Finite groups acting on homology spheres

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Problem: Which finite groups admit actions on integer and $\mathbb{Z}_2$-homology 3-spheres? (arbitrary action, not necessarily free)

Classical cases:

- actions on S^3
- free actions on integer homology 3-spheres

Some relations with knot theory:

- a manifold obtained by p/q-surgery on a knot in S^3, with p odd, is a $\mathbb{Z}_2$-homology 3-sphere
- the 2-fold cyclic branched covering of a knot in S^3 is a $\mathbb{Z}_2$-homology 3-sphere

1. Hyperbolic 2-fold branched coverings of knots and finite 2-groups acting on $\mathbb{Z}_2$-homology 3-spheres

Problem: How many inequivalent knots can have the same 2-fold branched covering?

In general the number of inequivalent knots with the same 2-fold cyclic branched covering can be arbitrary large (in [Montesinos (1976)] examples using Montesinos knots)

We consider the problem where the covering manifolds are hyperbolic.

Remark The number of the inequivalent knots with the same hyperbolic 2-fold branched covering M is bounded by the number of conjugacy classes of the involutions with nonempty fixed point set in the Sylow 2-subgroup of Iso$^+(M)$.

Problem: How many inequivalent knots can have the same hyperbolic 2-fold branched covering?

becomes

Problem: How many conjugacy classes of involutions with nonempty fixed point set can be contained in a finite 2-group acting on a $\mathbb{Z}_2$-homology 3-sphere?

[Reni 2000] using Finite Group Theory and low dimensional methods

Theorem 1. *Let S be a finite 2-group acting on a $\mathbb{Z}_2$-homology 3-sphere smoothly and orientation presevengly. Then one of the following cases occurs.*

- *S is cyclic, dihedral, quasidihedral or a quaternion group;*

- *S contains with index at most two the centralizer $C_S h$ of an involution h with nonempty fixed point set. The group $C_S h$ is a subgroup of a semidirect product $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m})$*

Theorem 2. *S has at most nine conjugacy classes of involutions with nonempty fixed point set.*

Corollary. There are at most nine inequivalent knots with the same hyperbolic 2-fold branched covering.

Different approach to analyze the 2-groups acting on $\mathbb{Z}_2$-homology 3-spheres

- Dotzel and Hamrick (1981) proved that any finite p-group acting on a $\mathbb{Z}_p$-homology n-sphere has a representation as a group of isometries of S^n (this representation preserves the dimension of the global fixed point set of any subgroup).

- The finite subgroups of $SO(4)$ are classified (see [Du Val(1974)]).

Complete solution of the topological problem

- We have at most nine inequivalent knots with the same hyperbolic 2-fold branched covering. ([Reni 2000])

- There exist examples of hyperbolic manifolds that are the common 2-fold branched covering of nine inequivalent knots. ([Akio Kawauchi (preprint 2005)].)

- The link version of the problem was completely solved ([M. and Reni 2002],[M. and Zimmermann 2004]).

2. Finite nonsolvable groups acting on $\mathbb{Z}_2$-homology 3-spheres

Preliminary definitions

- if G is a finite group, $\mathcal{O}(G)$ is the maximal normal subgroup of odd order of G

- $\mathrm{SL}(2,q)$ is the special linear group of 2×2 matrices of determinant one over the finite field with q element ($q = p^n$ a prime power)

- $\mathrm{PSL}(2,q) = \mathrm{SL}(2,q)/Z(\mathrm{SL}(2,q))$ is the projective linear group (simple for $q \geq 4$)

- $\widehat{\mathbb{A}}_7$ is the unique perfect central extension of the alternating group $\mathbb{A}_7$ with center of order two

- for q and q' odd, $\mathsf{SL}(2,q) \times_{\mathbb{Z}_2} \mathsf{SL}(2,q')$ is the central product of the two groups with the two central involutions identified

Theorem 3. [*M. and Zimmermann (2004)*]

Let G be a finite group of orientation preserving diffeomorphisms of a $\mathbb{Z}_2$-homology 3-sphere. If G is not solvable, then G has a normal subgroup N containing $\mathcal{O}(G)$ such that the factor group $N/\mathcal{O}(G)$ is isomorphic to one of the following groups:

$$\mathrm{PSL}(2,q) \quad \text{or} \quad \mathrm{PSL}(2,q) \times \mathbb{Z}_2,$$

$$\widehat{\mathbb{A}}_7 \quad \text{or} \quad \mathrm{SL}(2,q) \times_{\mathbb{Z}_2} \mathrm{SL}(2,q'),$$

$$\mathrm{SL}(2,q) \times_{\mathbb{Z}_2} C$$

where C is solvable with a unique involution, and q and q' are odd prime powers greater than four. Moreover the factor group G/N is abelian or a 2-fold extension of an abelian group.

Theorem 4. *If G is a nonabelian simple finite group of orientation preserving diffeomorphisms of a $\mathbb{Z}_2$-homology 3-sphere then G is isomorphic to $\mathrm{PSL}(2,q)$ with q an odd prime powers greater than four.*

(a shorter proof in M.and Zimmermann - preprint 2004)

Theorem 5. [*Reni and Zimmermann (2001-2004)*] *Let G be a finite group of orientation preserving diffeomorphisms of an integer homology 3-sphere. Then either G is solvable, or G is isomorphic to one of the following groups.*

$$\mathbb{A}_5 \quad \text{or} \quad \mathbb{A}_5 \times \mathbb{Z}_2, \quad \mathbb{A}_5^* \times_{\mathbb{Z}_2} C, \quad \mathbb{A}_5^* \times_{\mathbb{Z}_2} \mathbb{A}_5^*$$

where C is solvable with a unique involution. Each involution of $\mathbb{A}_5$ has nonempty fixed point set, and each of the factors $\mathbb{A}_5^*$ and C acts freely.
($\mathbb{A}_5^* \cong \mathrm{SL}(2,5)$ is the binary dodecaedral group)

Problem: Which finite (simple) groups of list in Theorem 3 really act on a $\mathbb{Z}_2$-homology 3-sphere?

Zimmermann produces (using computer calculations with GAP) examples of $\mathbb{Z}_2$-homology 3-spheres with

- $\mathrm{PSL}(2,q)$-actions for various small values of q (preprint 2004)
- $\mathrm{SL}(2,q)$-actions for various small values of q
- actions of some odd order groups which can not act on any integer homology 3-sphere

Conjecture: For all q odd, the group $\mathrm{PSL}(2,q)$ (resp. $\mathrm{SL}(2,q)$) acts on a $\mathbb{Z}_2$-homology 3-sphere.
Conjecture: All finite groups of odd order act on a $\mathbb{Z}_2$-homology 3-sphere.

What about $\widehat{\mathbb{A}}_7$? What about $\mathrm{SL}(2,q)\times_{\mathbb{Z}_2}\mathrm{SL}(2,q')$?

Sketch of the proof of Theorem 3

(Further motivation: these methods have been applied to study the isometry groups of the n-fold cyclic branched coverings of hyperbolic 2-bridge knots [Reni and Vesnin (2001)])

Let G be a finite group acting on a $\mathbb{Z}_2$-homology 3-sphere.

Basic facts:

1. G has sectional 2-rank smaller or equal to four, i.e. each 2-subgroup of G is generated by at most four elements.

2. if t is an involution of G with nonempty fixed point set $C_G(t)$ is a subgroup of $\mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_b)$.

The quasisimple case

A group Q is quasisimple if it is perfect (the abelianized group is trivial) and $Q/Z(Q)$ is a nonabelian simple group.

Lemma 1. *If G is quasisimple and G contains an involution with nonempty fixed point set then $G/Z(G)$ has only one conjugacy class of involutions.*

Proof of Lemma 1

- (sectional 2-rank of $G \leq 4$) $\Rightarrow$
  (sectional 2-rank of $G/Z(G) \leq 4$)

- we apply the Gorenstein-Harada classification of finite simple groups of sectional 2-rank at most four to obtain a list of possibilities for $G/Z(G)$

- we exclude the groups in the Gorenstein-Harada list that have two conjugacy classes of involutions analyzing the centralizer in the groups of the involutions (with nonempty fixed point set)

Lemma 2. *If G is quasisimple $G/Z(G)$ is isomorphic to* $\mathrm{PSL}(2, q)$*, for an odd prime power q greater than four, or to $\mathbb{A}_7$.*

Proof of Lemma 2

- by topological methods (except for one case) we deduce from Lemma 1 that the Sylow 2-subgroups of $G/Z(G)$ are dihedral

- we apply Gorenstein-Walter Theorem that classifies the simple groups with dihedral Sylow 2-subgroups

Remark: for simple groups we are able now to obtain that the Sylow 2-groups are dihedral avoiding the Classification of Simple Groups.

Lemma 3. *If G is quasisimple and $\mathcal{O}(G)$ is trivial. Then G is isomorphic to one of the following groups, for an odd prime power q greater than four:*

$$\mathsf{PSL}(2,q), \quad \mathsf{SL}(2,q) \quad or \quad \widehat{\mathbb{A}}_7.$$

Each involution in $\mathsf{PSL}(2,q)$ has nonemtpy connected fixed point set, the unique involutions of $\mathsf{SL}(2,q)$ and $\widehat{\mathbb{A}}_7$ act freely.

Remark: we are able to exclude $\mathbb{A}_7$ but its extension $\widehat{\mathbb{A}}_7$ remains

The general case

A group E is semisimple if it is perfect and the factor group $E/Z(E)$ is a direct product of nonabelian simple groups. A semisimple group is a central product of quasisimple groups.

$(\mathrm{SL}(2, q) \times_{\mathbb{Z}_2} \mathrm{SL}(2, q')$ are examples of semisimple groups)

Any finite group has a unique maximal semisimple normal subgroup $E(G)$ (may be trivial.)

First possibility: $E(G)$ is not trivial (G is not solvable)

Starting from Lemma 3 we prove that G is isomorphic to one of the nonsolvable groups presented in Theorem 3

Second possibility: $E(G)$ is trivial

In this case we prove that the group G is solvable. The key lemma is the following:

Lemma 4. *If G has a normal 2-subgroup containing an involution with nonempty connected fixed point set, then G is solvable.*

(the proof is based manly on topological arguments)

3. Nonsolvable groups acting on integer homology 4-spheres

Proposition 1. *A finite 2-group admitting an action on a $\mathbb{Z}_2$-homology 4-sphere has rank at most four.*

Proof uses Dotzel-Hamrick's result.

Theorem 6. *(for simple groups) M. and Zimmermann (Preprint 2004)*

The only finite nonabelian simple groups admitting an action on a integer homology 4-sphere are the alternating or linear fractional groups $\mathbb{A}_5 \cong \mathrm{PSL}(2,5)$ *and* $\mathbb{A}_6 \cong \mathrm{PSL}(2,9)$.