Numerical Study of Quantum Turbulence

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Thanks to W. F. Vinen

Ref. M. Kobayashi and M. Tsubota, Phys. Rev. Lett. 94, 065302 (2005)

Outline of this talk

1. Superfluid helium and quantized vortices

2. Classical turbulence

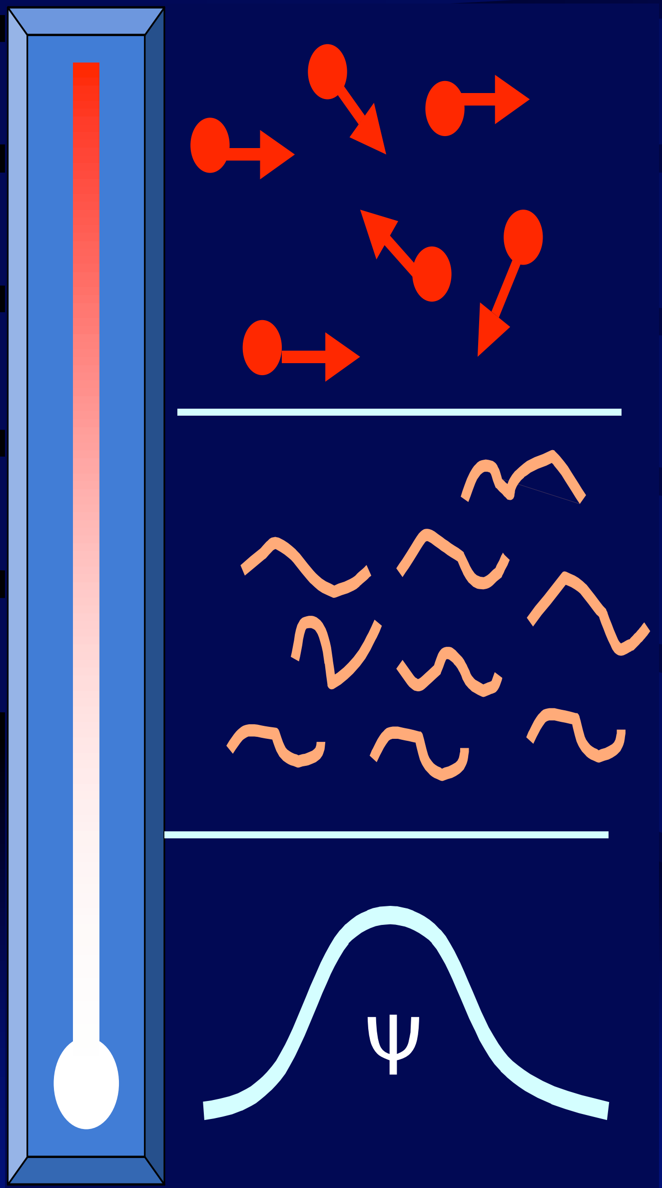
3. Quantum turbulence at very low temperatures
～Properties of the inertial range～

    3.1 Decaying turbulence

    3.2 Steady turbulence

4. Conclusions

1. Superfluid helium and quantized vortices



Duality of matter and wave

Each atom behaves as a particle at high temperatures.

Thermal de Broglie wave length
～ Mean distance between particles

Each atom behaves like a wave at low temperatures.

Bose-Einstein condensation

The matter waves become coherent, making a macroscopic wave function Ψ extending over the whole system.

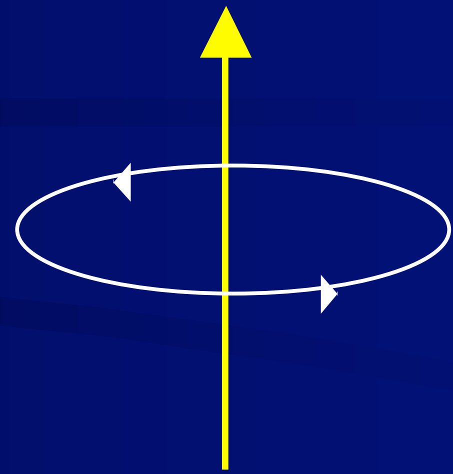
A quantized vortex of superflow is a topological defect in a Bose-Einstein condensate. Any rotational motion in superfluid is sustained by quantized vortices.

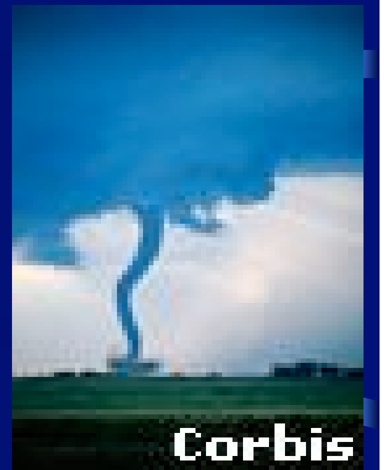
(i) The circulation is quantized.

$$\oint \mathbf{v}_s \cdot d\mathbf{s} = \kappa n \qquad (n = 0, 1, 2, \cdots)$$

$$\kappa = h/m = 9.97 \times 10^{-4}\ \mathrm{cm^2/s}$$

A vortex with $n \geqq 2$ is unstable.





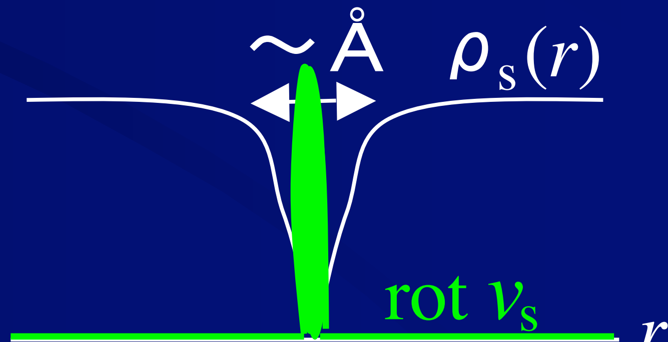
⇨ Every vortex has the same circulation.

(ii) The viscous diffusion of the vorticity does not work.

⇨ The vortex is stable.

(iii) The core size is very small.

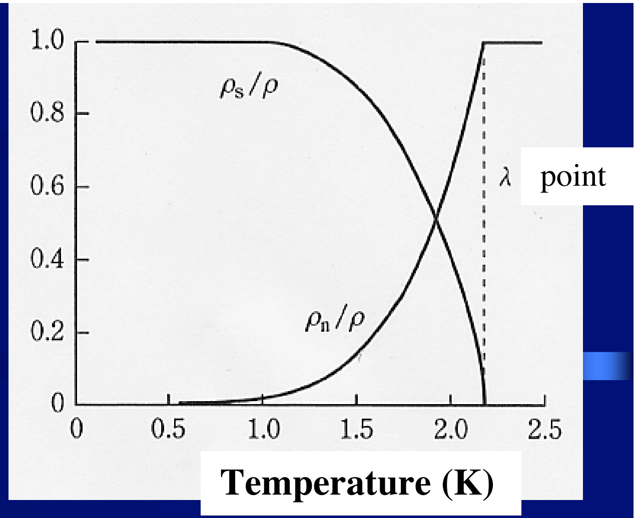
⇨ The core size is the order of the coherence length.



Superfluid helium

Liquid ^{4}He enters the superfluid state at 2.17K (λ point) with Bose condensation.

Its hydrodynamics is well described by the two fluid model.



Superfluid helium becomes dissipative when it flows above a critical velocity.

1955 **Feynman** Proposing "superfluid turbulence" consisting of a tangle of quantized vortices.

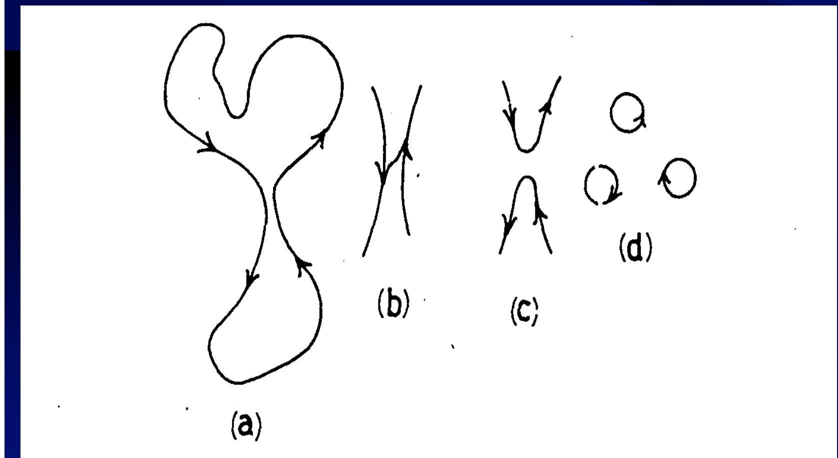


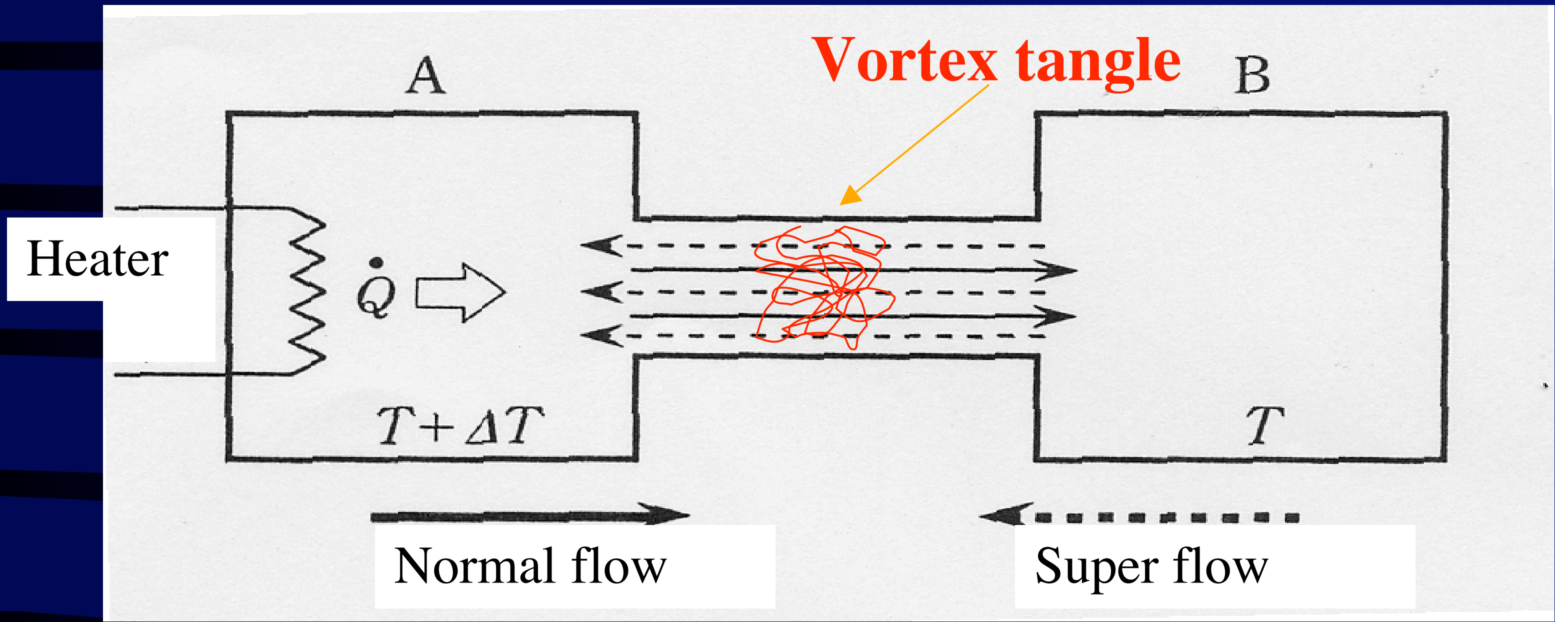
Fig. 10. A vortex ring (a) can break up into smaller rings if the transition between states (b) and (c) is allowed when the separation of vortex lines becomes of atomic dimensions. The eventual small rings (d) may be identical to rotons.

1955, 1957 **Hall and Vinen**

Observing "superfluid turbulence"

The mutual friction between the vortex tangle and the normal fluid causes that dissipation.

Lots of experimental studies were done chiefly for thermal counterflow of superfluid ^{4}He.



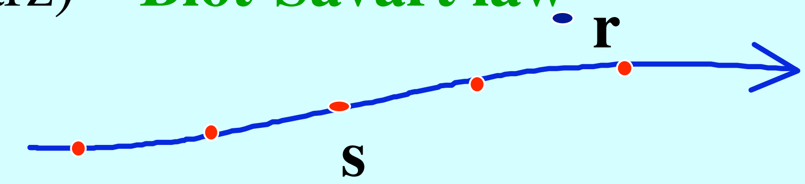
1980's **K. W. Schwarz** Phys.Rev.B38, 2398(1988)

Made the direct numerical simulation of the three-dimensional dynamics of quantized vortices and succeeded in explaining quantitatively the observed temperature difference $\triangle T$.

How to describe the vortex dynamics

Vortex filament formulation (Schwarz)

Biot-Savart law



$$\mathbf{v}_s(\mathbf{r}) = \frac{\kappa}{4\pi}\int \frac{(\mathbf{s}-\mathbf{r})\times d\mathbf{s}}{|\mathbf{s}-\mathbf{r}|^3}$$

A vortex makes the superflow of the Biot-Savart law, and moves with this local flow. At a finite temperature, the mutual friction should be considered.

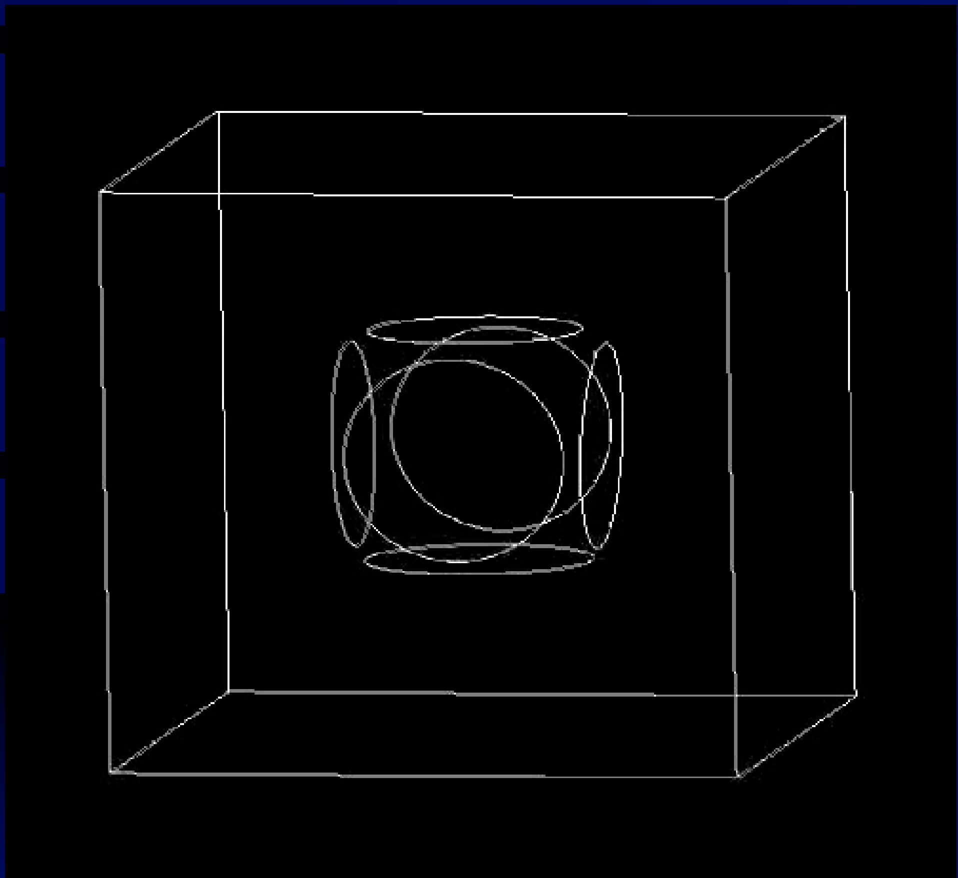
The Gross-Pitaevskii equation for the macroscopic wave function

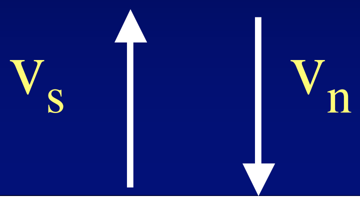
$$\Psi(r) = \sqrt{n_0(r)}\,e^{i\theta(r)}$$

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\mathrm{ext}}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t)$$

Development of a vortex tangle in a thermal counterflow

M.Tsubota, T.Araki, S.K.Nemirovskii , Phys. Rev. B62, 11751(2000).





Schwarz, Phys.Rev.B38, 2398(1988).

Schwarz obtained numerically the statistically steady state of a vortex tangle which is sustained by the competition between the applied flow and the mutual friction. The obtained vortex density $L(v_{ns}, T)$ agreed quantitatively with experimental data.

Most studies of superfluid tu
thermal counterflow.

⇨ No ana

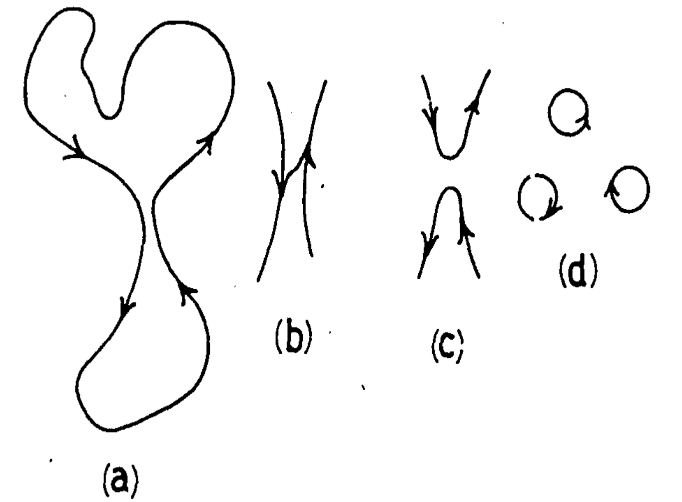


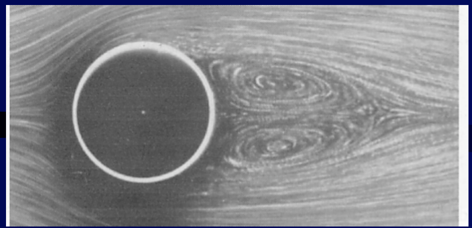
Fig. 10. A vortex ring (a) can break up into smaller rings if the transition between states (b) and (c) is allowed when the separation of vortex lines becomes of atomic dimensions. The eventual small rings (d) may be identical to rotons.

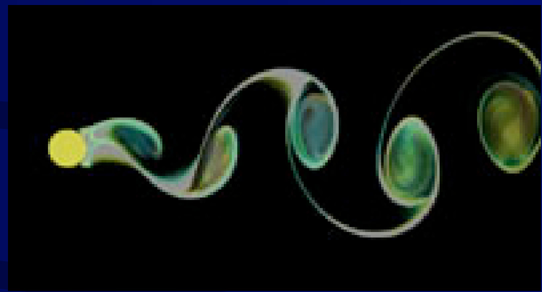
When Feynman showed the above figure, he thought of a cascade process in classical turbulence.

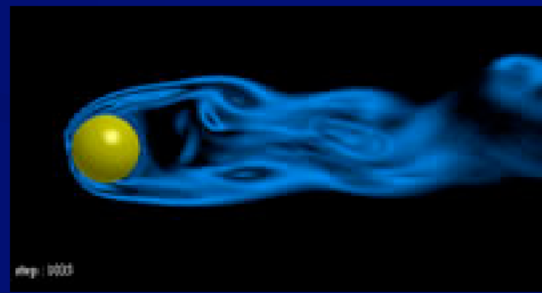
What is the relation between superfluid turbulence and classical turbulence ?

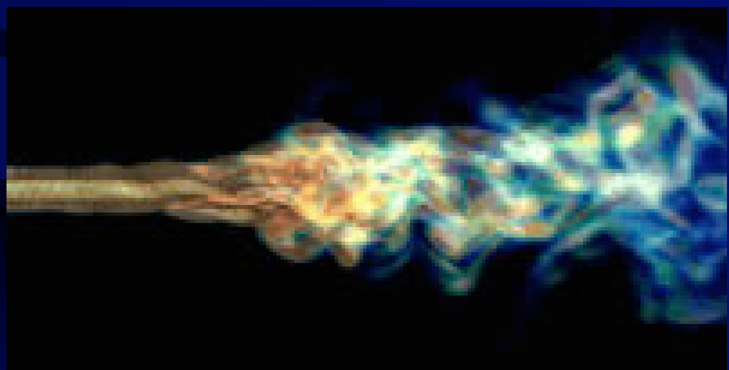
2. Classical turbulence

When we raise the flow velocity around a sphere,



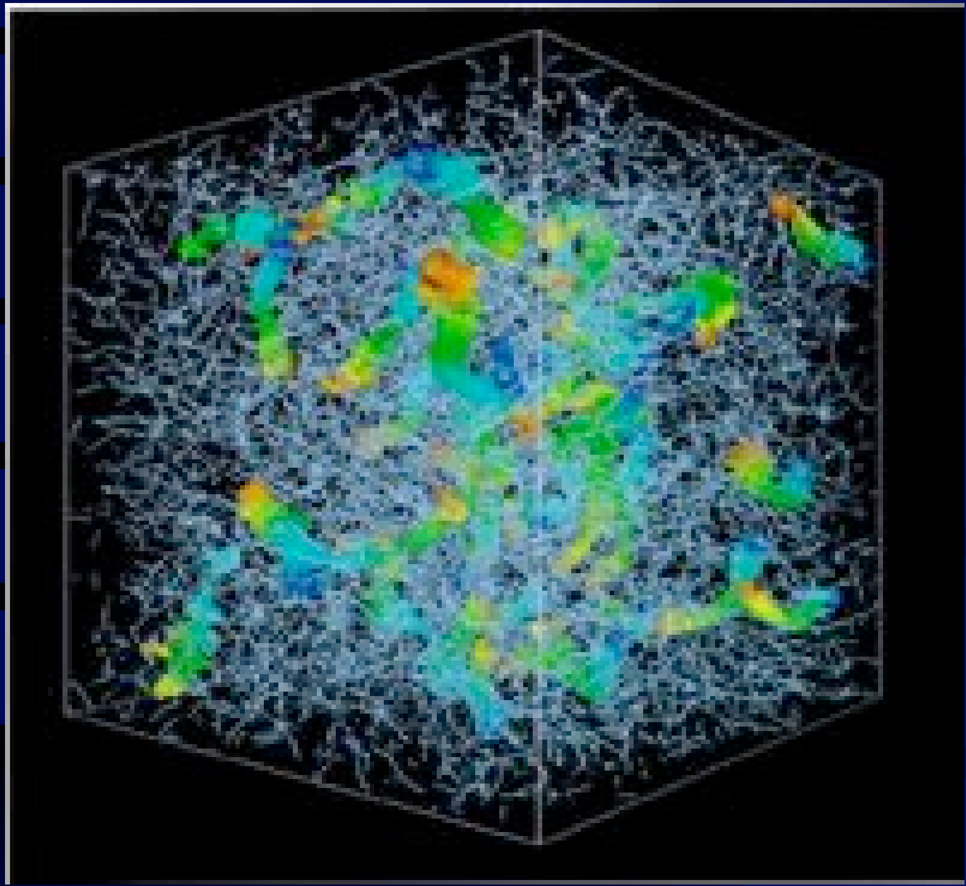




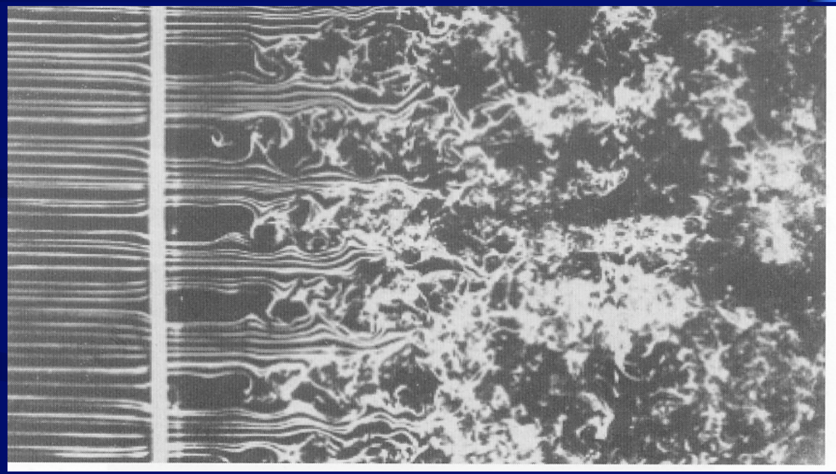


The understanding and control of turbulence have been one of the most important problems in fluid dynamics since Leonardo Da Vinci, but it is too difficult to do it.

Classical turbulence and vortices



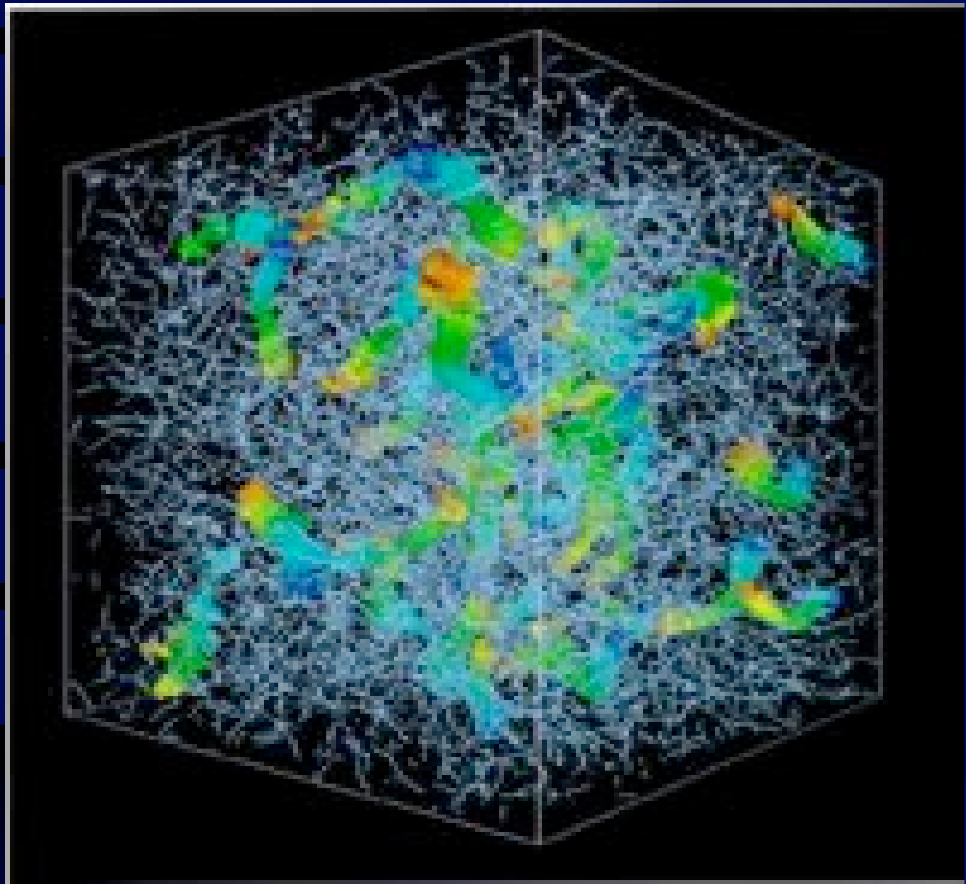
Grid turbulence



Numerical analysis of the Navier-Stokes equation made by Shigeo Kida

Vortex cores are visualized by tracing pressure minimum in the fluid.

Classical turbulence and vortices



・The vortices have different circulation and different core size.

・The vortices repeatedly appear, diffuse and disappear.

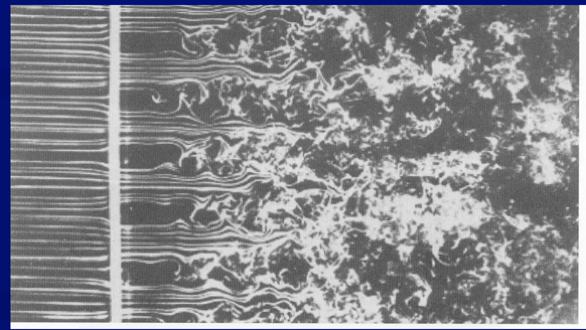
It is difficult to identify each vortex!

Compared with quantized vortices.

Numerical analysis of the Navier-Stokes equation made by Shigeo Kida

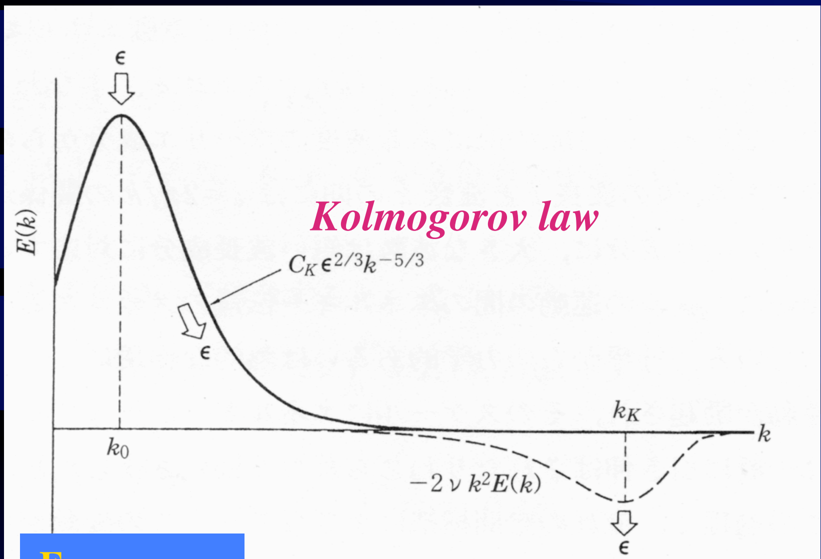
Vortex cores are visualized by tracing pressure minimum in the fluid.

Classical turbulence



Energy spectrum of the velocity field

$$E = \int \frac{1}{2} \mathbf{v}^2 \, d\mathbf{r} = \int E(k)\, dk$$



Energy-containing range

Inertial range

Energy-dissipative range

Energy spectrum of turbulence

Energy-containing range

The energy is injected into the system at $k \sim k_0 = 1/\ell_0$

Inertial range

Dissipation does not work. The nonlinear interaction transfers the energy from low *k* region to high *k* region.

Kolmogorov law : $E(k) = C\, \varepsilon^{2/3} k^{-5/3}$

Energy-dissipative range

Dissipation works at the Kolmogorov wave number $k_K = (\varepsilon / \nu^3)^{1/4}$.

Classical turbulence

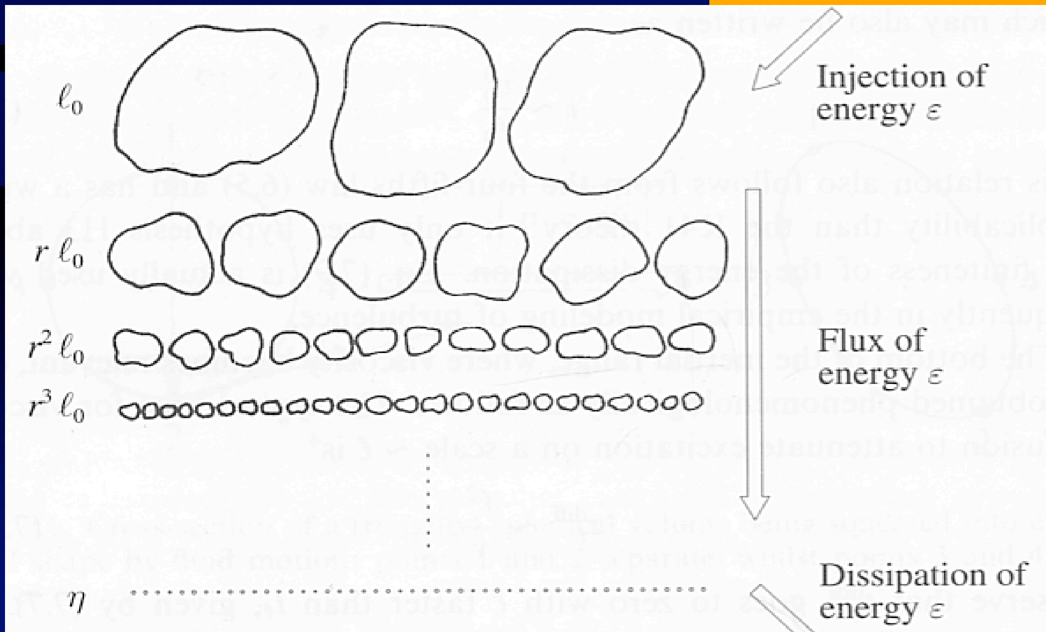
～Richardson cascade and the Kolmogorov law～

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

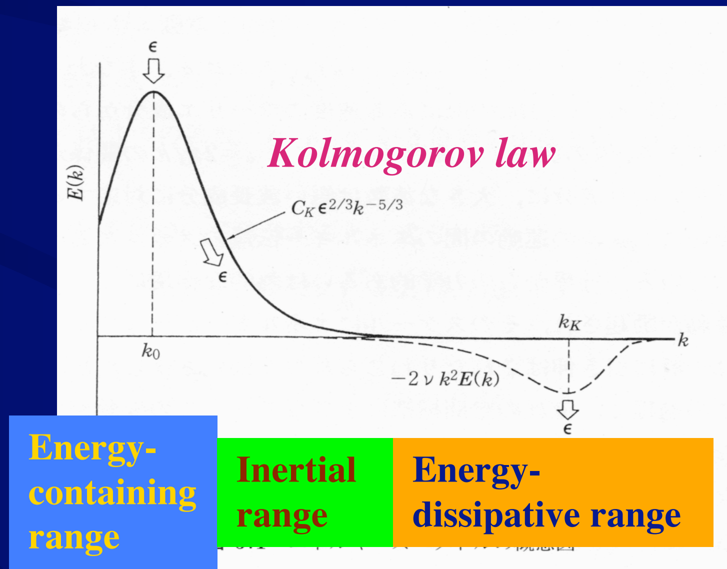
ε = Injected energy per time per unit mass in the Energy-containing range

= Flux of energy in the Inertial range **(Scale-invariance and localness of the interaction)**

= Dissipative energy in the Energy-dissipative range



Richardson Cascade



Does ST mimic CT or not?

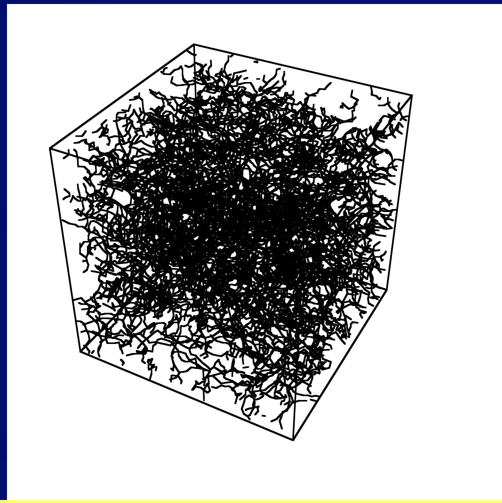
- **Tangle of quantized vortices**

<u>Characteristics of quantized vortices</u>

•Quantization of the circulation

•Very thin core

•No viscous diffusion of the vorticity



•A quantized vortex is a stable and definite topologocal defect, compared with vortices in a classical fluid. The only alive freedom is the topological configuration of its thin cores. **～Vortex skeletons as elementary vortices～**

•Because of superfluidity, some dissipation could work only at large wave numbers (at very low temperatures).

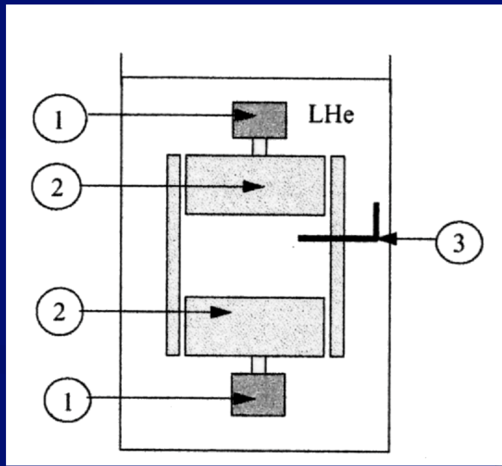
The tangle may give a prototype of turbulence, having the inertial range with the Kolmogorov law.

New study of superfluid turbulence (1)

- Maurer and Tabeling, Europhysics. Letters. 43, 29(1998)

  $1.4\mathrm{K} < T < T_\lambda$

  Measurements of local pressure in flows driven by two counterrotating disks finds the Kolmogorov spectrum.



- Stalp, Skrbek and Donnely, Phys.Rev.Lett. 82, 4831(1999)

  $1.4 < T < 2.15\mathrm{K}$ **Decay of grid turbulence**

  The data of the second sound attenuation was consistent with a classical model with the Kolmogorov spectrum.

New study of superfluid turbulence (2)

- Vinen, Phys.Rev.B61, 1410(2000)

  Considering the relation between ST and CT

  The Oregon's result is understood by the coupled dynamics of the superfluid and the normal fluid due to the mutual friction.

  Length scales are important, compared with the characteristic vortex spacing in a tangle.

- Kivotides, Vassilicos, Samuels and Barenghi, Europhysics Lett. 57, 845(2002) **When superfluid is coupled with the normal-fluid turbulence that obeys the Kolmogorov law, its spectrum follows the Kolmogorov law too.**

What happens at very low temperatures?
Is there still the similarity or not?

3. Superfluid turbulence at very low temperatures

～ Energy spectrum of a vortex tangle without mutual friction ～

Decaying Kolmogorov turbulence in a model of superflow

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)

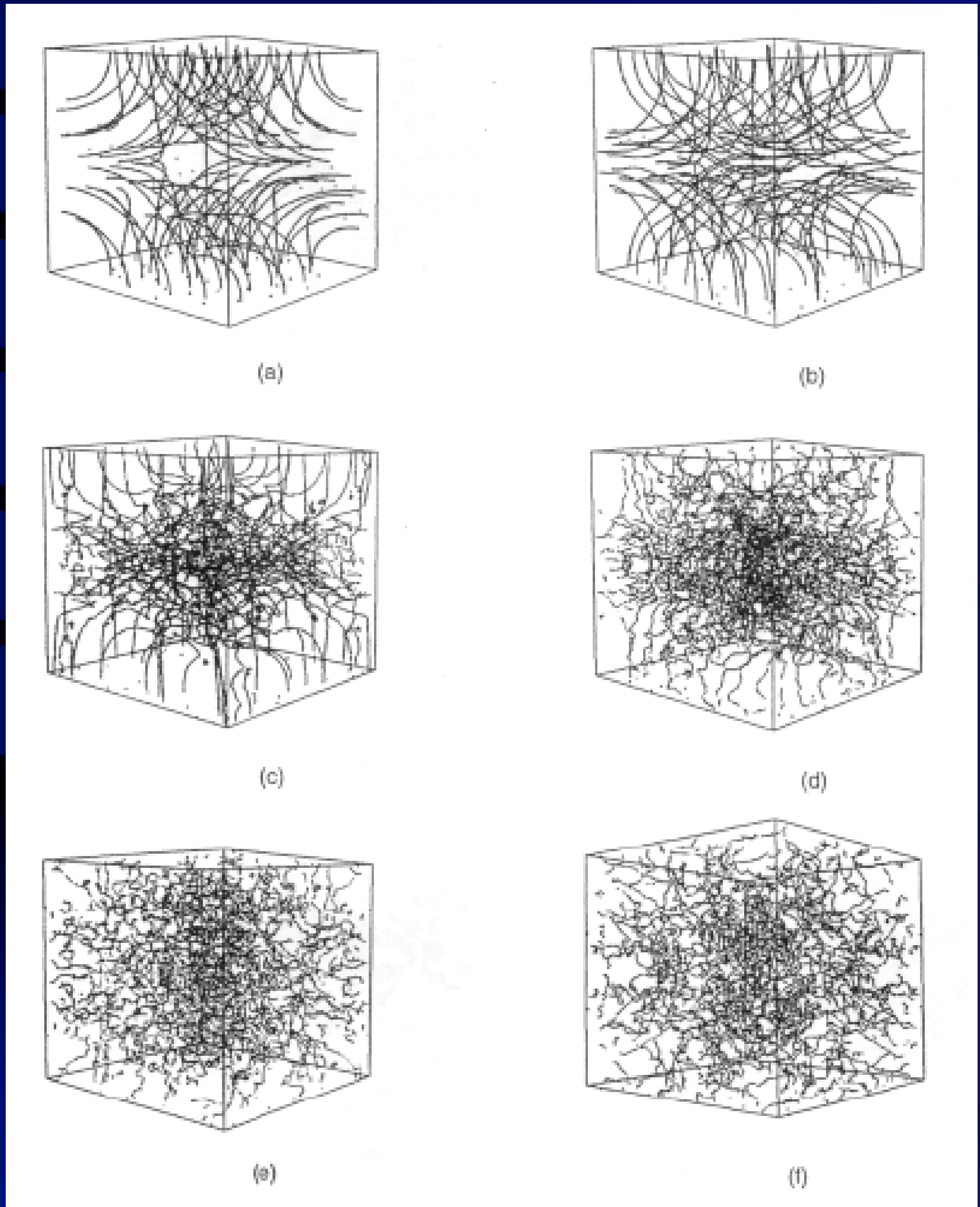
By solving the Gross-Pitaevskii equation, they studied the energy spectrum of a Tarlor-Green flow. The spectrum shows the -5/3 power on the way of the decay, but the acoustic emission is concerned and the situation is complicated.

Energy Spectrum of Superfluid Turbulence with No Normal-Fluid Component

T. Araki, M.Tsubota and S.K.Nemirovskii, Phys.Rev.Lett.89, 145301(2002)

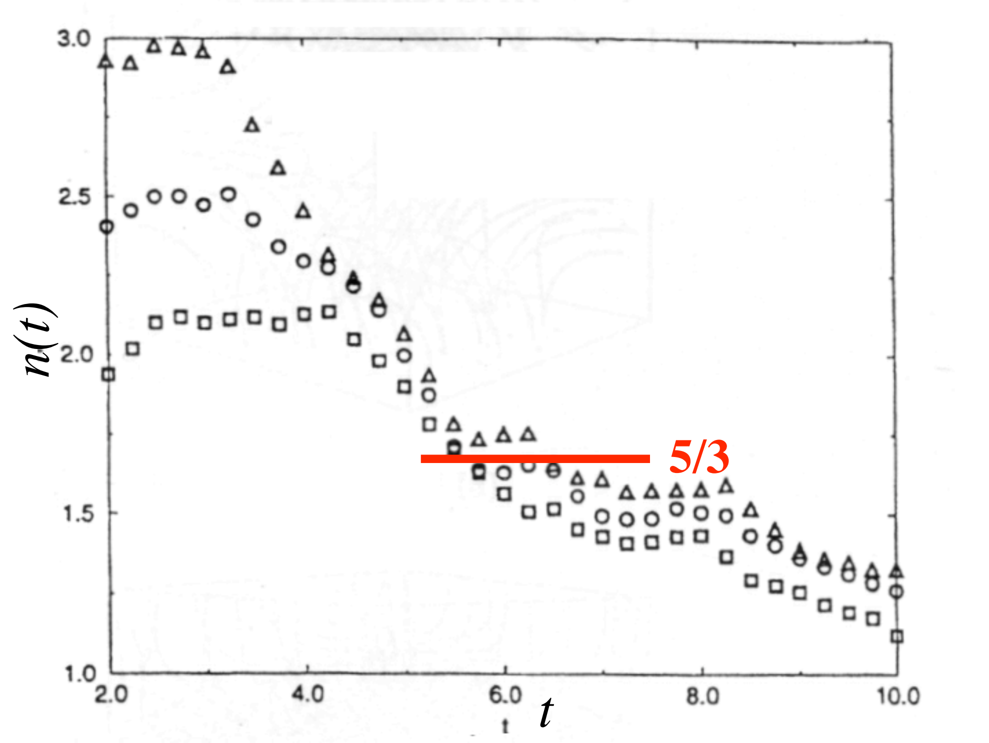
The energy spectrum of a Taylor-Green vortex was obtained under the vortex filament formulation. The absolute value with the energy dissipation rate was consistent with the Kolmogorov law, though the range of the wave number was not so wide.

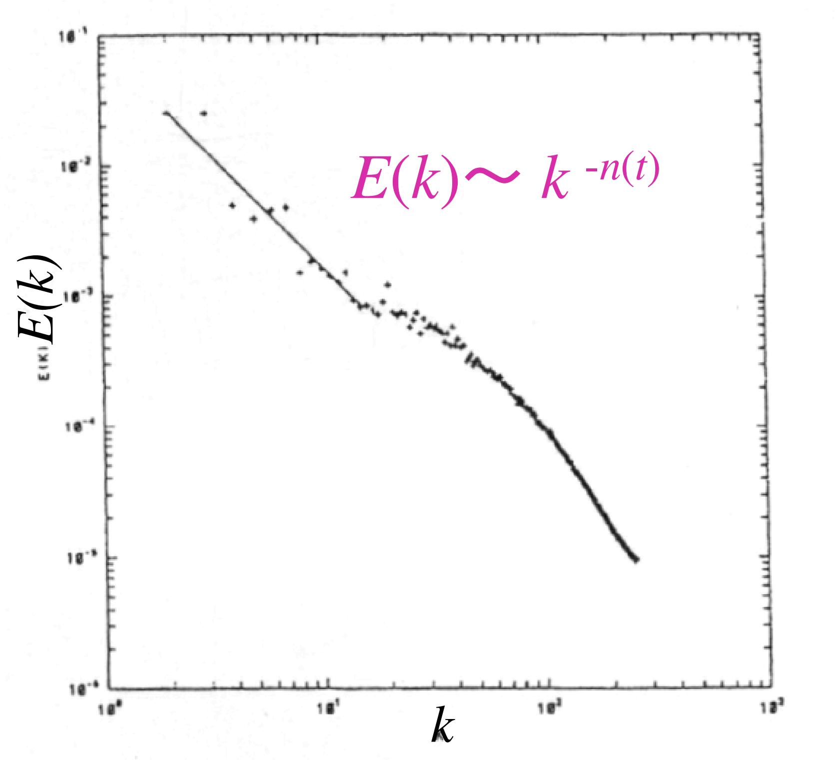
C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)



Although the total energy is conserved, its incompressible kinetic energy is changed to the compressible one with emission of sound waves.

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)





The right figure shows the energy spectrum at a moment. The left figure shows the development of the exponent $n(t)$. The exponent $n(t)$ goes through 5/3 on the way of the dynamics.

In the late stage, the sound waves resulting through vortex reconnections disturb the inertial range and deviate the exponent from 5/3.

In order to overcome this difficulty, we have studied the turbulence of the Gross-Pitaevskii model by introducing the small-scale dissipation.

Decaying turbulence

M. Kobayashi and M. Tsubota, PRL94, 065302 (2005)

We obtained the Kolmogorov spectrum more clearly.

Steady turbulence

This study (in preparation)

By introducing both small-scale dissipation and large-scale excitation, we obtained the steady state of the turbulence. In the inertial range, we confirmed the Kolmogorov spectrum, the energy flux and the vortex size distribution etc.

3.1 Decaying turbulence

Kolmogorov spectrum of superfluid turbulence
M. Kobayashi and M. Tsubota, Phys. Rev. Lett. 94, 065302 (2005)

1. We solved the GP equation in the wave number space in order to use the fast Fourier transformation.

2. We introduced a dissipative term which dissipates the Fourier component of the very high wave number, namely, phonons of short wave length.

The GP equation in the Fourier space

$$i\frac{\partial}{\partial t}\Phi(\mathbf{k},t) = \left(k^{2} - \mu\right)\Phi(\mathbf{k},t) + \frac{g}{V^{2}}\sum_{\mathbf{k}_1,\mathbf{k}_2}\Phi(\mathbf{k}_1,t)\Phi^{*}(\mathbf{k}_2,t)\Phi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,t)$$

$\xi^{2} = 1/g|\Phi|^{2}$ healing length giving the vortex core size

To solve the GP equation numerically with high accuracy, we use the Fourier spectral method in space with the periodic boundary condition in a cube.

The GP equation with the small scale dissipation

$$\{i - \gamma(k)\}\frac{\partial}{\partial t}\phi(\mathbf{k},t) = \left(k^2 - \mu\right)\phi(\mathbf{k},t)$$

$$+\frac{g}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\phi(\mathbf{k}_1,t)\phi^*(\mathbf{k}_2,t)\phi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,t)$$

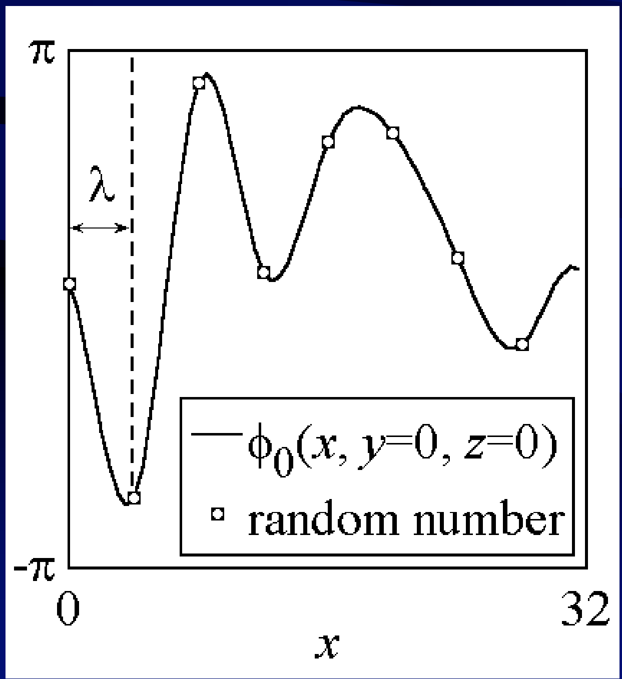
$\xi^2 = 1/g|\phi|^2$:healing length giving the vortex core size

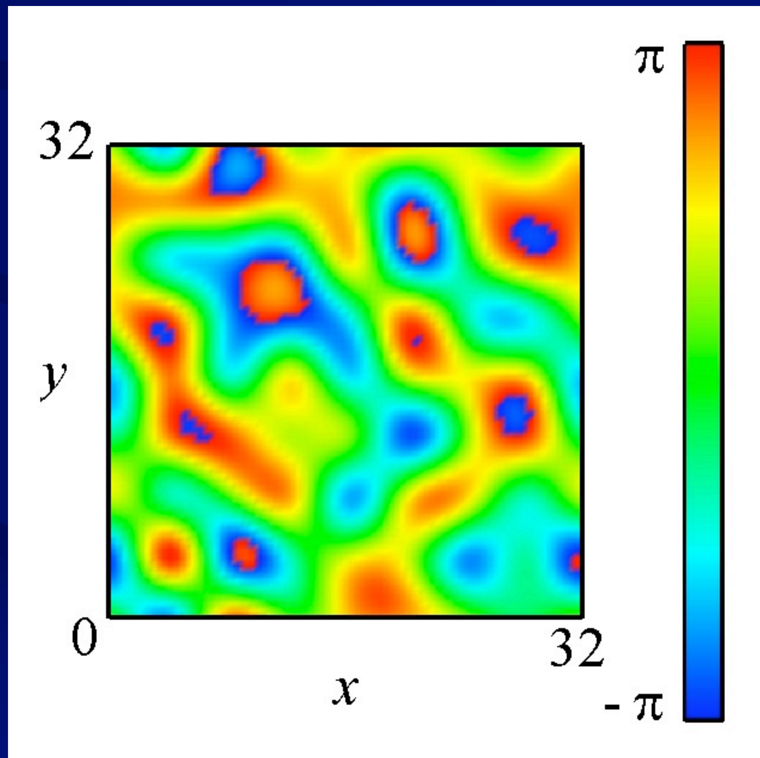
$$\gamma(k) = \gamma_0\theta(k - 2\pi/\xi)$$

We introduce the dissipation that works only in the scale smaller than ξ .

Initial condition

To obtain a vortex tangle, we start from an initial state where the condensate density is uniform and the phase is random.





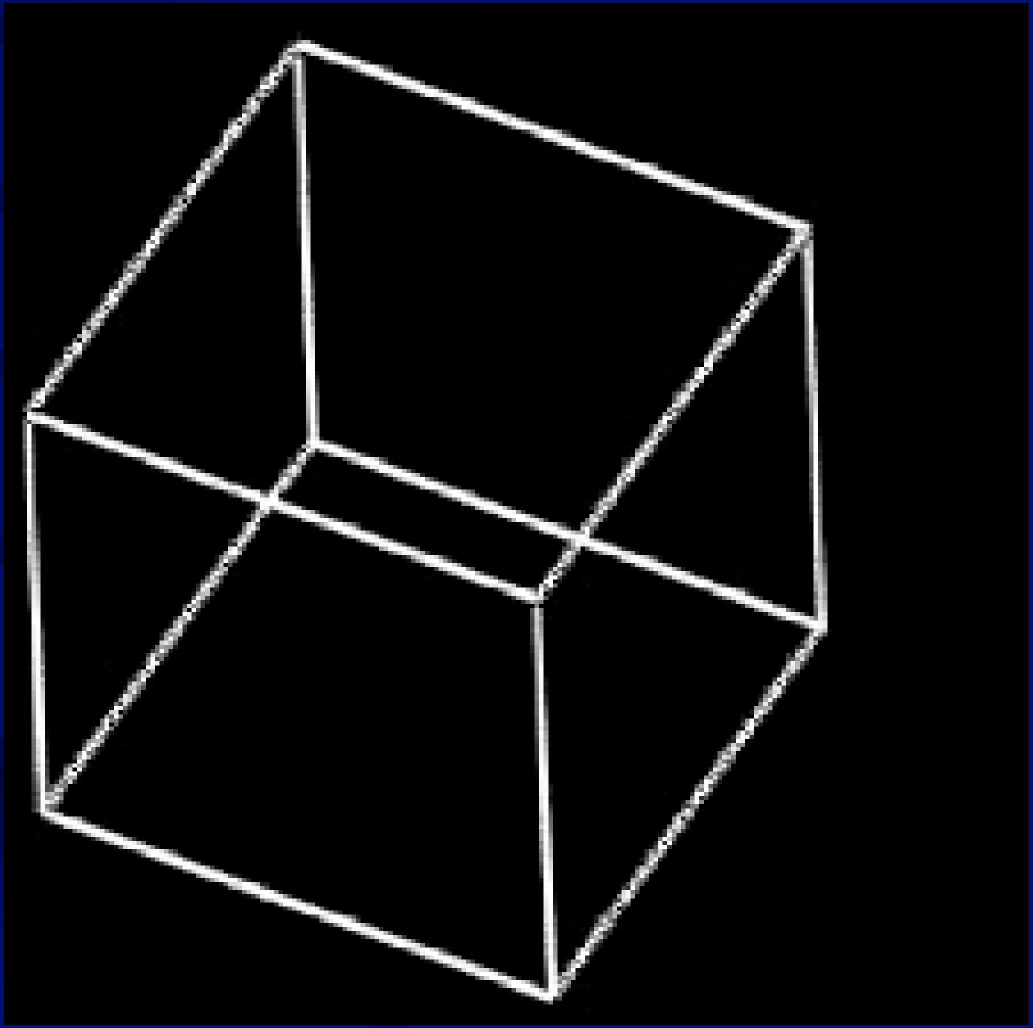
A vortex tangle obtained from an initial state

Quantized vortices are visualized by the contour of vorticity.

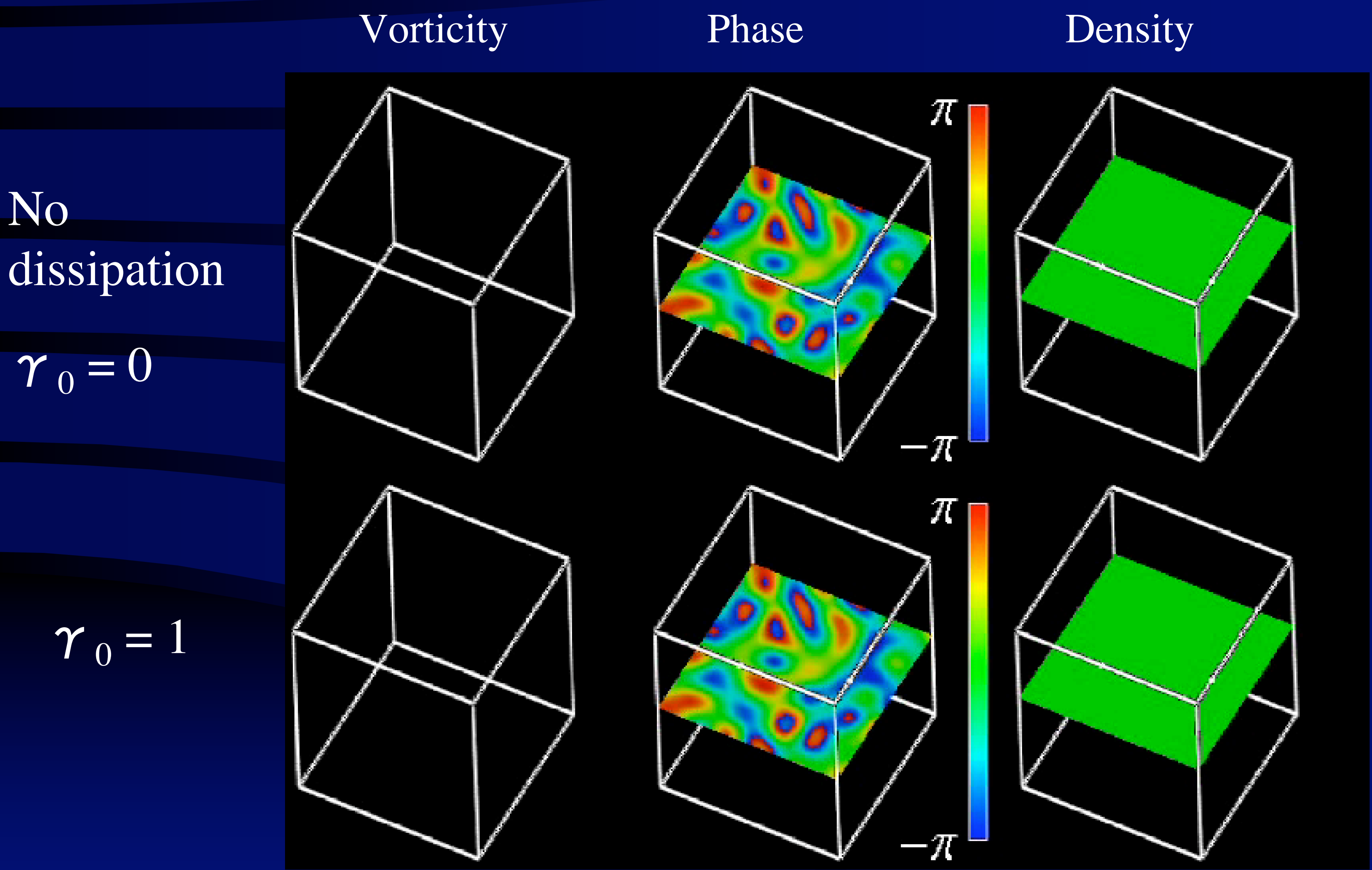
$0 < t < 5.76$

$\gamma = 1$

256^3 grids



Effect of the dissipative term



Small-scale fluctuations are certainly removed by the dissipation.

In order to study the Kolmogorov spectrum, we decompose the total energy into some components. (Nore *et al.*, 1997)

Total energy $E = \frac{1}{\int d\mathbf{x}\,\rho} \int d\mathbf{x}\, \psi^{*}\left[-\nabla^{2} + \frac{g}{2}|\psi|^{2}\right]\psi \qquad \psi = \sqrt{\rho}\exp(i\phi)$

$$E = E_{\rm int} + E_{q} + E_{kin}$$

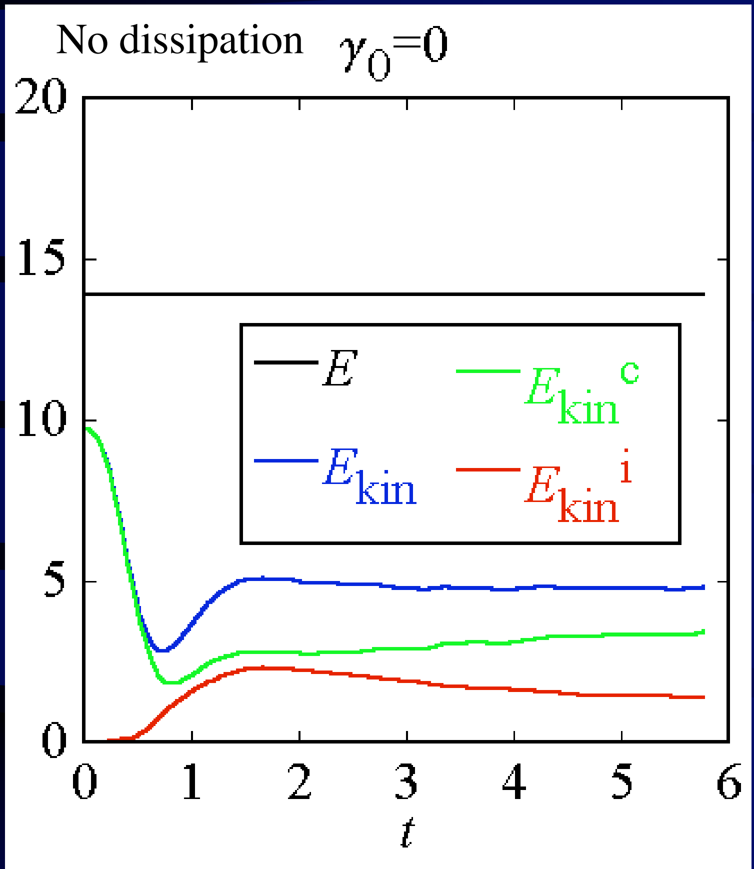
The kinetic energy $E_{kin} = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left(|\psi|\nabla\phi\right)^{2}$ **is divided into**

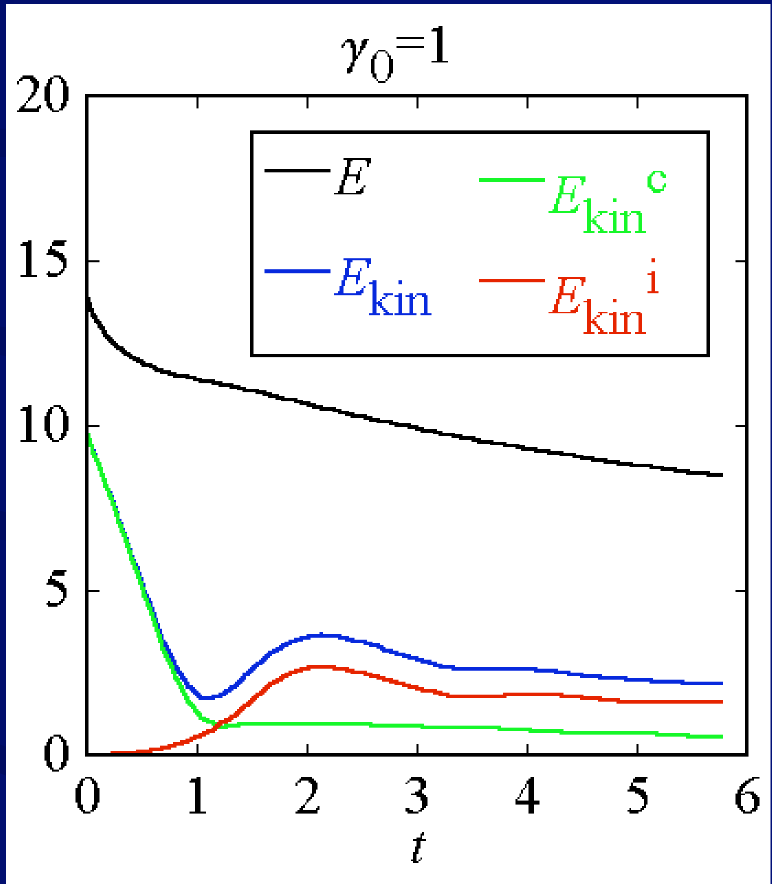
the compressible part $E_{kin}{}^{c} = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left[\left(|\psi|\nabla\phi\right)^{c}\right]^{2}$ **with** $\mathrm{rot}\left(|\psi|\nabla\phi\right)^{c} = 0$

and

the incompressible part $E_{kin}{}^{i} = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left[\left(|\psi|\nabla\phi\right)^{i}\right]^{2}$ **with** $\mathrm{div}\left(|\psi|\nabla\phi\right)^{i} = 0$.

Development of each component

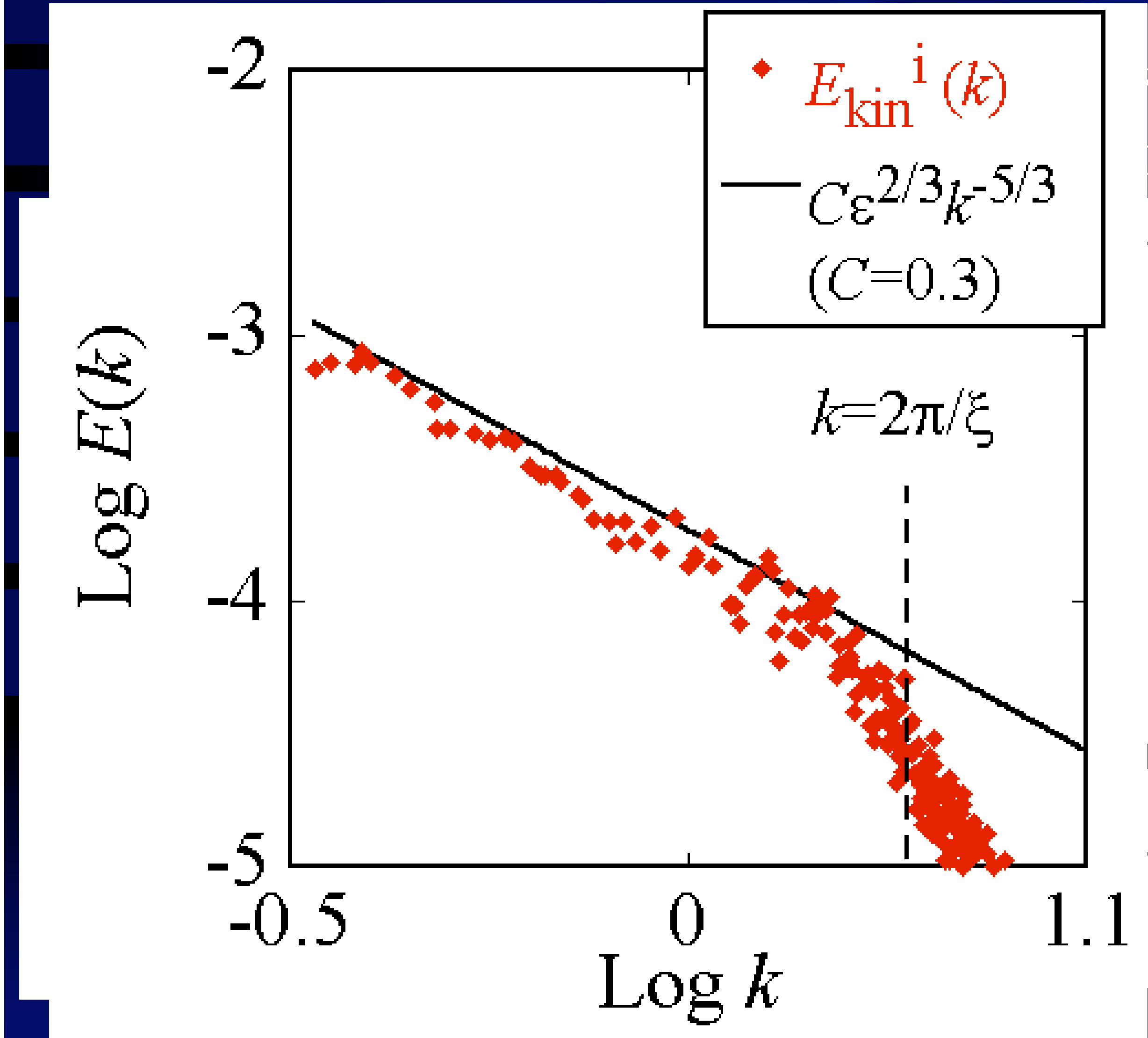




Sound waves are created through reconnections, increasing the compressible kinetic energy.

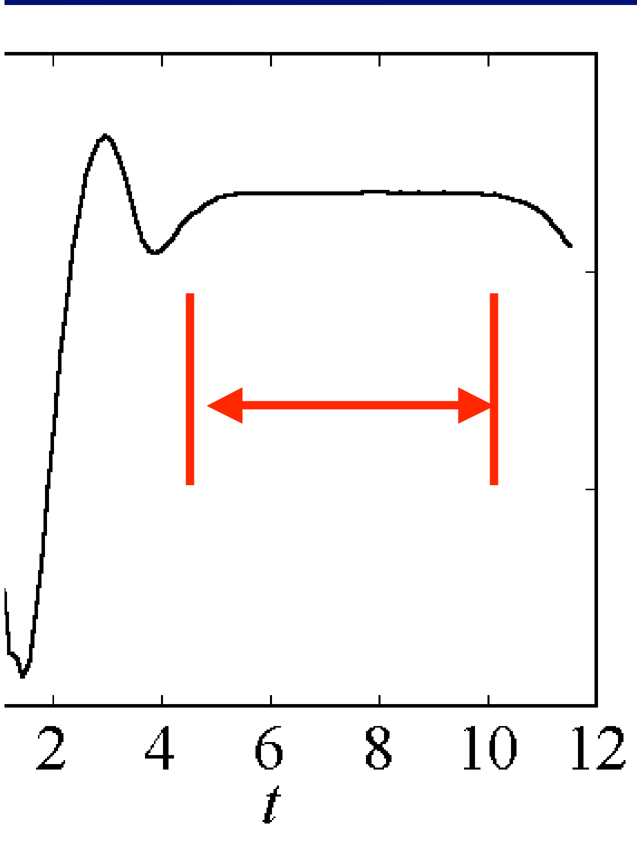
The dissipation removes the sound waves to make the incompressible kinetic energy dominant.

Energy spectrum of the incompressible kinetic energy



…velopment of the

…on $\varepsilon = -\partial E_{kin}^{i} / \partial t$



The exponent η is about 5/3 and ε is constant for $4 < t < 10$!

3.2 Steady turbulence

In order to study the inertial range, we make steady turbulence by introducing large-scale excitation as well as small-scale dissipation.

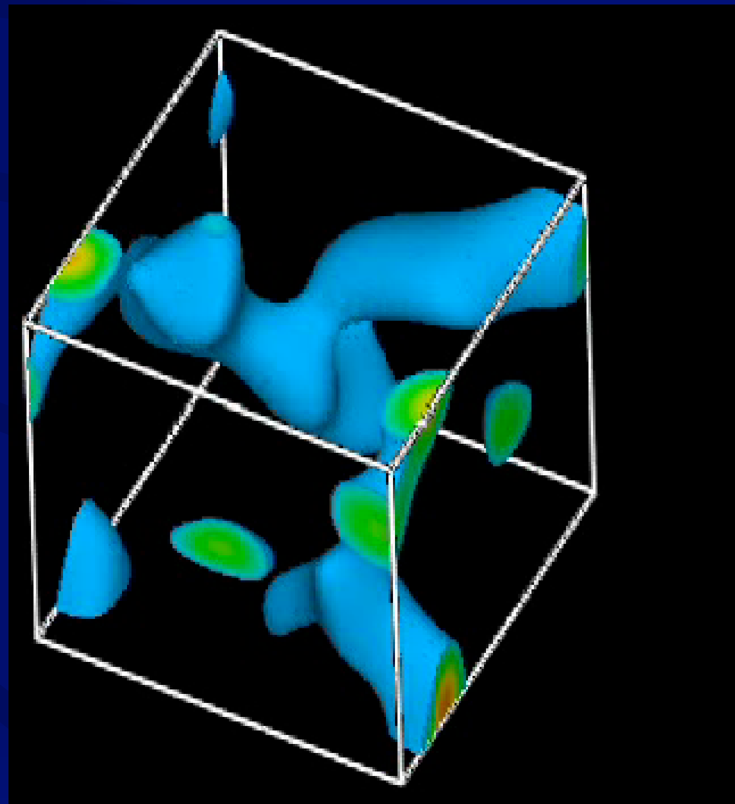
How to excite the system at large scales?

This is done by moving the random potential satisfying the space-time correlation:

$$\langle V(\mathbf{x},t)V(\mathbf{x}',t')\rangle = V_0^2 \exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{2X_0^2} - \frac{(t-t')^2}{2T_0^2}\right]$$

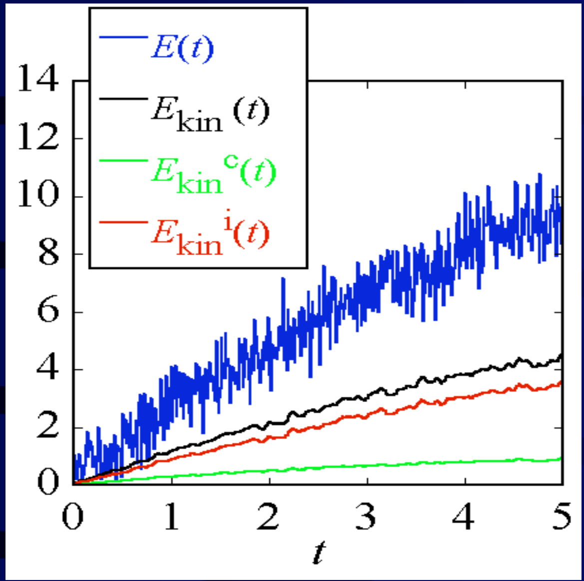
The variable X_0 determines the scale of the energy-containing range.

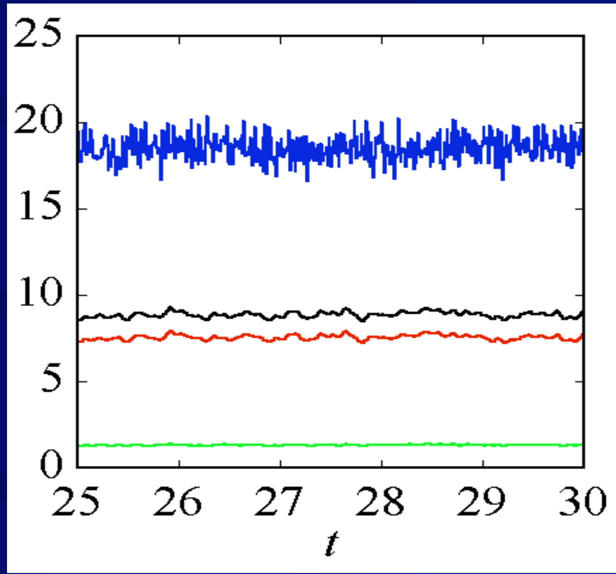
V_0=50, X_0=4 and T_0=6.4×10^{-2}

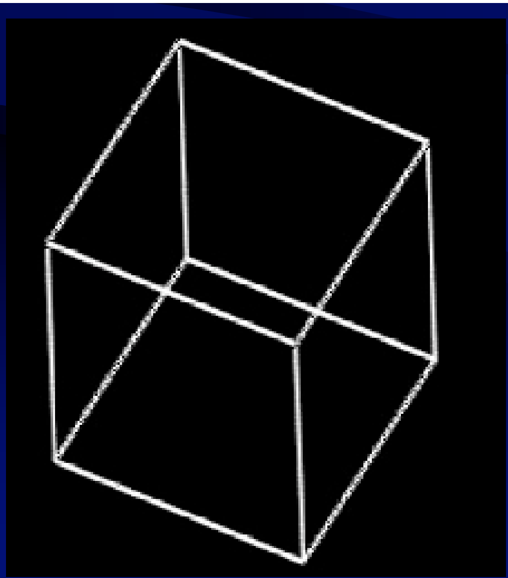


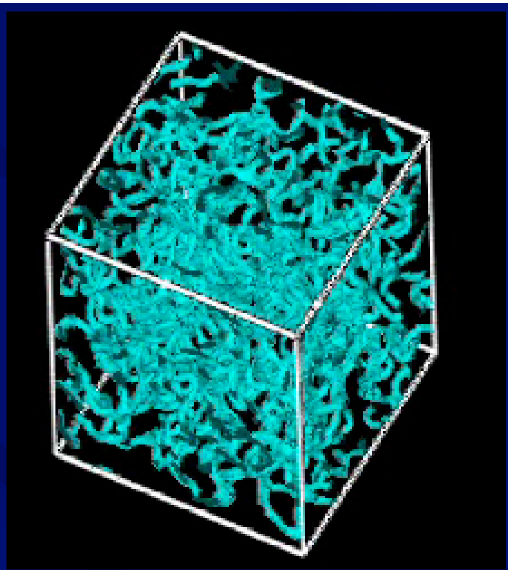
Thus steady turbulence was obtained.(1)

Time development of each energy component

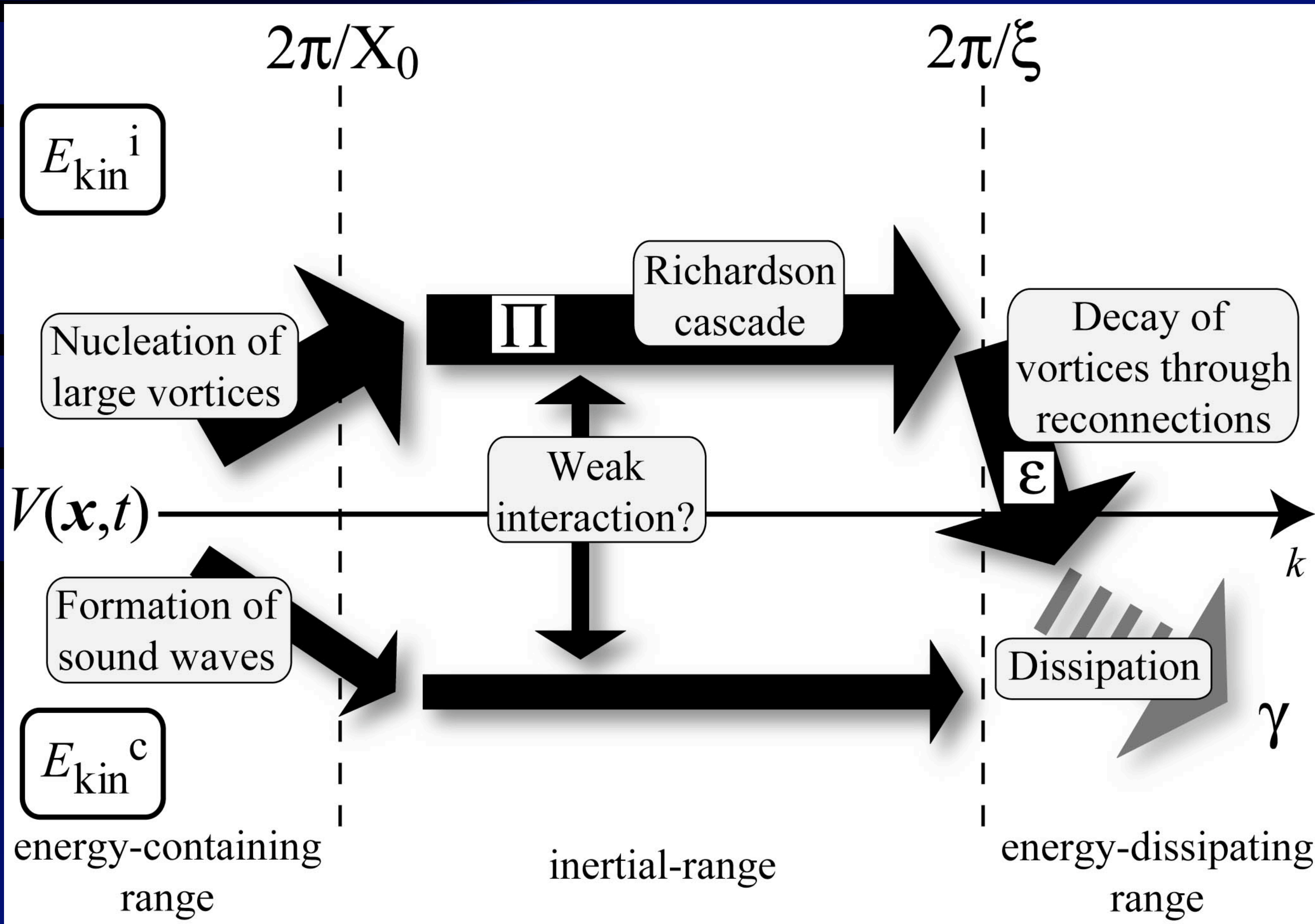








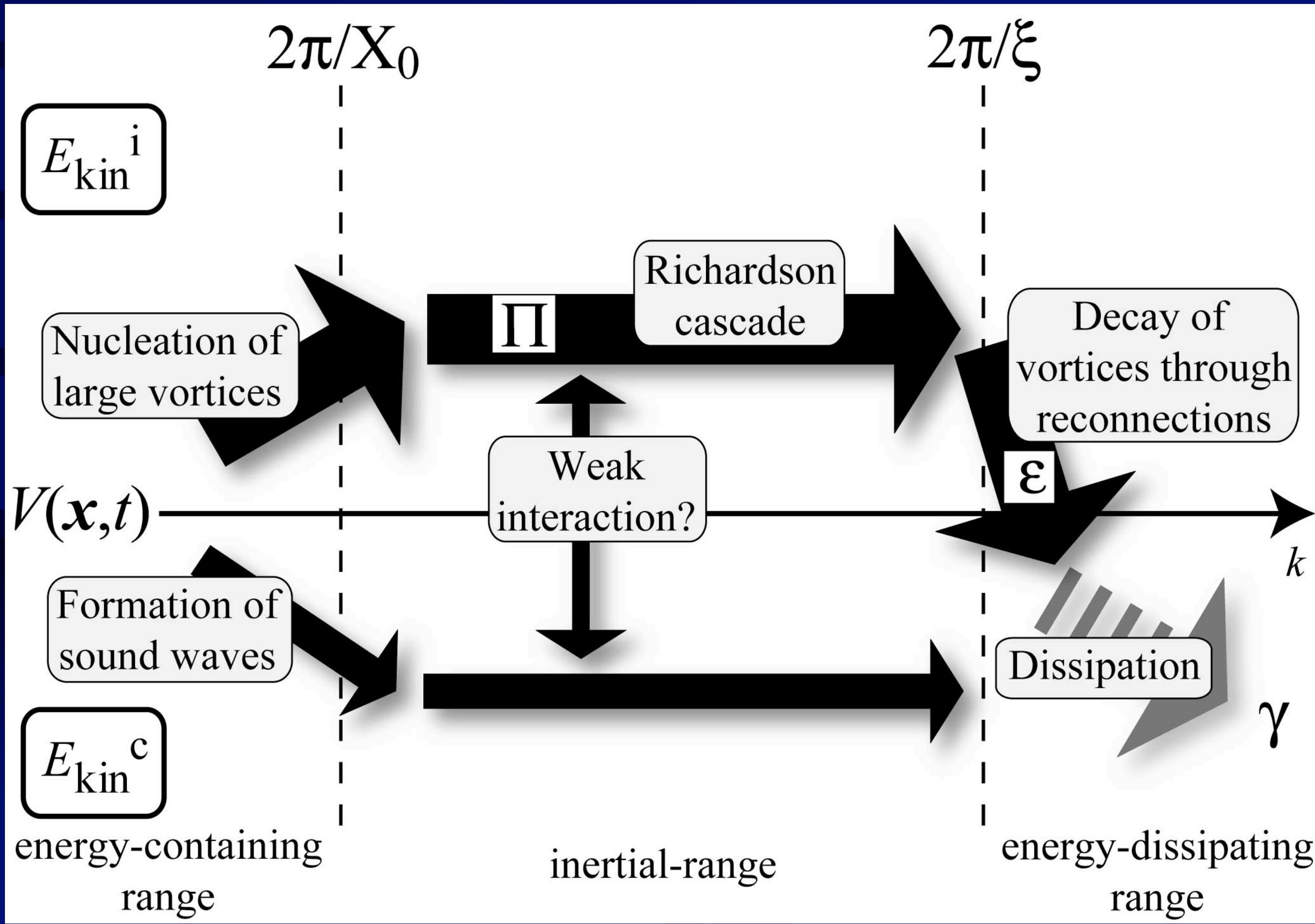
Picture of the cascade process



In order to confirm this picture, we calculate

(1) the energy dissipation rate ε of E_{kin}^{i}

(2) the energy flux Π of the Richardson cascade.



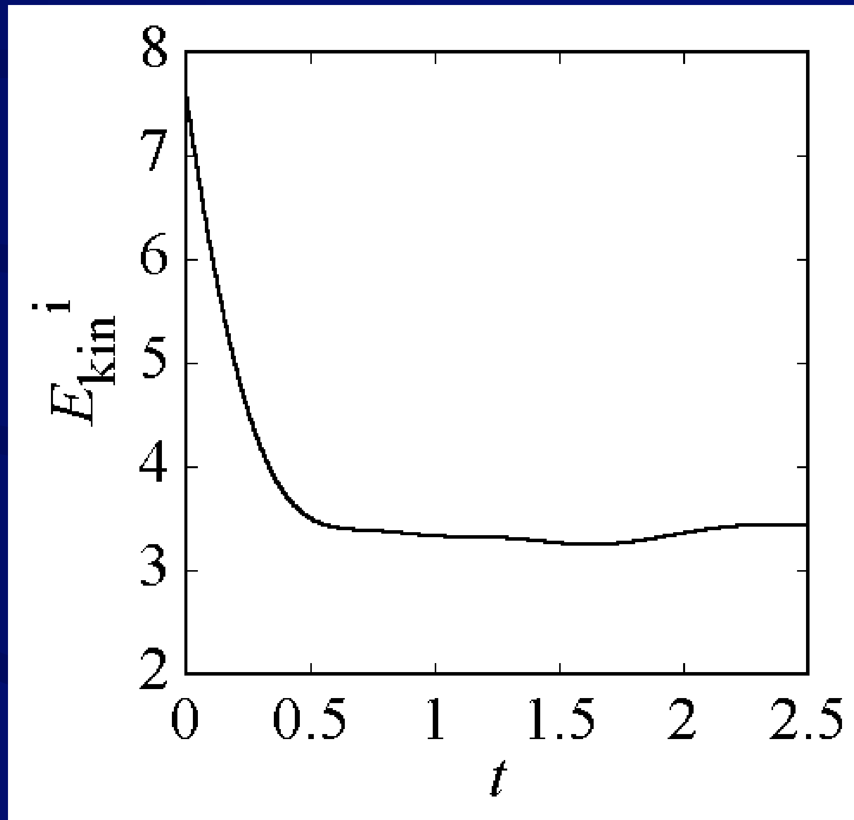
(1) the energy dissipation rate ε of $E_{\mathrm{kin}}^{\mathrm{i}}$

In the steady state, we turn off the large-scale excitation suddenly and monitor the time development of $E_{\mathrm{kin}}^{\mathrm{i}}$.

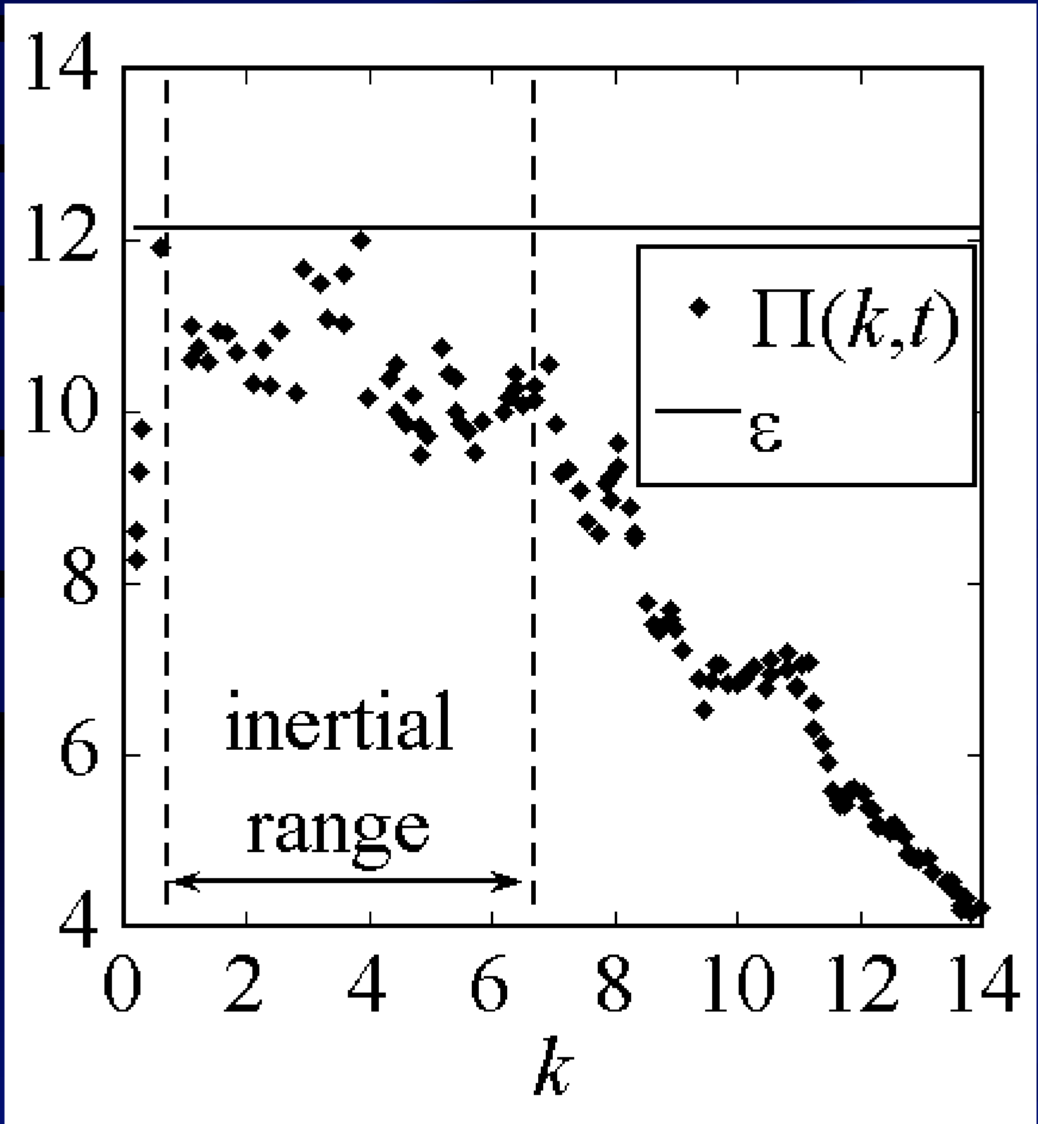
Thus we obtain

$$\varepsilon = -\frac{d E_{\mathrm{kin}}^{\mathrm{i}}}{dt} \approx 12.5 \pm 2.3$$

from the decay.



(2) the energy flux Π of the Richardson cascade



Ensemble averaged over 50 states.

← Dissipation rate ε ～12.5

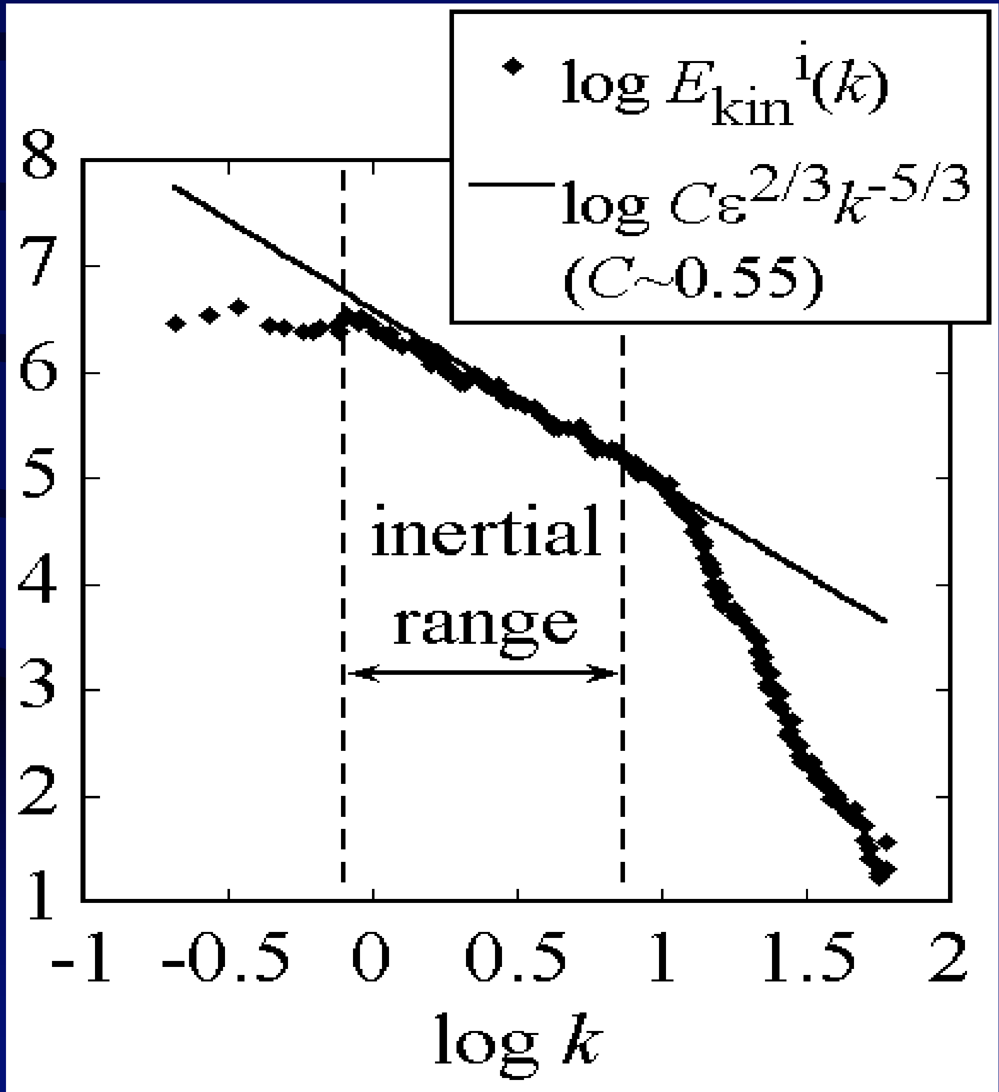
1. Π is about constant in the inertial range.

2. Π is comparable to ε.

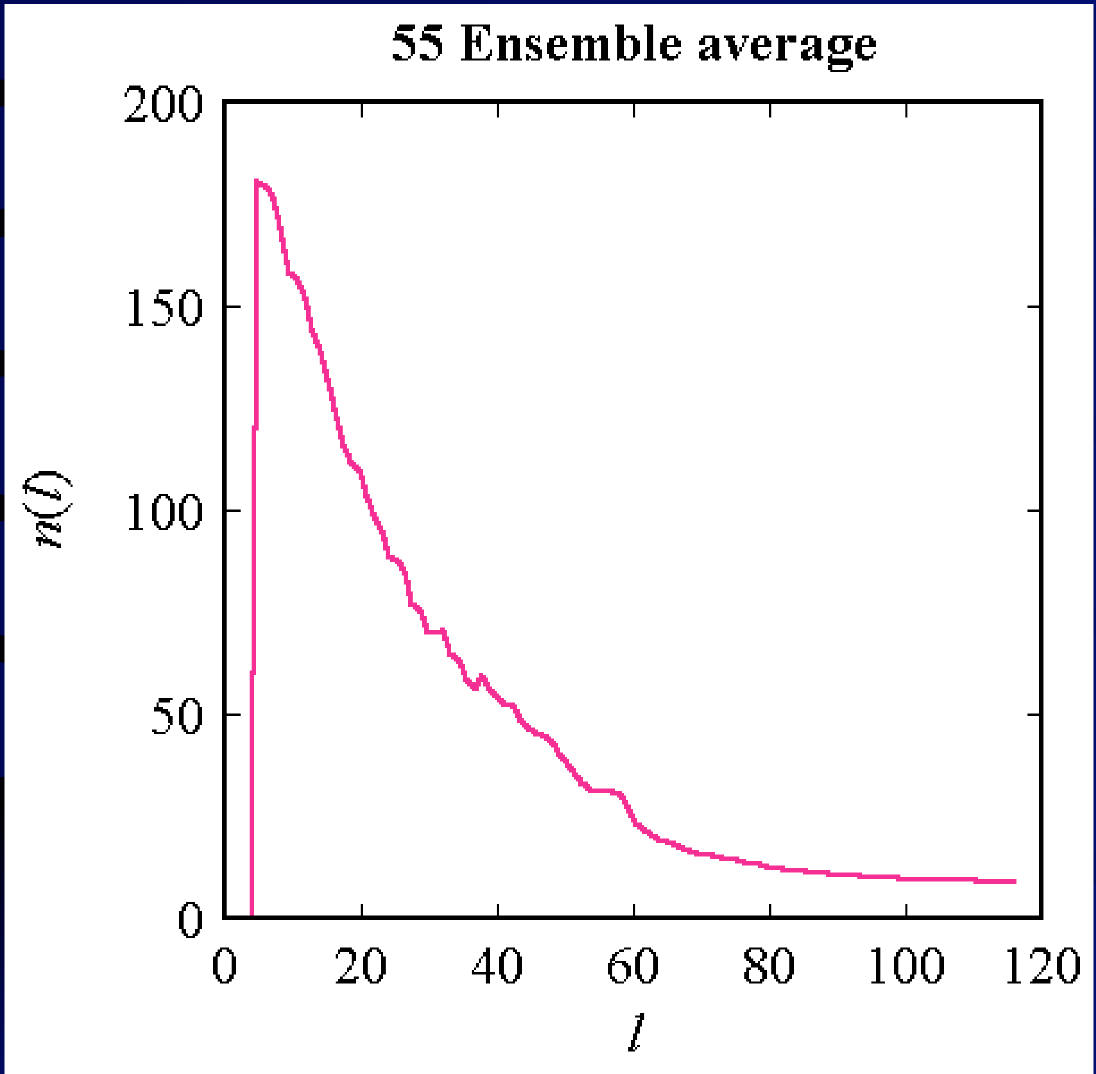
They confirm strongly the picture of the inertial range.

Energy spectrum of the steady turbulence

The energy spectrum takes the Kolmogorov form.



We obtained the vortex size distribution.



When we assume $n(l) \sim l^{-x}$, the power x is about 1.4.

This distribution should be concerned with the Richardson cascade and the Kolmogorov spectrum!

5. Conclusions

We showed the recent motivation and research activity of quantum turbulence.

Quantum turbulence made of quantized vortices can give a prototype of turbulence, especially at very low temperatures.

By the numerical analysis of the Gross-Pitaevski model, we confirmed the picture of the inertial range of quantum turbulence for both decaying and steady cases.