftrightarrow -p$ (!)

WE CAN CONSTRUCT A SMOOTH FIELD $\underset{\sim}{B}_0(\underset{\sim}{x})$ WITH THE TOPOLOGY THAT INTERESTS US. IN GENERAL,

$$\nabla \times (\underset{\sim}{j}_0 \times \underset{\sim}{B}_0) \neq 0$$

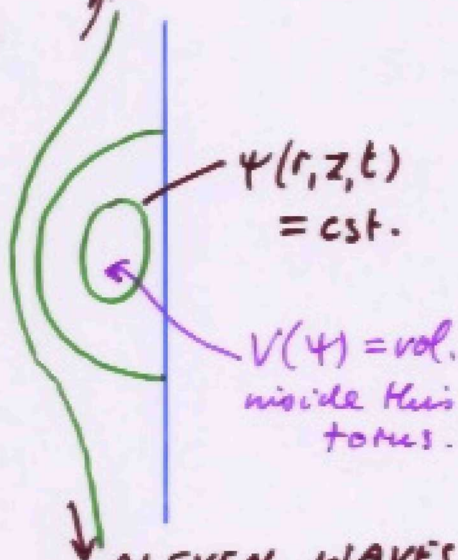
∴ FLUID WILL MOVE AND WILL CARRY MAGNETIC FIELD ACCORDING TO INDUCTION EQN. OF MHD

$$\frac{\partial B}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \qquad \underset{\sim}{v} = \text{vel}^y$$



TOPOLOGY OF $\underset{\sim}{B}(\underset{\sim}{x}, t)$ IS CONSERVED !

WHAT IS THE END-STATE ?



With $\underset{\sim}{B}(\underset{\sim}{x},t) = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$,

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \text{curl}\,(\underset{\sim}{u} \times \underset{\sim}{B})$$

$$\Rightarrow \frac{D\psi}{Dt} = 0 \qquad \left(\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}.\nabla\right)$$

ALFVEN WAVES (DAMPED BY VISCOSITY) MAY PROPAGATE TO $\pm\infty$

i.e. MAGNETIC SURFACES $\psi = \text{cst.}$ MOVE WITH FLUID.

$\therefore$ Volume inside $\psi = \Psi$ remains constant

$\therefore$ SIGNATURE $V(\Psi)$ is invariant during relaxation

THIS IS REALLY A TOPOLOGICAL INVARIANT SINCE IT TELLS US THAT NESTED TORI RETAIN THEIR ORDERING.

SO WE ARE DEALING WITH A PROBLEM OF RELAXATION UNDER TOPOLOGICAL CONSTRAINTS

WE NEED ALSO AN ENERGY EQUATION.

Navier-Stokes model (not the only possibility)

$$\left.\begin{aligned} \rho \frac{D\underset{\sim}{v}}{Dt} &= \boxed{-\nabla p + \underset{\sim}{j} \times \underset{\sim}{B}} + \rho \nu \nabla^2 \underset{\sim}{v} \\ \frac{\partial \underset{\sim}{B}}{\partial t} &= \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \\ \nabla \cdot \underset{\sim}{v} &= \nabla \cdot \underset{\sim}{B} = 0 \end{aligned}\right\} \quad (*)$$

At $t=0$, $\underset{\sim}{B}(\underset{\sim}{x},0) = \underset{\sim}{B}_0(\underset{\sim}{x})$ with given $V(+)$
and satisfying $\underset{\sim}{B}_0(\underset{\sim}{x}) \sim \underset{\sim}{B}_\infty$ at ∞.
$\underset{\sim}{v}(\underset{\sim}{x},0) = 0$.

For $t>0$, we maintain outer cond^{n}: $\underset{\sim}{B}(\underset{\sim}{x},t) \sim \underset{\sim}{B}_\infty$ at ∞.
and $\underset{\sim}{v} \to 0$ as $|\underset{\sim}{x}| \to \infty$.

Let $\underset{\sim}{B} = \underset{\sim}{B}_\infty + \underset{\sim}{b}(\underset{\sim}{x},t)$

Eqns. (*) are compatible with outer cond^{ns}.

$$|\underset{\sim}{v}| = O(|\underset{\sim}{x}|^{-2}), \quad |\underset{\sim}{b}| = O(|\underset{\sim}{x}|^{-3}), \quad p - p_\infty = O(|\underset{\sim}{x}|^{-3})$$

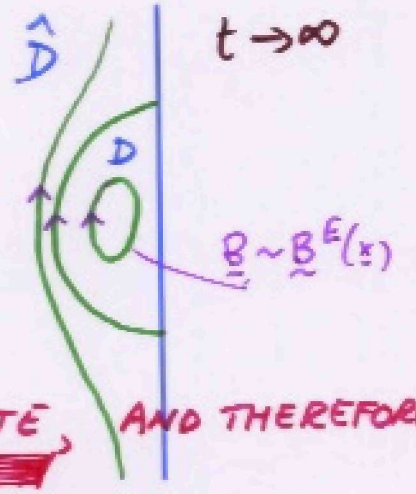
at ∞.

We may then easily obtain

$$\frac{d}{dt}\left[M(t) + K(t)\right] = -\Phi(t) \qquad \textit{Energy eqn.}$$

where $M(t) = \frac{1}{2}\int \underset{\sim}{b}^2\, dV$, $K(t) = \frac{1}{2}\rho \int \underset{\sim}{v}^2\, dV$

and $\Phi(t) = \rho\nu \int (\nabla \times \underset{\sim}{v})^2\, dV$



$$\frac{d}{dt}\left(M(t) + K(t)\right) = -\Phi(t)$$

$\therefore$ $M(t) + K(t)$ is monotonic decreasing
$\therefore$ tends to a limit (since ≥ 0)

$\therefore$ $\Phi(t) \to 0$ as $t \to \infty$.

MOREOVER $\underset{\sim}{B}$ CERTAINLY REMAINS FINITE AND THEREFORE CONTINUOUS (!) (the surfaces $\psi = $ cst. remain separate for all t)

$\therefore$ NO SINGULARITIES IN $\underset{\sim}{v}$ CAN DEVELOP
(the process can be as viscous as we like — even low Re)

So $\underset{\sim}{v} \to 0$ for all $\underset{\sim}{x}$

and $\underset{\sim}{B}(\underset{\sim}{x}, t) \to \underset{\sim}{B}^E(\underset{\sim}{x})$ satisfying

$$\underset{\sim}{j}^E \times \underset{\sim}{B}^E = \nabla p^E \qquad \underset{\sim}{B}^E = \underset{\sim}{B}_{\infty} + \underset{\sim}{b}^E$$

($\underset{\sim}{b}^E$ cannot be zero since $\underset{\sim}{B}^E$ has same signature $V(\Psi)$ as the initial field $\underset{\sim}{B}_0(\underset{\sim}{x})$).

Also $\underset{\sim}{j}^E \equiv 0$ outside domain D of closed $\underset{\sim}{B}^E$-lines

Proof: $\underset{\sim}{B}^E . \nabla p^E = 0$ $\therefore$ $p^E = $ cst. on $\underset{\sim}{B}^E$-lines $= p_{\infty}$ in $\hat{D}$
$\therefore$ $\underset{\sim}{j}^E \times \underset{\sim}{B}^E = 0$ in $\hat{D}$ $\therefore$ $\underset{\sim}{j}^E = \alpha(x)\, \underset{\sim}{B}^E$ in $\hat{D}$
where $\underset{\sim}{B}^E . \nabla\alpha = 0$ $\therefore$ $\alpha = $ cst. on $\underset{\sim}{B}^E$-lines
but $\alpha = 0$ at ∞ $\therefore$ $\alpha \equiv 0$ in $\hat{D}$
$\therefore$ $\underset{\sim}{j}^E \equiv 0$ in $\hat{D}$ QED

BY VIRTUE OF THE ANALOGY WITH THE STEADY EULER EQUNS., WE MAY NOW SIMPLY REPLACE $\underline{B}^E$ by $\underline{u}^E$, $\underline{B}_\infty$ by $\underline{U}$, AND WE HAVE IN EFFECT CONSTRUCTED A VORTEX RING WITH THE ARBITRARILY PRESCRIBED SIGNATURE $V(\psi)$.

CONSTRUCTION OF INITIAL FLUX FUNCTION $\psi_0(r,z)$ WITH A GIVEN SIGNATURE $V(\psi)$

Let $\psi = \Psi(V)$ be fn. inverse to $V(\psi)$

Let $\psi_0(r,z) = K\big(\underbrace{\psi_H(r,z)}_{\text{Hill's vortex}}\big)$ $\quad$ K(ψ) to be detd

Then surfaces $\psi_H = \text{cst.}$ coincide with surfaces $\psi_0 = \text{cst.}$

$$\therefore V_H(\psi) = \text{vol. inside } \psi_H = \psi$$

$$= \text{vol. inside } \psi_0 = K(\psi)$$

$$= V(K(\psi))$$

$$\therefore K(\psi) = \Psi\big(V_H(\psi)\big)$$

$$\therefore \boxed{\psi_0(r,z) = \Psi\left\{V_H\left[\psi_H(r,z)\right]\right\}} \text{ gives reqd. flux fn.}$$

VORTEX RINGS WITH SWIRL

Suppose now that

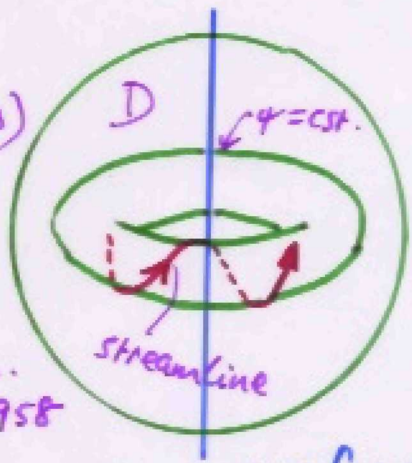
$$\underset{\sim}{u} = \left(\frac{1}{r}\frac{\partial \psi}{\partial z},\ u_\varphi(r,z),\ -\frac{1}{r}\frac{\partial \psi}{\partial r} \right)$$

Under steady conditions,

$$u_\varphi = \frac{1}{r} G(\psi), \qquad \frac{\omega_\varphi}{r} = F(\psi) - \frac{G(\psi)\, G'(\psi)}{r^2}$$

A particular family of solutions is known for case when $G(\psi) = \alpha \psi$, $F(\psi) = \beta \psi$, α, β cst.

Streamlines are helices on the tori $\psi =$ cst. (ergodic or torus knots)
These solutions are described in Moffatt 1969, JFM, $\underline{35}$, 117
– analogous to magnetostatic sol$^{\underline{n}}$! obtained by Chandrasekhar, Prendergast, ... ~ 1958



(probably known to Lamb, Rayleigh or Kelvin!)

$$\mathcal{H} = \int_D \underset{\sim}{u} \cdot \underset{\sim}{\omega}\, dV \neq 0$$

CAN WE OBTAIN MORE GENERAL SOLN$^{\underline{S}}$ FOR WHICH $F(\psi)$ and $G(\psi)$ ARE NOT SIMPLE LINEAR FUNCTIONS?

YES
(i) GUESS A REASONABLE FIELD $\underline{u}_0(\underline{x})$
(ii) REPLACE $\underline{u}_0(\underline{x})$ by $\underline{B}_0(\underline{x})$
(iii) LET $\underline{B}_0(\underline{x})$ RELAX TO EQM. $\underline{B}^E(\underline{x})$
(iv) REPLACE $\underline{B}^E(\underline{x})$ by $\underline{u}^E(\underline{x})$

INVARIANTS DURING RELAXATION

(i) $V(\psi)$ is invariant as before.

(ii) Also $W(\psi) = \int_A B_\varphi \, dA = \text{cst.}$

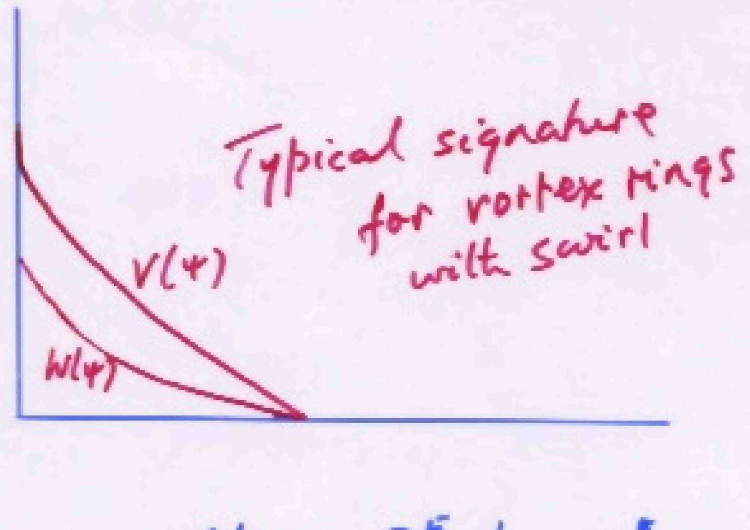
[Andrew Gilbert]

where A = cross-section in (r, z) plane of torus $\psi(r, z) = \text{cst.}$ [Kruskal & Kulsrud 1958]

(since flux of $\underline{B}$ through any moving circuit is conserved – Alfven's theorem)

So the signature is now the pair of functions

$$\{V(\psi), W(\psi)\}$$



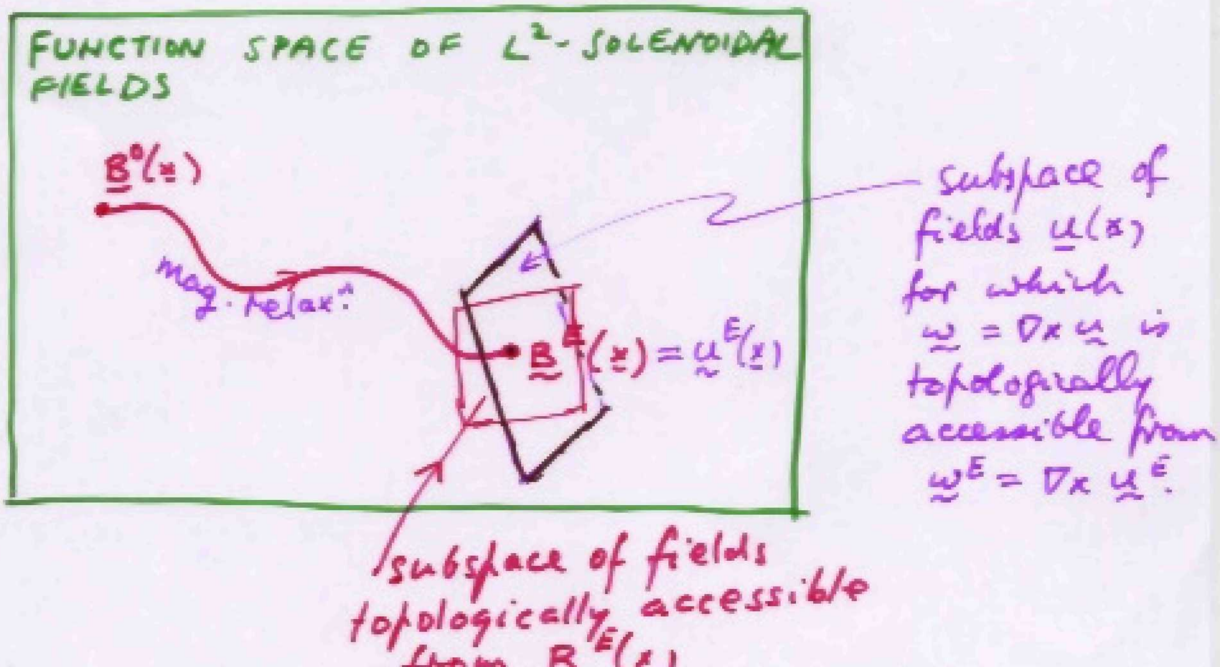
When we replace $\underline{B}^E$ by $\underline{u}^E$, $W(\psi)$ becomes the azimuthal flux (flow rate) within the torus $\psi = \text{cst.}$

COMMENT ON STABILITY

BY VIRTUE OF THE RELAXATION PROCESS, THE MAGNETOSTATIC EQUILIBRIA $\underline{B}^E(\underline{x})$ WILL BE GENERALLY STABLE (energy norm)

– BUT THIS TELLS US NOTHING ABOUT THE STABILITY OF THE ANALOGOUS EULER FLOWS $\underset{\sim}{u}^E(\underline{x})$

stability
Moffatt + Moore
JFM 1979



Stability of Euler flow requires consideration of perturbations in a different subspace.

[Arnold 1966 J. Méc. $\underset{\sim}{5}$, 29-43 (in French)

Moffatt: 1986 JFM $\underset{\sim}{166}$ 359-378].

MAGNETOHYDRODYNAMICS (IDEAL)

$$\left.\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= \mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{B} - \nabla\left(\tfrac{p}{\rho} + \tfrac{1}{2}\mathbf{u}^2\right)\\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla\wedge(\mathbf{u}\wedge\mathbf{B}) \qquad (\mathbf{j} = \nabla\wedge\mathbf{B})\end{aligned}\right\}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0$$

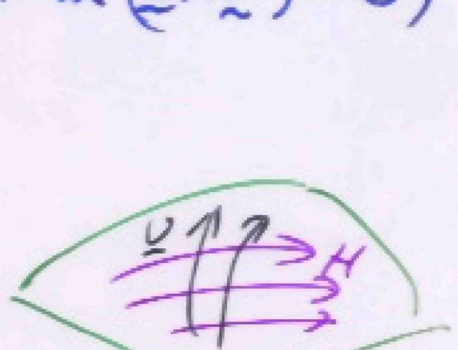
STRUCTURE OF STEADY STATES (FIXED POINTS OF DYNAMICAL SYSTEM)

Let
$$\left.\begin{aligned}\mathbf{u} &= \mathbf{U}(\mathbf{x}), \quad \mathbf{B} = \mathbf{H}(\mathbf{x})\\ \boldsymbol{\omega} &= \boldsymbol{\Omega}(\mathbf{x}), \quad \mathbf{j} = \mathbf{J}(\mathbf{x})\end{aligned}\right\}$$

$$\mathbf{U}\wedge\mathbf{H} = \nabla\varphi \qquad (\text{since } \nabla\wedge(\mathbf{U}\wedge\mathbf{H}) = 0)$$

$$\mathbf{U}\cdot\nabla\varphi = 0, \quad \mathbf{H}\cdot\nabla\varphi = 0$$

$\mathbf{U}$-lines and $\mathbf{H}$-lines lie on surfaces $\varphi = \text{cst}$.



HOWEVER, IT MAY HAPPEN THAT $\varphi \equiv \text{cst.}$ in some $\mathcal{D}_1 \subseteq \mathcal{D}$

Then $\mathbf{U}\wedge\mathbf{H} = 0$ $\qquad$ $\mathbf{H} = \alpha(\mathbf{x})\,\mathbf{U}$ in $\mathcal{D}_1$

$\mathbf{H}\cdot\nabla\alpha = \mathbf{U}\cdot\nabla\alpha = 0$ $\qquad$ $\mathbf{U}$-lines & $\mathbf{H}$-lines lie on surfaces $\alpha = \text{cst.}$

HOWEVER, IT MAY HAPPEN THAT

$\alpha \equiv$ cst. in some $\mathcal{D}_2 \subseteq \mathcal{D}_1$

Then $\underset{\sim}{H} = \alpha \underset{\sim}{U}$ in $\mathcal{D}_2$ (α = cst.)

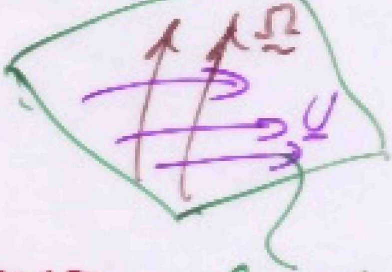
$\therefore \underset{\sim}{J} = \alpha \underset{\sim}{\Omega}$

so $\underset{\sim}{J} \wedge \underset{\sim}{H} = -\alpha^2 \underset{\sim}{U} \wedge \underset{\sim}{\Omega}$

so $(1-\alpha^2)(\underset{\sim}{U} \wedge \underset{\sim}{\Omega}) = \nabla h$ $\quad h = p/\rho + \frac{1}{2}U^2$

$\therefore \underset{\sim}{U} \cdot \nabla h = \underset{\sim}{\Omega} \cdot \nabla h = 0$

$\therefore$ $\underset{\sim}{U}$-lines and $\underset{\sim}{\Omega}$-lines lie on surfaces h = cst.



HOWEVER, IT MAY HAPPEN THAT

$h \equiv$ cst. in some $\mathcal{D}_3 \subseteq \mathcal{D}_2$

Then $\beta \underset{\sim}{U} = \underset{\sim}{\Omega}$ $\qquad \underset{\sim}{U} \cdot \nabla\beta = \underset{\sim}{\Omega} \cdot \nabla\beta = 0$

$\underset{\sim}{U}$-lines & $\underset{\sim}{\Omega}$-lines lie on surfaces β = cst.

HOWEVER, IT MAY HAPPEN THAT

$\beta \equiv$ cst. in some $\mathcal{D}_4 \subseteq \mathcal{D}_3$

Then $\underset{\sim}{\Omega} = \beta \underset{\sim}{U}$ in $\mathcal{D}_4$ $\quad \beta$ = cst.

BELTRAMI FLOW

& only then can $\underset{\sim}{U}$-lines be chaotic in $\mathcal{D}_4$!

INVARIANTS OF IDEAL MHD EQUATIONS

1. Energy : $E = \frac{1}{2}\int_{\mathcal{D}} (\boldsymbol{u}^2 + \boldsymbol{b}^2)\, dV$

2. Magnetic Helicity :

$$\mathcal{H}_M = \int_{\mathcal{D}} \boldsymbol{B} \cdot \mathrm{curl}^{-1} \boldsymbol{B}\, dV$$

[$\boldsymbol{B}$-lines are frozen in the fluid and $\int_S \boldsymbol{B} \cdot \boldsymbol{n}\, dS$ is conserved for every material surface S

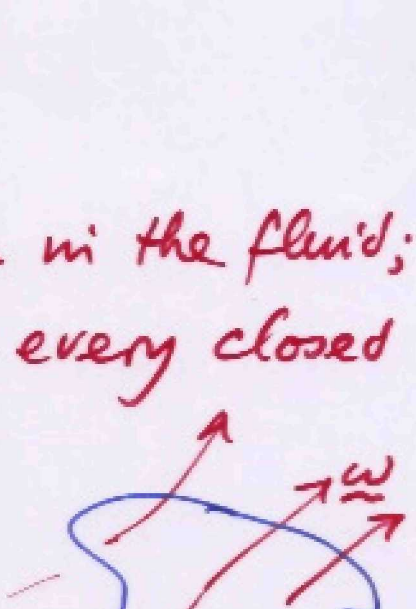
]

3. Cross Helicity

$$\mathcal{H}_C = \int_{\mathcal{D}} \boldsymbol{u} \cdot \boldsymbol{B}\, dV$$

[$\boldsymbol{\omega}$-lines are not frozen in the fluid; but flux of $\boldsymbol{\omega}$ through every closed $\boldsymbol{B}$-line is conserved

$$\Gamma_B = \oint_{C_B} \boldsymbol{u} \cdot d\boldsymbol{x} = \mathrm{cst}.$$



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ISOMAGNETOVORTICAL FOLIATION OF FUNCTION SPACE $\{\underset{\sim}{u}(\underline{x}), \underset{\sim}{B}(\underline{x})\}$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\boxed{\underset{\sim}{u}} \wedge \underset{\sim}{B})$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} = \boxed{\underset{\sim}{u}} \wedge \underset{\sim}{\omega} + \boxed{\underset{\sim}{j}} \wedge \underset{\sim}{B} - \nabla h$$

$$\underset{\sim}{\omega} = \nabla \wedge \underset{\sim}{u} \,, \quad \underset{\sim}{j} = \nabla \wedge \underset{\sim}{B}$$

REPLACE $\underset{\sim}{u}$ by $\underset{\sim}{v}(\underline{x}, t)$ $\quad \nabla \cdot \underset{\sim}{v} = 0$

$\underset{\sim}{j}$ by $\underset{\sim}{c}(\underline{x}, t)$ $\quad \nabla \cdot \underset{\sim}{c} = 0$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{B}) = [\underset{\sim}{v}, \underset{\sim}{B}] \quad \text{Commutator}$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} = \underset{\sim}{v} \wedge \underset{\sim}{\omega} + \underset{\sim}{c} \wedge \underset{\sim}{B} - \nabla h$$

$$\nabla \cdot \underset{\sim}{u} = \nabla \cdot \underset{\sim}{B} = 0$$

THESE MODIFIED EQUATIONS STILL CONSERVE $\mathcal{H}_M$ and $\mathcal{H}_C$ for arbitrary choice of $\underline{v}(\underline{x}, t)$ and $\underline{c}(\underline{x}, t)$!

BUT THEY DO NOT CONSERVE

$$E = \frac{1}{2} \int (\underline{u}^2 + B^2) \, dV.$$

In fact

$$\frac{dE}{dt} = \int_{\mathcal{D}} \left(\underset{\sim}{u} \cdot \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{B} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} \right) dV$$

$$= \int \Big\{ \underset{\sim}{u} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{\omega}) + \underset{\sim}{u} \cdot (\underset{\sim}{c} \wedge \underset{\sim}{B}) - \underset{\sim}{u} \cdot \nabla h$$

$$+ \underbrace{\underset{\sim}{B} \cdot \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{B})}_{\downarrow \; \underset{\sim}{j} \cdot (\underset{\sim}{v} \wedge \underset{\sim}{B})} \Big\} \, dV$$

$$= -\int \left(\underset{\sim}{v} \cdot (\underset{\sim}{u} \wedge \underset{\sim}{\omega} + \underset{\sim}{j} \wedge \underset{\sim}{B}) + \underset{\sim}{c} \cdot (\underset{\sim}{u} \wedge \underset{\sim}{B}) \right) dV$$

(this vanishes when $\underset{\sim}{v} = \underset{\sim}{u}$ and $\underset{\sim}{c} = \underset{\sim}{j}$)

Choose $\underset{\sim}{v}$ and $\underset{\sim}{c}$ so that E decreases

$$\underset{\sim}{v} = \underset{\sim}{u} \wedge \underset{\sim}{\omega} + \underset{\sim}{j} \wedge \underset{\sim}{B} - \nabla \alpha$$

$$\underset{\sim}{c} = \underset{\sim}{u} \wedge \underset{\sim}{B} - \nabla \beta$$

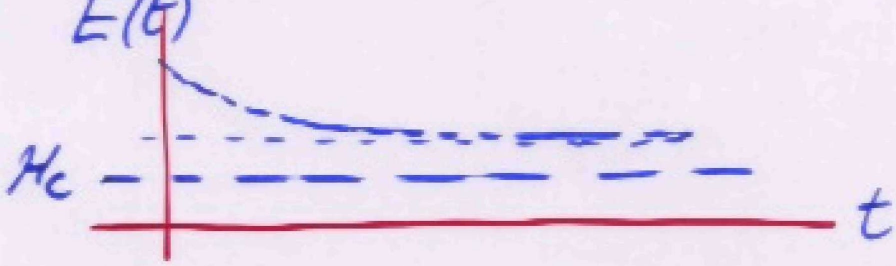
$$\int \underset{\sim}{v} \cdot \nabla \alpha = \int_{\partial \mathcal{D}} \underset{\sim}{n} \cdot \underset{\sim}{v} \, \alpha \, dS = 0 \text{ etc.}$$

$$\Rightarrow \frac{dE}{dt} = -\int \left(\underset{\sim}{v}^2 + \underset{\sim}{c}^2 \right) dV$$

so E is monotonic decreasing

BUT E has a lower bound:

$$\int (\underset{\sim}{u}^2 + \underset{\sim}{B}^2)\, dV \geq 2\left|\int \underset{\sim}{u}\cdot\underset{\sim}{B}\, dV\right| = 2\mathcal{H}_c$$
$$= \underline{\underline{cst.}}$$



So $E(t) \to$ limit as $t \to \infty$

$\frac{dE}{dt} \to 0 \quad \therefore \quad \underset{\sim}{v}^2 + \underset{\sim}{c}^2 \to 0$ everywhere

(point singularities ??)

In the limit

$$\underset{\sim}{u} \to \underset{\sim}{U}^E(\underline{x}), \quad \underset{\sim}{\omega} \to \underset{\sim}{\Omega}^E(\underline{x})$$
$$\underset{\sim}{j} \to \underset{\sim}{J}^E(\underline{x}), \quad \underset{\sim}{B} \to \underset{\sim}{B}^E(\underline{x})$$

with
$$\left.\begin{aligned} \underset{\sim}{U}^E \wedge \underset{\sim}{\Omega}^E + \underset{\sim}{J}^E \wedge \underset{\sim}{B}^E &= \nabla\alpha \\ \underset{\sim}{U}^E \wedge \underset{\sim}{B}^E &= \nabla\beta \end{aligned}\right\}$$

and (i) ~~all~~ topology of $\underset{\sim}{B}^E$ is same as that of $B_0(\underline{x})$ arbitrary

(ii) all cross-helicity invariants are same for $\{\underset{\sim}{B}^E, \underset{\sim}{U}^E\}$ as for $\{\underset{\sim}{B}_0, \underset{\sim}{U}_0\}$

Hence we have a Theorem:

There exists a steady solution of the (ideal) MHD equations having arbitrarily prescribed topology of $\underset{\sim}{B}$ and having arbitrarily prescribed mutual linkage of $\underset{\sim}{\Omega}$ and $\underset{\sim}{B}$ fields.

MOREOVER:

SUCH SOLUTIONS ARE STABLE BECAUSE

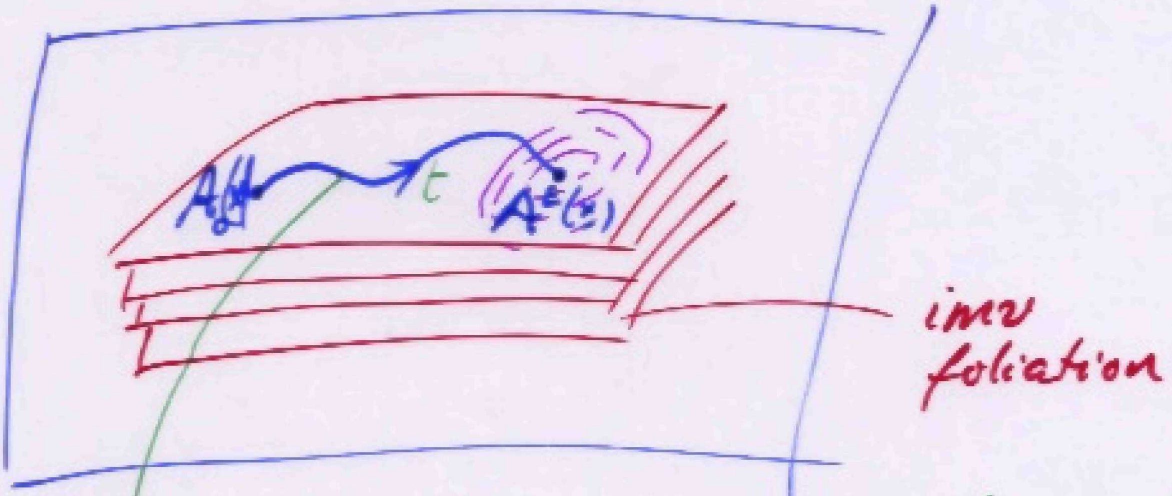
E is MINIMAL ON THE

ISOMAGNETOVORTICAL (IMV) FOLIUM

(THE SET OF FIELDS ACCESSIBLE FROM $\{\underline{B}_0(\underline{x}), \underline{U}_0(\underline{x})\}$ via "DEFORMATION FIELDS" $\{\underline{v}(\underline{x},t), \underline{c}(\underline{x},t)\}$.)

$\mathcal{F}$ = FUNCTION SPACE OF FIELDS

$$\mathcal{A}(\underset{\sim}{x}) = \{\underset{\sim}{B}(\underset{\sim}{x}), \underset{\sim}{U}(\underset{\sim}{x})\}$$



relaxation on the inv folium through $A_0(x)$
E monotonic decreasing

"Curves" E = cst. on inv folium near $A^E(x)$ are elliptic.

WHY DOES THIS NOT WORK WHEN $\underset{\sim}{B}_0(\underset{\sim}{x}) \equiv 0$?

BECAUSE THEN $\mathcal{H}_c = \int \underset{\sim}{u} \cdot \underset{\sim}{B} \, dV = 0$

AND WE LOSE THE POSITIVE LOWER BOUND

$E = \frac{1}{2} \int \underset{\sim}{u}^2 dV$ can go to zero

No guarantee of existence of steady stable 3D Euler flows.