The Density Conjecture for Bounded Geometry Kleinian Groups

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Definitions:

A *Kleinian group* Γ is a discrete (torsion free) subgroup of $PSL_2\mathbb{C}$.

The quotient $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold.

The *convex core* $CC(M)$ is the smallest convex subset of M whose inclusion is a homotopy equivalence.

Γ (or M) is *geometrically finite* if $CC(M)$ has finite volume.

Conjecture 1 (Bers-Sullivan-Thurston) *Every finitely generated Kleinian group is an algebraic limit of geometrically finite Kleinian groups.*

A sequence of Kleinian groups Γ_i converges to Γ *algebraically* if there exists isomorphisms

$$\rho_i : G \longrightarrow \Gamma_i$$

and

$$\rho : G \longrightarrow \Gamma$$

such that $\rho_i(g) \to \rho(g)$ for all $g \in G$.

M has ϵ-bounded geometry if the length of any closed geodesic in M has length $\geq \epsilon$.

History

- Bounded geometry, freely indecomposable fundamental group and no parabolics - Minsky

- Unbounded geometry, freely indecomposable fundamental group and no parabolics, B, Brock-B

- Complete Conjecture - Many people!

Outline of proof

- Tameness: Associate end-invariants to a hyperbolic 3-manifold - Thurston, Bonahon, Canary, Agol, Calegari-Gabai

- Limit theorems: Show that a manifold with a given set of end-invariants can be realized as the algebraic limit of geometrically finite manifolds - Thurston, Ohshika, Brock, Kleineidam-Souto, Kim-Ohshika, Lecuire, Namazi-Souto

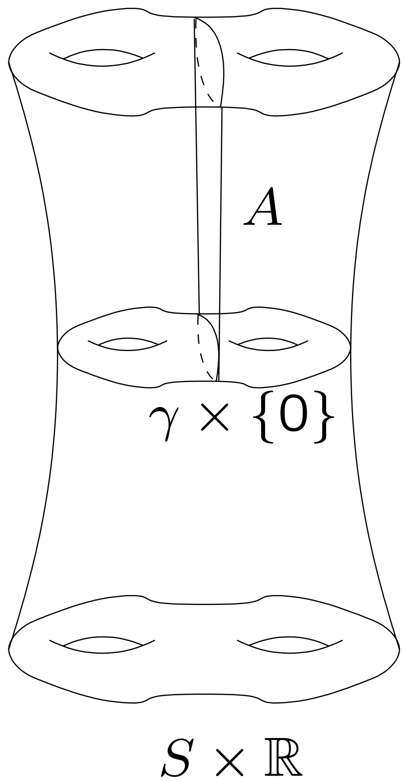
- Ending lamination conjecture: Show that a manifold is uniquely determined by its end-invariants: Minsky, Minsky-Masur, Brock-Canary-Minsky

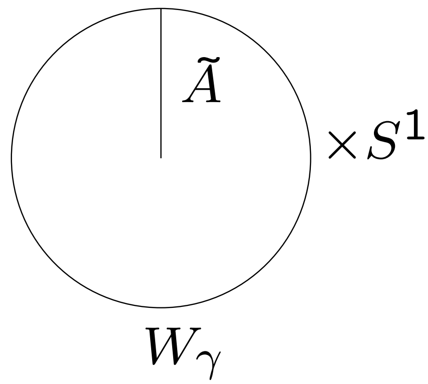
We'll only discuss the case where Γ is isomorphic to $\pi_1(S)$ for a surface S of genus $g \geq 2$. We'll further assume that Γ does not contain parabolics.

In this case if Γ is geometrically finite then $CC(M)$ is compact. Such manifolds and groups are called *quasifuchsian*.

If Γ is geometrically infinite but $CC(M) \neq M$ then M is *singly degenerate*.

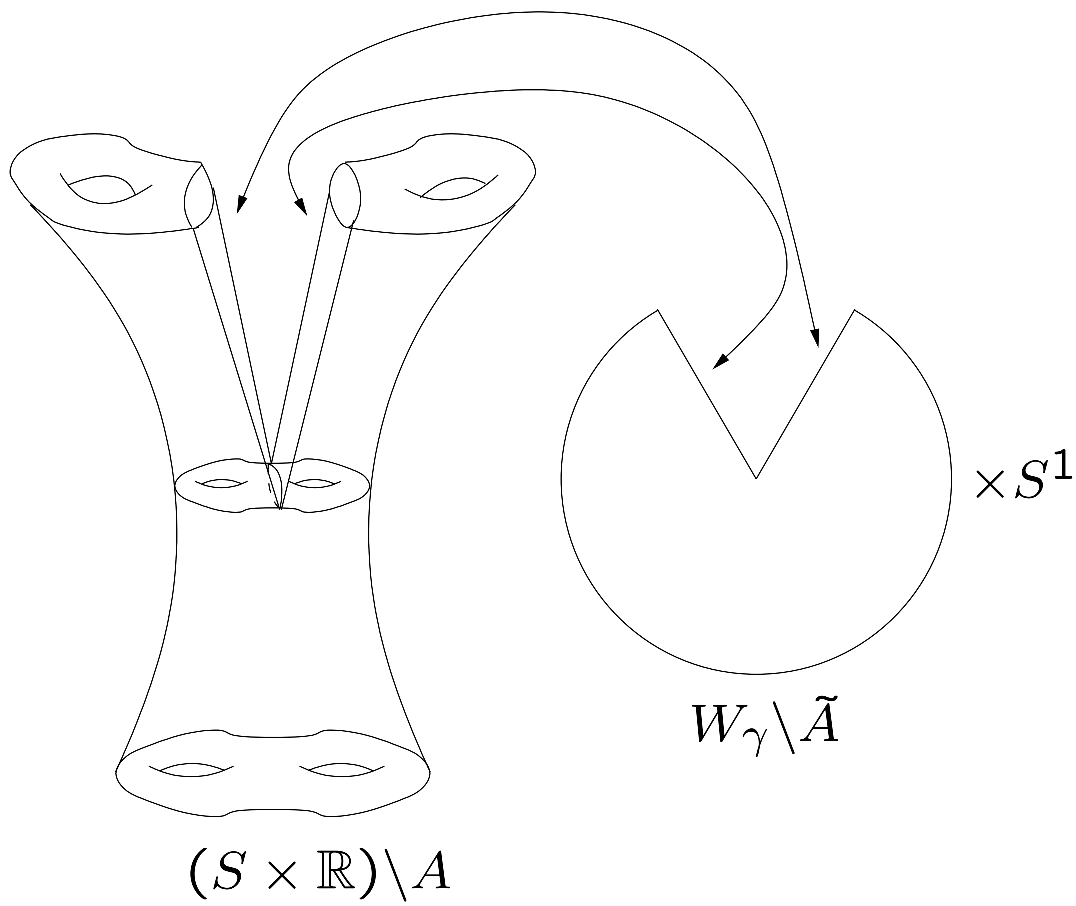
Grafting





S is a closed surface and γ a simple closed curve on S. A is the annulus $\gamma \times [0, \infty)$ in $S \times \mathbb{R}$.

W_γ is the cover of $S \times \mathbb{R}$ coming from the subgroup generated by γ. A lifts homeomorphically to $\tilde{A}$ in W_γ.



We form a new copy of $S \times \mathbb{R}$ by gluing $(S \times \mathbb{R}) \backslash A$ to $W_\gamma \backslash \tilde{A}$.

Define a map

$$\pi_\gamma : S \times \mathbb{R} \longrightarrow S \times \mathbb{R}$$

by setting π_γ to be the identity on $(S \times \mathbb{R}) \backslash A$ and the covering map on $W_\gamma \backslash \tilde{A}$.

If (M, g) is a complete hyperbolic metric and $\gamma \times \{0\}$ is a geodesic then $g_\gamma = \pi_\gamma^* g$ is a *hyperbolic cone-metric*.

Theorem 2 (B, Brock-B) *The grafted end is geometrically finite.*

A geodesic is *unknotted* if it is isotopic to a simple closed curve on $S \times \{0\}$. Otal showed that a sufficiently short geodesic is unknotted.

Therefore if (M, g) has unbounded geometry there are a sequence of geodesic on which we can perform grafting. This gives a sequence of geometrically finite hyperbolic cone-manifolds converging to (M, g).

We then use the deformation theory of hyperbolic cone-manifolds developed by Hodgson and Kerckhoff to find smooth geometrically finite manifolds converging to (M, g).

Theorem 3 (Thurston) *Let g_i be a family of hyperbolic metrics on $M \cong S \times \mathbb{R}$ with ϵ bounded geometry and assume that x_i are contained in the convex core of (M, g_i). Then there exist a hyperbolic metric (M, g_∞) with ϵ-bounded geometry and a basepoint $x \in M$ such that (after passing to a subsequence) $((M, g_i), x_i)$ converges to $((M, g_\infty), x_\infty)$ in the pointed bi-Lipschitz topology. That is for any compact set $K \subset M$ with $x \in K$ and any $L > 1$ there exists smooth L-bi-Lipschitz embeddings*

$$\phi_i : ((K, g_\infty), x_\infty) \longrightarrow ((M, g_i), x_i)$$

for i sufficiently large.

Theorem 4 *Given ϵ, L, R there exist n and K such that for every ϵ-bounded geometry manifold $M \cong S \times \mathbb{R}$ and a every closed geodesic γ of length $\leq L$ there exists a cover $\pi : \widehat{M} \longrightarrow M$ such that the following holds.*

1. *The geodesic γ lifts homeomophically to an unknotted simple closed geodesic $\widehat{\gamma}$.*

2. *The tube radius of $\widehat{\gamma}$ is $\geq R$.*

3. *The degree of the cover is $\leq n$.*

4. *Let $\hat{Z}$ be the conformal boundary obtained from grafting along $\widehat{\gamma}$. There exists an $X \in \mathcal{T}(S)$ such that $d_{\mathcal{T}(S)}(\hat{Z}, \pi^*(X)) \leq K$.*

Proof. Assume not. For any metric there is some finite degree cover where (1) and (2) and hold. Assume that we have a sequence of examples of manifolds (M, g_i) and curves γ_i where the degree of this cover must limit to infinity. Extract a geometric limit of the (M, g_i) with the basepoints on γ_i. In the limit we only need to take a finite degree cover. By pulling this cover back to the approximates we obtain a contradiction.

Repeat this process for (4).

Theorem 5 (Minsky) *If* (M, g) *is singly degenerate with bounded geometry then it is an algebraic limit of geometrically finite groups.*

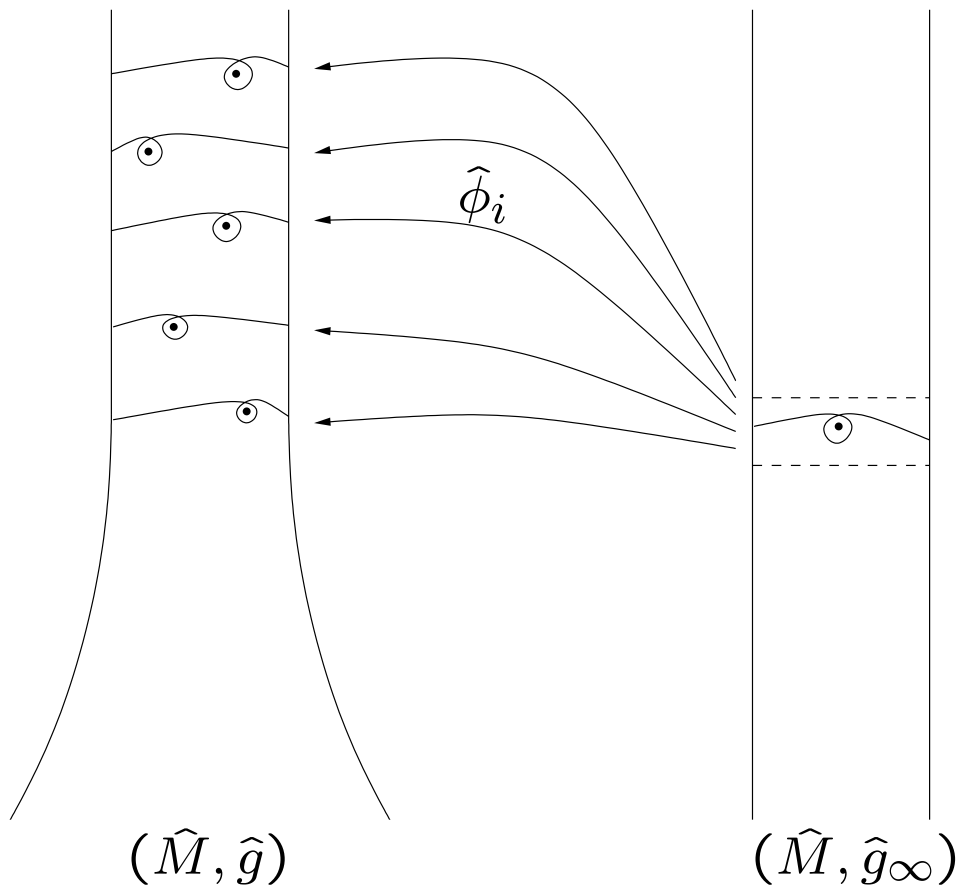
Proof.(B-Souto)

There exists a sequences of closed geodesics γ_i of length $\leq L$ exiting the degenerate end of (M, g). By Theorem 4 there exists an n such that for each i there is cover (M_i, g_i) of degree $\leq n$ such that γ_i lifts homeomorphically to an unknotted geodesic $\widehat{\gamma}_i$ with tube radius $\geq R$.

There are only finitely many covers of degree $\leq n$ so we can pass to a subsequence such that each (M_i, g_i) is a fixed cover $(\widehat{M}, \widehat{g})$. Graft $(\widehat{M}, \widehat{g})$ along $\widehat{\gamma}_i$ and then smooth using Hodgson-Kerckhoff cone deformation theory to obtain quasifuchsian manifolds $Q(\hat{Z}_i, \hat{Y})$ converging to $(\widehat{M}, \widehat{g})$.

By (4) of Theorem 4 there exists a K such that for each i there is an $X_i \in \mathcal{T}(S)$ with $d_{\mathcal{T}}(\hat{Z}_i, \pi^*(X_i)) \leq K$. The compactness of K-quasiconformal maps and Sullivan rigidity imply that the sequences $Q(\hat{Z}_i, \hat{Y})$ and $Q(\pi^*(X_i), \hat{Y})$ have the same limits.

Each $Q(\pi^*(X_i), \hat{Y})$ will isometrically cover $Q(X_i, Y)$ and therefore $Q(X_i, Y)$ will converge to (M, g).



$\widehat{\phi}_i$
$(\hat{M}, \hat{g})$
$(\hat{M}, \hat{g}_\infty)$