



SMR1662/2

Summer School and Conference on Geometry and Topology of 3-manifolds

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Lectures on geometric 3-manifolds

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1. Some general results on spaces of constant curvature, group actions and geometric manifolds

Thm. A simply-connected complete Riemannian manifold of constant curvature 1, 0 or -1 is isometric to

S^n , n-sphere	\mathbb{R}^n , euclidean n-space	\mathbb{H}^n , hyperbolic n-space
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Cor. If M is a complete Riemannian manifold of constant curvature 1, 0 or -1, then the universal covering of M is

$X = S^n, \mathbb{R}^n$ or \mathbb{H}^n ,
and $M = X/G$ where G is the universal covering group: a group of isometries of X which acts freely and (properly) discontinuously on X .

Such a manifold is called a (complete) spherical, euclidean or hyperbolic manifold

Viceversa, if $G \subset \text{Iso}(X)$ acts freely and properly discontinuously, then

$p: X \rightarrow X/G$ is a covering and $M = X/G$ a spherical, euclidean or hyperbolic manifold.

free action: every non-trivial element has empty fixed point set

(properly) discontinuous action:

for every compact subset C of X ,

$\{g \in G : g(C) \cap C \neq \emptyset\}$ is finite

(in particular: points $x \in X$ have finite stabilizers $G_x = \{g \in G : g(x) = x\}$)

motivation for the definition:

Prop. ([Thurston 1, 3.5.7])

Let Γ be a group acting on a manifold X . The quotient space X/Γ is a mfd. with $X \rightarrow X/\Gamma$ a covering projection if and only if Γ acts freely and discontinuously

(here manifolds are supposed to be Hausdorff spaces!)

Thurston calls an action wandering

If every $x \in X$ has a neighbourhood U such that $\{g \in \Gamma : g(U) \cap U \neq \emptyset\}$ is finite. Then the Proposition holds for free wandering actions (instead of free discontinuous) but the quotient X/Γ need not be Hausdorff, in general

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \subset \mathbb{R}^{n+1}$$

a Riemannian submanifold

$$\text{Iso}(S^n) = O(n+1)$$

(the isometry group) the orthogonal group of orth
 $(n+1) \times (n+1)$ matrices (a Lie-group)

(an isometry is determined by the action in one point and its tangent space)

\mathbb{R}^n euclidean n -space, with the euclidean metric $ds^2 = dx_1^2 + \dots + dx_n^2$

$$\text{Iso}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n) \quad (\text{Lie-group})$$

translations | iso. fixing 0
 with semidirect product,
 with normal subgroup \mathbb{R}^n

ex. S^n , $\mathbb{RP}^n = S^n / \{\pm \text{id}\}$ are spherical mfds,
 real projective space

the n -torus $\mathbb{R}^n / \mathbb{Z}^n \cong (S^1)^n$
 is euclidean

Prop. ([Thurston, 3.5.11])

Suppose G is a Lie group and X a mfd. on which G acts transitively and with compact stabilizers. Then any discrete subgroup of G acts discontinuously on X .

(Also, discontinuous \Rightarrow discrete (which is easy); in general, the converse is not true, e.g. for actions on compact spaces)

transitive action: $\forall x_1, x_2 \in X, \exists g \in G : g(x_1) = x_2$

Remark. If a Lie group G acts transitively on mfd. M^n with compact stabilizers, then there is a G -invariant Riemannian metric on X , so G acts by isometries (by G -averaging the scalar-product at each point: summation (if G is finite) or integration).

Viceversa, if G acts by isometries then the stabilizer of each point is a subgroup of $O(n)$; if a stabilizer contains $SO(n)$ then all sectional curvatures in flat points are equal. For $n > 2$ this implies that X has constant curvature everywhere, for $n=2$ it follows from the transitivity

If G is a discontinuous (or discrete) group of isometries of $X = S^4, \mathbb{R}^4$ or \mathbb{H}^4 , then X/G is called a spherical, euclidean or hyperbolic orbifold;

The singular set of the orbifold is the projection of the fixed point sets of the non-trivial elements of G .

Each $x \in X$ has a neighbourhood U (a closed or open ball) s.t.

$$g(U) = U \text{ for } g \in G_x \quad (\text{the finite stabilizer of } x)$$

$$g(U) \cap U = \emptyset \text{ for } g \notin G_x,$$

so the orbifold X/G contains U/G_x as a subset.

(X, G) -Structures

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Let X be a connected manifold and G a group of diffeomorphisms of X .
A manifold M has an (X, G) -structure

(is modelled on (X, G)) if M has coordinate maps to X and the transition functions are restrictions of elements in G .

Suppose that X and G are real-analytic.
Let \tilde{M} be the universal covering of M , with covering group Γ , so $M = \tilde{M}/\Gamma$. Starting with a single coordinate map, by analytic continuation one obtains a map

$$D : \tilde{M} \longrightarrow X$$

("developing map"); if $\gamma \in \Gamma$,

$$D(\gamma(x)) = H(\gamma)(D(x)),$$

for an element $H(\gamma) \in G$ which does not depend on x , and we get a homomorphism of groups

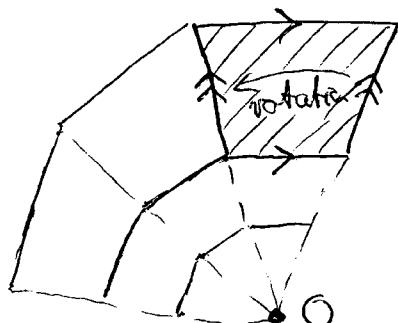
$$H : \Gamma \longrightarrow G$$

the holonomy of M , well-defined up to ϵ
 conjugation; $H(\Gamma)$ is the holonomy group
 of M .

Def. M be complete (XG) -manifold if

$D: \tilde{M} \rightarrow X$ is a covering;
 in particular, if X is simply connected,
 D is a homeomorphism, and \tilde{M} can be
 identified with X and the universal covering
 group with the holonomy group.

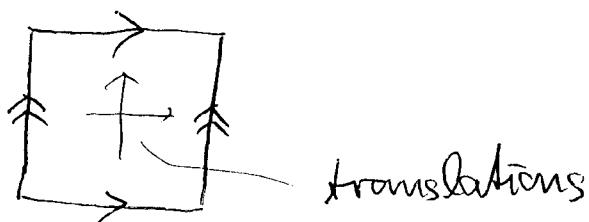
example. $X = \mathbb{R}^2$, $G =$ group of similarities
 of \mathbb{R}^2



$$\text{1. } \alpha \in \mathbb{R}_+$$

a similarity
torus which is
 not complete
 (the developing map
 avoids O)

Remark. such a similarity torus is
 complete iff it is euclidean:



Prop ([Thurston 1, 34, 15])

Let G be transitive group with compact stabilizers of real-analytic diffeomorphisms of X . Fix a G -invariant metric on X , and let M be a (X, G) -manifold with the metric induced from X . Then M is a complete (X, G) -manifold iff M is complete as a metric space.

In particular, every closed (X, G) -mfld is complete.

geometric manifolds

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A geometric structure on a manifold M is a complete locally homogeneous

Riemannian metric on M (i.e., every 2 points have isometric neighbourhoods).

Let $X := \tilde{M}$ and $G := \text{Iso}(X)$, then G has compact stabilizers and acts transitively on X (X is homogeneous: by a result of Singer, b.c. homog + simply connected \Rightarrow homog.)

Hence, the geometric manifolds are exactly the (X, G) -mfds where X is simply connected and G acts transitively with compact stabilizers on X ;

in fact, a G -invariant Riemannian metric on X is complete (easy to prove);

X has constant curvature iff stabilizers of points contain $\text{SO}(n) \subset \text{O}(n)$.

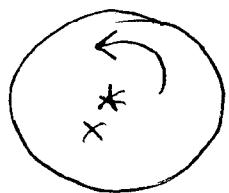
One says that M has (X, G) -geometry

2. The 2-dimensional case

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Let G be a discontinuous group of or.-pres. isometries of $X = S^2, \mathbb{R}^2$ or H^2 with compact quotient (a "cocompact action");

then each stabilizer $G_x = \{g \in G : g(x) = x\}$ is a finite subgroup of $\text{SO}(2) \cong S^1$ and $G_x \cong \mathbb{Z}_n$ acts by rotations on a disk-around of x ;



it follows that X/G is an or. closed surface with finitely many singular points: a 2-orbifold

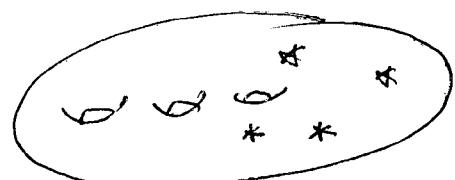
of type or signature

$$F_g(u_1, \dots, u_\alpha)$$

$$(g; u_1, \dots, u_\alpha)$$

genus
of X/G

branch orders



$$\chi(g; u_1, \dots, u_\alpha) := 2 - 2g - \sum_{i=1}^{\alpha} \left(1 - \frac{1}{u_i}\right)$$

is the orbifold Euler characteristic

Euler characteristics behave multiplicatively under finite coverings

$$-\left(1 - \frac{1}{n_i}\right) = -1 + \frac{1}{n_i}$$

delete singular vertex

add weighted singular vertex

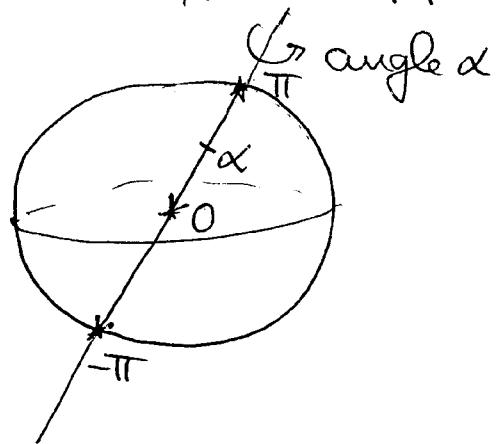
2.1 spherical geometry

$$\text{Iso}_+(S^2) = \text{SO}(3) - \text{rotations around axes in } \mathbb{R}^3$$

$$\text{SO}(3) \cong \text{RP}^3$$

real projective 3-space

$$\cong S^3 / \text{antipodal revolution}$$



$$\cong \text{unit tangent bundle of } S^2$$

a spherical 3-mfd; also, a Seifert fiber space fibered by circles

the spherical 2-mfds are S^2 and $\text{RP}^2 = S^2/\{\pm \text{id}\}$.

the finite subgroups of $SO(3)$ are

cyclic \mathbb{Z}_n
quotient is $(0, n, n)$

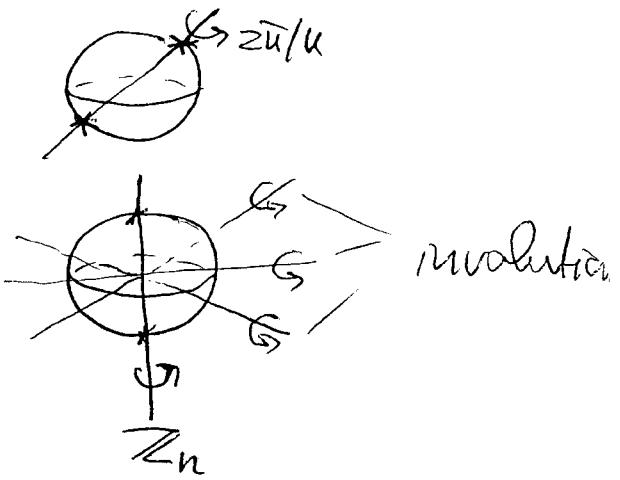
dihedral D_{2n}
 $(0, 2, 2, n)$

tetrahedral
 $(0; 2, 3, 3)$

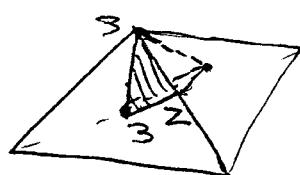
octahedral
 $(0; 2, 3, 4)$

dodecahedral
 $(0; 2, 3, 5)$

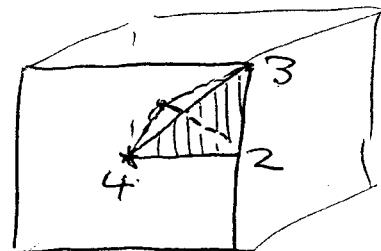
angs
 $\pi/n_1, \pi/n_2, \pi/n_3$



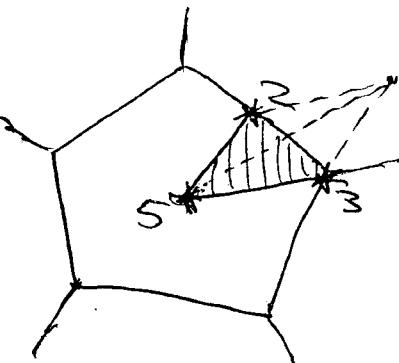
A_4



S_4



A_5



gives tessellations
of S^2 by
spherical
triangles

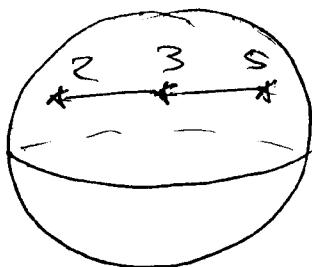
The above groups are the or. pres. subgroups
of index 2, of the groups generated by the
reflections in the sides of these triangles

example.

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$$S^2 \longrightarrow S^2/A_5 \text{ dodecahedral group}$$

u



(0, 2, 3, 5)

a "complex" with 3 vertices,
2 edges and 1 face

$$\begin{aligned} \chi(S^2) &= 2 - 2g = 2 \\ \text{Euler characteristic} &= 60 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 2 + 1 \right) \\ &\quad \text{weighted vertices} \quad \text{edges} \quad \text{face} \end{aligned}$$

$$= 60 \chi(0, 2, 3, 5)$$

(\Rightarrow Classification of or spherical
2-orbifolds)

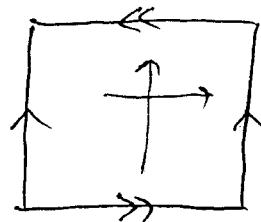
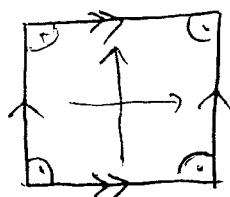
$$\pi - \frac{\pi}{2} - \frac{\pi}{3} - \frac{\pi}{5} = \pi \chi(0, 2, 3, 5)$$

is the area of the spherical triangle
with angles $\pi/2, \pi/3, \pi/5$.
 \rightarrow $\pi/2 \rightarrow \alpha_1 \rightarrow \pi/3 \rightarrow \alpha_2 \rightarrow \pi/5 \rightarrow \alpha_3$ the manifold

2 euclidean case

$$\text{Iso}(\mathbb{R}^2) = \mathbb{R}^2 \times O(2)$$

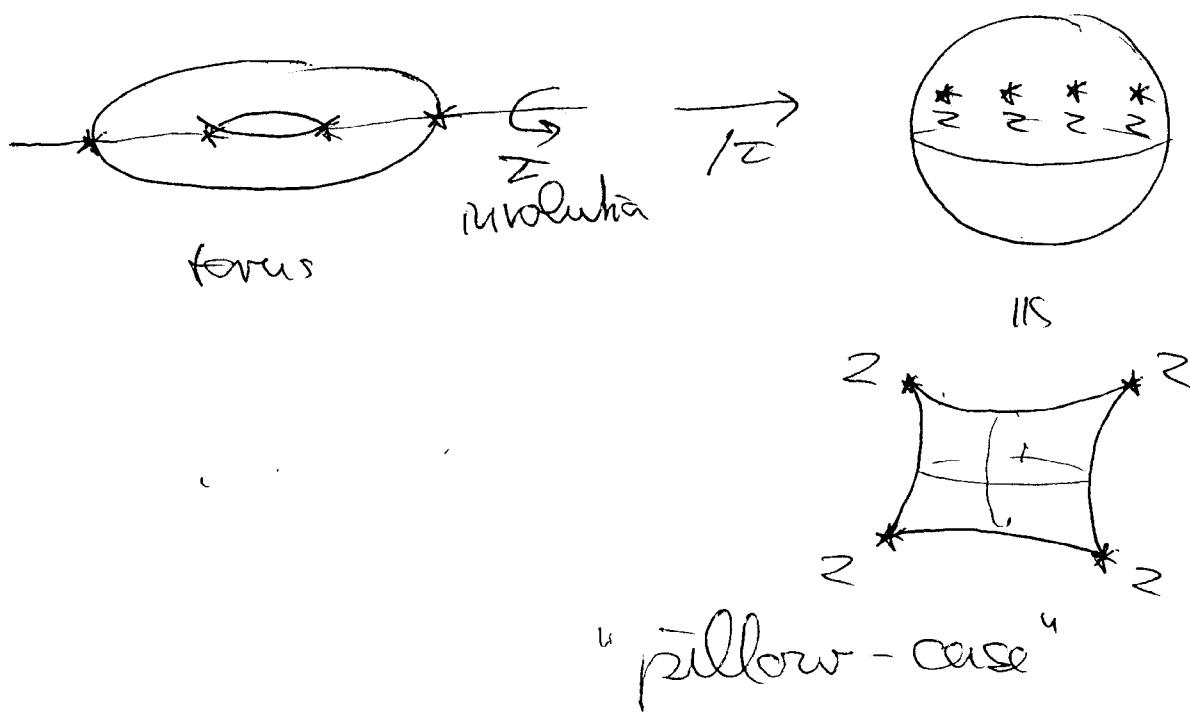
the closed spherical 2-mfds are
the torus and the Klein-bottle



the orientable euclidean 2-orbifolds
are

$(0; 3,3,3)$, $(0; 2,4,4)$, $(0; 2,3,6)$ and
 $(0; 2,2,2,2)$

all Euler-characteristics are 0



2.3 Hyperbolic case

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$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$$

special linear group

$$\boxed{PSL(2, \mathbb{C})} = SL(2, \mathbb{C}) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

projective
special linear group,
group of or.-pres.
linear fractional group, Möbius transformation

$$PSL(2, \mathbb{C}) \text{ acts on } \mathbb{CP}^1 = \left\{ [z_1 : z_2] , z_1, z_2 \in \mathbb{C}, \text{ not both } 0 \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

$$\mathbb{CP}^1 \cong \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$$

$$[z_1 : z_2] \rightarrow \frac{z_1}{z_2} \quad \text{Riemann sphere}$$

$$z_2 \neq 0 : \quad z = \frac{z_1}{z_2} \rightarrow \frac{az+b}{cz+d} \in \overline{\mathbb{C}}$$

$$z_2 = 0 : \quad z_1/0 = \infty \rightarrow a/c$$

$$PSL(2, \mathbb{C}) = \left\{ g : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}} : \right.$$

$$g(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, ad - bc = 1$$

any element of $\boxed{PSL(2, \mathbb{C})}$ has

Remarks. i) Any element of $PSL(2, \mathbb{C})$ is a composition of elements of the following kinds:

$$\begin{cases} z \rightarrow z+b, \quad b \in \mathbb{C} & ("parabolic") \\ z \rightarrow az = \frac{\sqrt{a}z}{\overline{\sqrt{a}}}, \quad a \in \mathbb{R}_+ & ("hyperbolic") \\ z \rightarrow az = \frac{e^{ip}z}{e^{-ip}}, \quad a = e^{\frac{z+i\pi}{2}}, |a|=1 & ("elliptic") \\ z \rightarrow -\frac{1}{z} & ("elliptic") \end{cases}$$

These fix ∞

Pf. $f(z) = \frac{az+b}{cz+d}$; assume $c \neq 0$:

$$f_1(z) = z - c^{-1}d, \quad f \circ f_1(z) = a' + \frac{b''}{z},$$

$$f_2(z) = -\frac{1}{z}, \quad ((f \circ f_1 \circ f_2)(z)) = a' - b''z, \text{ etc.}$$

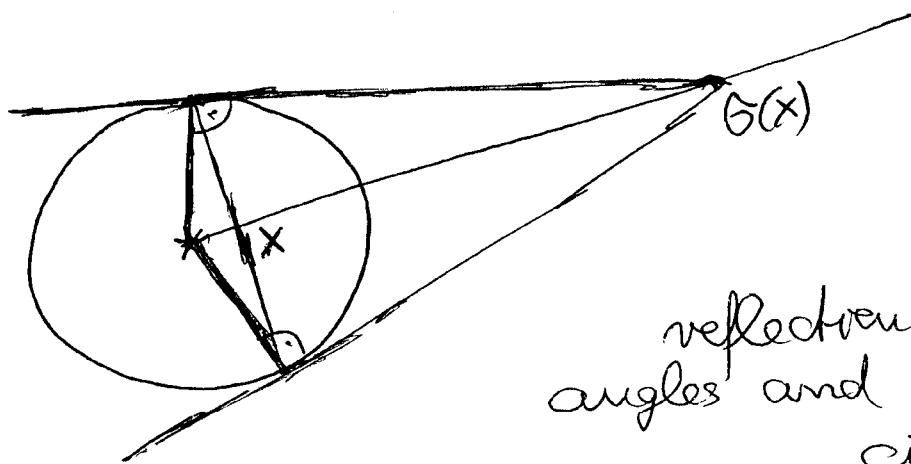
ii) any element of $PSL(2, \mathbb{C})$ is conjugate to an element of the form

$$z \rightarrow z+b \quad \text{or} \quad z \rightarrow az$$

Pf. By conjugation, move one fixed point of the element to ∞ ; if it has a second fixed point, move this to 0 .

Remarks. i) The group $PSL(2, \mathbb{C})$ is exactly the group of conformal (angle and orientation-preserving) homeomorphisms of $\overline{\mathbb{D}}$

- ii) The elements of $PSL(2, \mathbb{C})$ map generalized circles (circles and lines $\cup \{\infty\}$) to generalized circles (which are the sets $\{z \in \mathbb{C} : az\bar{z} + bz + \bar{b}\bar{z} + c = 0,$
 $a, c \in \mathbb{R}, b \in \mathbb{C}, b\bar{b} - ac > 0\}$, plus ∞ if $a = 0$),
- iii) Any element of $PSL(2, \mathbb{C})$ is a composition of reflections in generalized circles; the group generated by such reflections is the group of Möbius transformations of \mathbb{D} .



reflections preserve
angles and generalized
circles

models for the hyperbolic plane H^2

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$$B^2 := \{ z \in \mathbb{C} : |z| = 1 \} \subset \overline{\mathbb{C}}$$

the subgroup of $\text{PSL}(2, \mathbb{C})$ which maps B^2 to itself is

$$:= \left\{ z \rightarrow \frac{az+b}{bz+a}, a, b \in \mathbb{C}, a\bar{a} - b\bar{b} = 1 \right\}$$

which acts transitively on $\overset{\circ}{B^2}$,

the subgroup which maps $\overset{\circ}{B^2}$ to itself is $\{ z \rightarrow a'z, |a'| = 1 \} = \text{SO}(2)$
(rotations of B^2)

so I has compact stabilizers

$\Rightarrow \overset{\circ}{B^2}$ has a I -invariant

Riemannian metric, necessarily of constant curvature, and I is the group of isometries (or. -pres.)

The metric is

$$ds = \frac{2|dz|}{1-|z|^2}$$

area:

$$\frac{4 dx_1 dx_2}{(1-|z|^2)^2}$$

$$ds^2 = \frac{4(dx_1^2 + dx_2^2)}{(1-(x_1^2+x_2^2))^2}$$

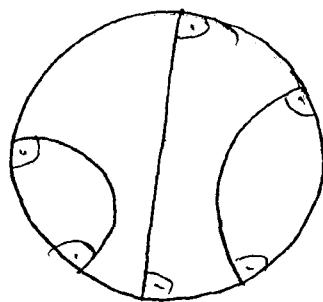
This is the Poincaré disk-model

of the hyperbolic plane H^2

geodesic are

half-circles

orthogonal to $\partial B^2 = S^1$



upper half-space model of H^2

$$\mathbb{C}_+ = \mathbb{R}_+^2 = \{x_1 + ix_2 : x_2 > 0\}$$

metric $ds^2 = \frac{1}{x_2^2} (dx_1^2 + dx_2^2)$

$$ds = \frac{|dz|}{x_2}$$

area $\frac{dx_1 dx_2}{x_2^2}$

or pres. Isometry group is

$$[\overline{\text{PSL}(2, \mathbb{R})}] \subset \text{PSL}(2, \mathbb{C})$$

acting on $\mathbb{RP}^1 \cong \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1$,

$$\partial \mathbb{R}_+^2 = \overline{\mathbb{R}}$$

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Remarks i) Let $z(t)$, $0 \leq t \leq 1$, be a curve in \mathbb{C}_+ with $z(0) = \bar{z}$ and $z(1) = az$. Then the length of $z(t)$ is (with $z(t) = x(t) + iy(t)$)

$$\begin{aligned} \int_0^1 \frac{|dz(t)|}{y(t)} &= \int_0^1 \frac{1}{y(t)} \left| \frac{dz(t)}{dt} \right| dt \\ &= \int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt \\ &\geq \int_0^1 \frac{1}{y(t)} \left| \frac{dy}{dt} \right| dt \\ &= \log y(1) - \log y(0) = \log a, \end{aligned}$$

with equality iff $\frac{dx}{dt} = 0$, that is if $z(t)$ is the segment between \bar{z} and $a\bar{z}$, hence the segment is the shortest curve.

ii) $w = -\frac{1}{z}$ is an isometry of \mathbb{C}_+ :

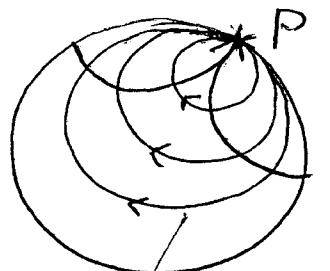
$$w = u + iv = -\frac{1}{x+iy} = \frac{-x+iy}{x^2+y^2}$$

$$\begin{aligned} \frac{|dw|}{v} &= \frac{|d(-\frac{1}{z})|}{v} = \frac{\left| -\frac{1}{z^2} \right| |dz|}{v} \\ &= \frac{\frac{1}{(x^2+y^2)}}{\frac{y}{x^2+y^2}} |dz| = \frac{|dz|}{y} \end{aligned}$$

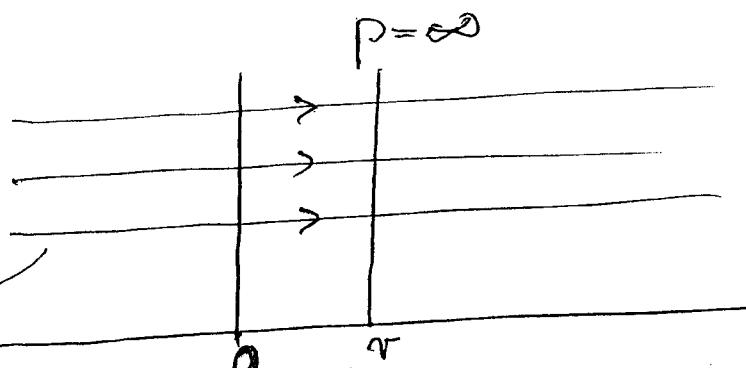
types of isometries: $z \rightarrow \frac{az+b}{cz+d} = g(z)$

parabolic ($a+d = \pm 2$)

one fixed point P which is in $\partial \mathbb{H}^2$



"horospheres"



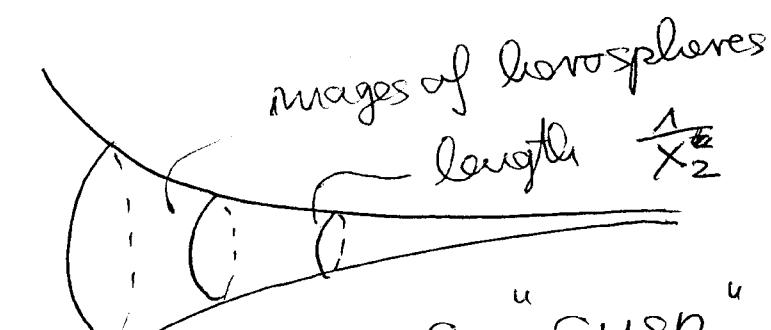
translation

$$z \rightarrow z + r,$$

$$r \in \mathbb{R} :$$

all parabolic elements are conjugate
in $\text{Iso}(\mathbb{H}^2)$, for example
conjugate to $z \rightarrow z + 1$

$$\mathbb{H}^2 / \{z \rightarrow z + 1\} \cong$$



images of horospheres

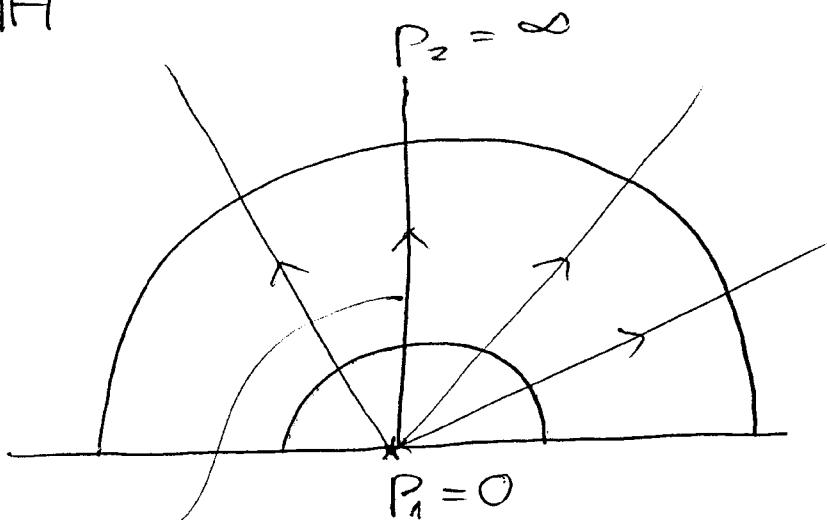
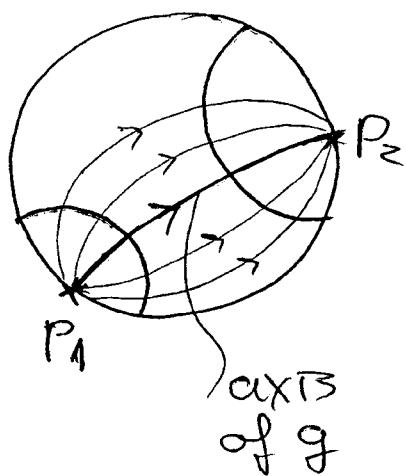
length $\frac{1}{x_2}$

a "cusp" or "punctum"
this part has
finite volume
(area)

hyperbolic:

$$a+d \in \mathbb{R}, |a+d| > 2$$

2 fixed points P_1, P_2 ,
 $P_1, P_2 \in \partial \mathbb{H}^2$



axis
all conjugate to
 $z \rightarrow nz, n \in \mathbb{R}_+$

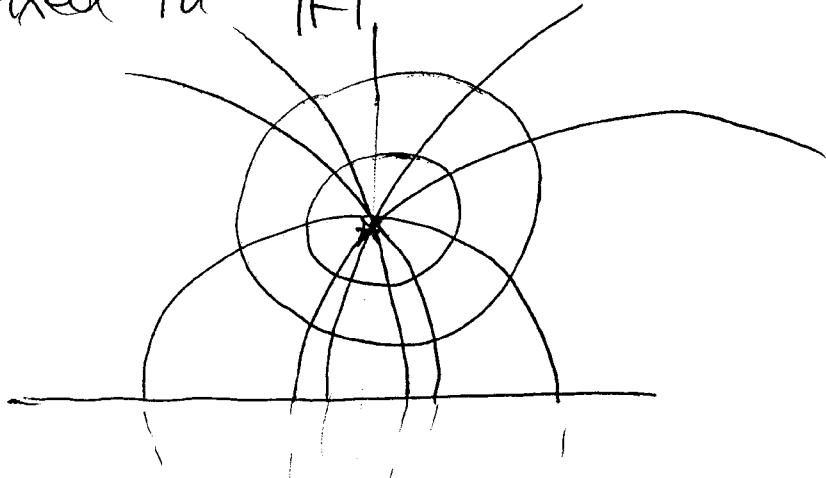
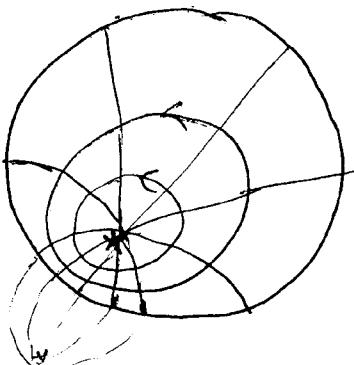
$$\frac{\sqrt{n} z}{\sqrt{n-1}}$$

elliptic

$$|a+d| < 2$$

$$a+d \in \mathbb{R}$$

2 fixed points in $\overline{\mathbb{C}}$
 1 fixed in \mathbb{H}^2



"rotation": the only linear \mathbb{H}^2 -wave fermi order

a discrete or properly discontinuous group G of ^{ori.-pres.} isometries of \mathbb{H}^2 is called a Fuchsian group;

If the orbifold \mathbb{H}^2/G is compact then it is of type

$$(g; n_1, \dots, n_\alpha)$$

$$\chi(g; n_1, \dots, n_\alpha) := 2 - 2g - \sum_{i=1}^{\alpha} \left(1 - \frac{1}{n_i}\right)$$

orbifold Euler characteristic

We interrupt for a general very useful result:

Theorem (Selberg's Lemma)

Every finitely generated subgroup of $GL(n, \mathbb{C})$ has a torsion-free normal subgroup of finite index.

It follows that every ^{f.g.} Fuchsian group G has such a normal subgroup $N \trianglelefteq G$ (a "surface group").

\mathbb{H}^2/N is a closed or. surface of some genus \tilde{g} ,

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$$\mathbb{H}^2/N \longrightarrow \mathbb{H}^2/G$$

is a branched covering, and

$$\begin{aligned}\chi(\mathbb{H}^2/N) &= 2 - 2\tilde{g} = [G:N] \chi(\mathbb{H}^2/G) \\ &= [G:N] \chi(g; u_1, \dots, u_\alpha) \\ &= [G:N] \left(2 - 2g - \sum_{i=1}^{\alpha} \left(1 - \frac{1}{u_i}\right)\right)\end{aligned}$$

(Formula of Riemann-Hurwitz)

A Fuchsian group G of signature (quotient type) $(g; u_1, \dots, u_\alpha)$ has a presentation ("orbifold fundamental group" of \mathbb{H}^2/G)

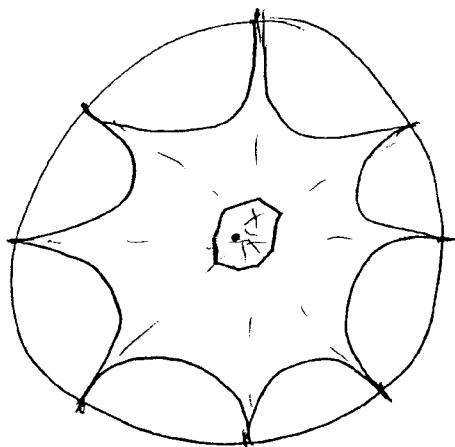
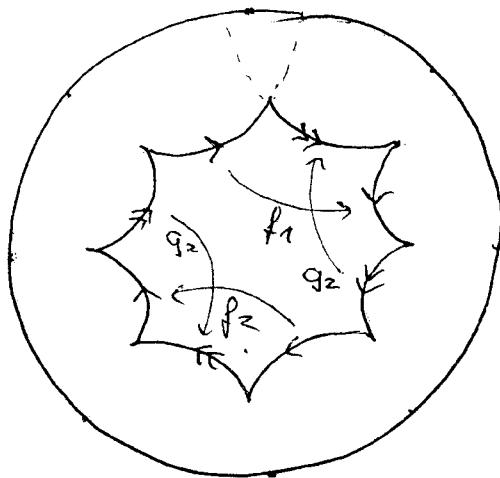
$$\begin{aligned}G &= \langle a_1, b_1, \dots, a_g, b_g, s_1, \dots, s_\alpha \mid \\ &\quad \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^\alpha s_j = 1, \\ &\quad s_j^{n_j} = 1, j=1, \dots, \alpha \rangle\end{aligned}$$

(apply Van Kampen's theorem to the quotient, replacing a disk neighborhood of each fixed point of an element of order n_j by n_j disks which get cyclically permuted: this gives a regular)

examples.

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i)



Identify using
hyperbolic isometries f_1, g_1, g_2
choose octagon such that internal angles
are $2\pi/8$: then the result is

a surface F of genus 2 with a
Riemannian metric which is locally
hyperbolic; in fact, as the surface
is compact, the hyperbolic
structure is complete

H^2 is the universal covering
of F , and $G = \langle f_1, g_1, g_2 \rangle$ the
covering group, with the octagon
("Poincaré's
polyhedron")

Theorem on
fundamental polyhedra)

ii) Let $u_1, u_2, u_3 \in \mathbb{Z}_+$ with

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$$1 - \frac{1}{u_1} - \frac{1}{u_2} - \frac{1}{u_3} > 0,$$

then there is a hyp. triangle Δ with angles $\pi/u_1, \pi/u_2$ and π/u_3 (unique up to isometry); its area is

$$\pi - \pi/u_1 - \pi/u_2 - \pi/u_3 = \pi \left(1 - \frac{1}{u_1} - \frac{1}{u_2} - \frac{1}{u_3}\right)$$

Let \tilde{G} be the group of isometries of \mathbb{H}^2 generated by the reflections on the sides of Δ , and G its or.-pres. subgroup, of index 2; then \mathbb{H}^2/G is an orbifold of type $(0; u_1, u_2, u_3)$ and hyp. volume $2\pi \left(1 - \frac{1}{u_1} - \frac{1}{u_2} - \frac{1}{u_3}\right)$.

Thm. (Gauß-Bonnet) Let M be a closed or. Riemannian 2-manifold with curvature function k ; then

$$\int_M k \frac{da}{\text{area}} = 2\pi \chi(M).$$

special case:

$k = 1, 0$ or -1
 (constant curvature); then

$$k \text{ area}(M) = 2\pi \chi(M)$$

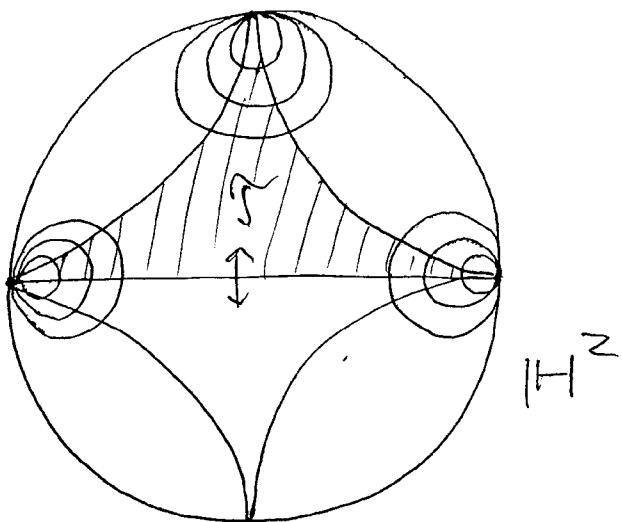
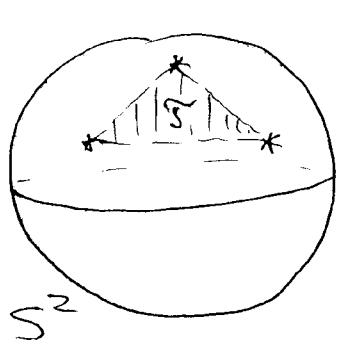
Pf. decompose M into finitely many geodesic hyp. triangles,
 with angles $\alpha_i, \beta_i, \gamma_i$, of area
 (volume) $\pi - \alpha_i - \beta_i - \gamma_i$, $i=1, \dots, n$

$$\begin{aligned} \text{then } \text{area}(F) &= \sum_{i=1}^n (\pi - \alpha_i - \beta_i - \gamma_i) \\ &= n\pi - v \sqrt{2\pi} \\ &= 2\pi \left(\frac{n}{2} - v \right) \text{ number of vertices} \end{aligned}$$

$$\begin{aligned} \chi(F) &= v - e + n, \quad 2e = 3n \\ &= v - \frac{n}{2} \end{aligned}$$

$$\Rightarrow k \text{ area}(F) = \chi(F) 2\pi$$

iii) The 2-sphere minus 3 points

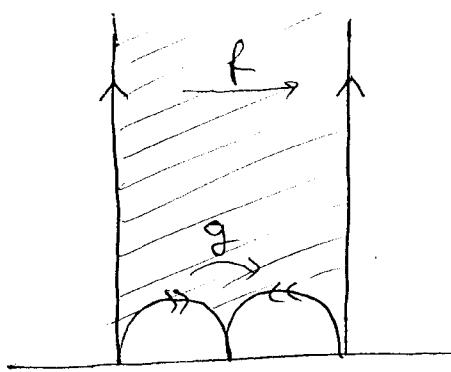
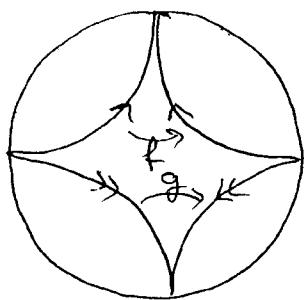


Let Γ be the group generated by the reflections in the sides of an ideal (vertices at infinity) hyperbolic triangle T , and Γ_+ its or.-pres. subgroup (of index 2). The reflections preserve horospheres around the ideal vertices, hence

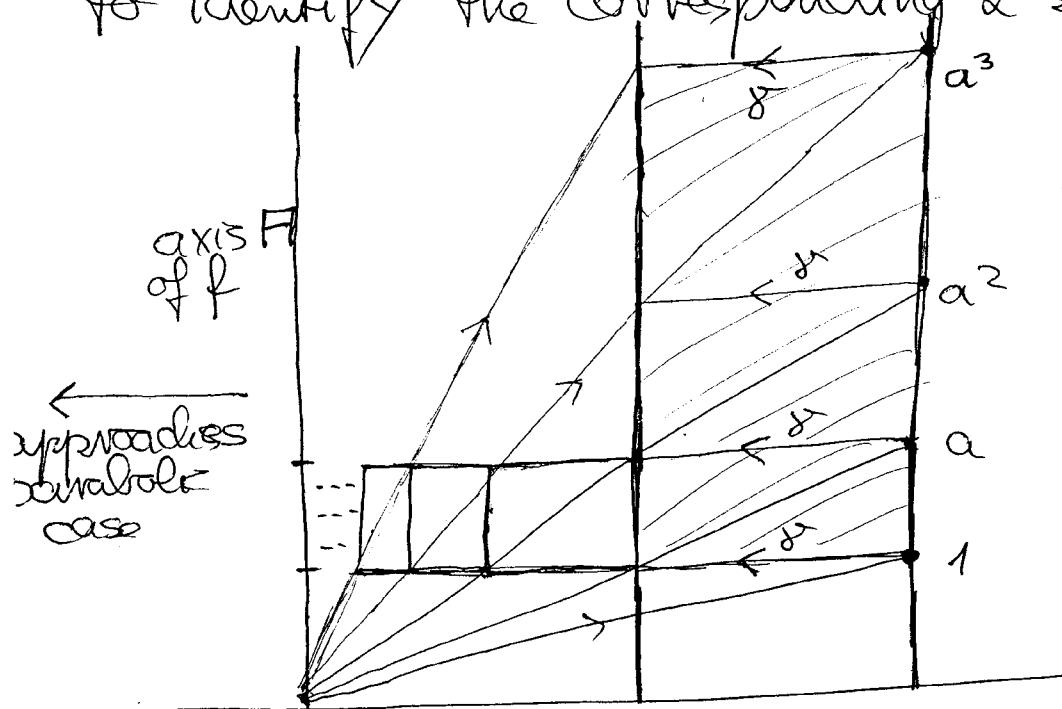
$$\mathbb{H}^2/\Gamma = \text{[diagram of a hyperbolic triangle]} \quad \text{and}$$

$$\mathbb{H}^2/\Gamma_+ = \text{[diagram of a punctured torus]} \approx \text{2-sphere - 3 points}$$

is a complete hyp. 2-orbifold resp.
2-manifold

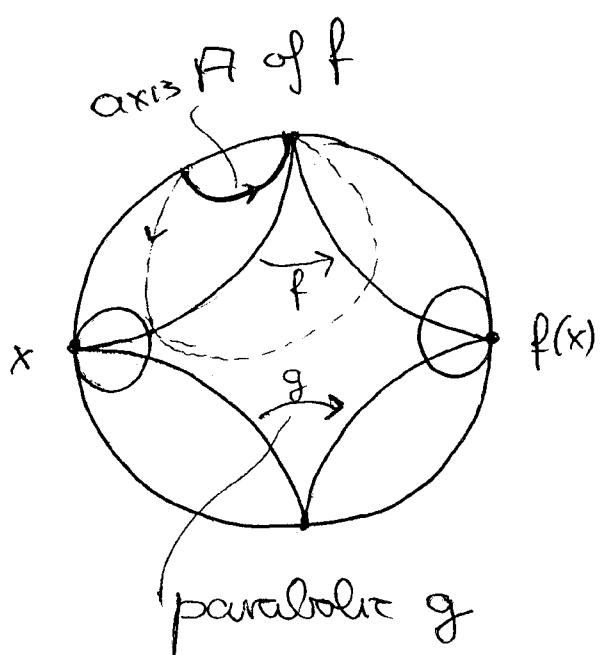
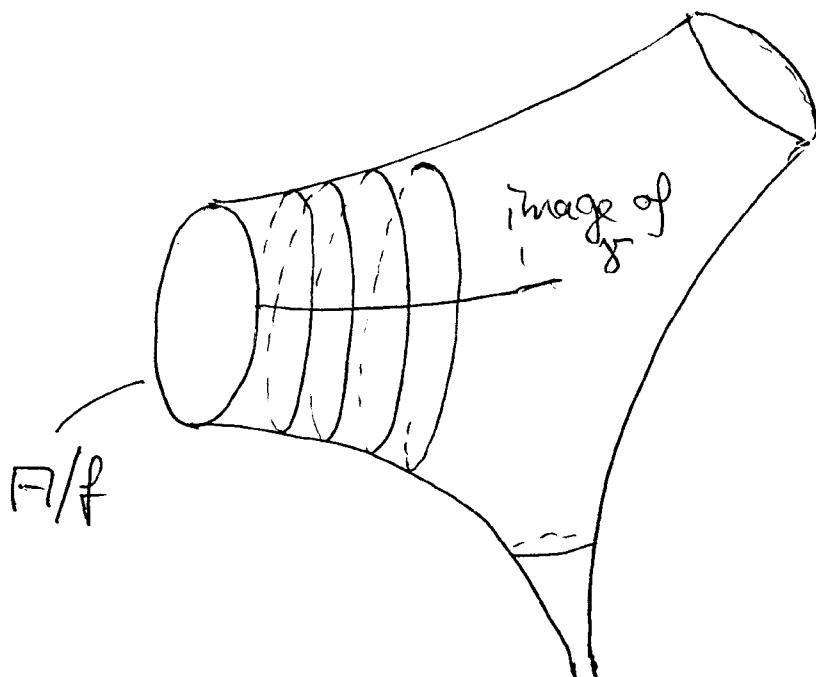


The parabolic transformations f and g generate \mathbb{P}^1 . Now, instead of a parabolic element, use a hyperbolic transformation f to identify the corresponding 2 sides of \mathbb{P}^1 :

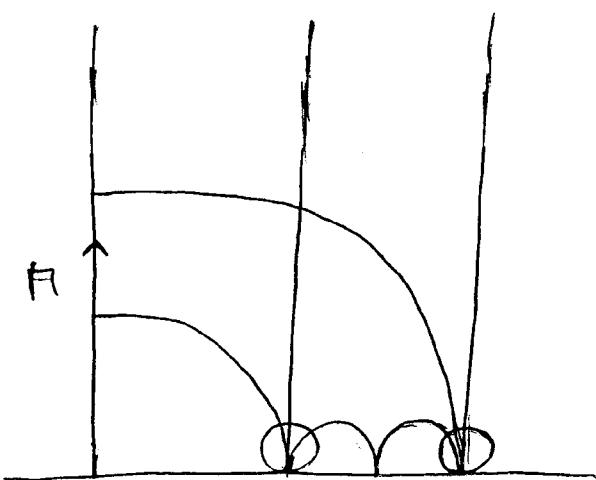


$f = \text{multiplication by } a \in \mathbb{R}_+, a > 1$

after identification, the curve γ has finite length, and the resulting hyperbolic 2-manifold is not complete; it can be compactified (around the image of ∞) by adding the circle \mathbb{P}^1/f :



$(g^{-1} \circ f)(x) = x$
 but does not
 preserve horocycles
 around x , so
 one can, compactly,
 again by a circle



fundamental domains
 of f on the
 left hand side of A

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next interesting cases, for a Fuchsian group G :

- \mathbb{H}^2/G of finite volume
- G finitely generated

2.4 The role of parabolic elements

Prop. Let G be a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Iso}_+(\mathbb{H}^2)$ containing the parabolic element $z \rightarrow z+1$, and

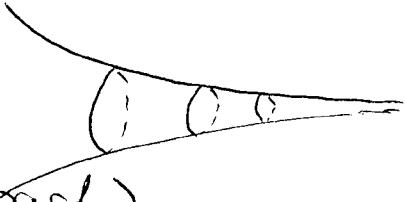
$$\Sigma := \{x + iy \in \mathbb{H}^2 = \mathbb{C}_+: y > 1\}$$

(an open horoball, bounded by the horosphere $y = 1$). Then, for each $g \in G$, either

- $g(\infty) = \infty$, $g(\Sigma) = \Sigma$ and g is parabolic
- or
- $g(\infty) \neq \infty$, $g(\Sigma) \cap \Sigma = \emptyset$.

Cor. The subgroup P of elements of G fixing ∞ consists of parabolic elements, and is infinite cyclic. The hyperbolic 2-orbifold \mathbb{H}^2/G contains the cusp (or puncture) Σ/P (in particular, \mathbb{H}^2/G is not compact).

(a parabolic element moves points arbitrarily small distances \mathbb{H}^2/G ...)



Lemma. Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent elements in G . Then $c=0$ or $|c| \geq 1$.

Pf. Let $V_1 := V$, $V_{n+1} = V_n^{-1}PV_n \in G$

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \Rightarrow$$

$$\begin{aligned} V_{n+1} &= \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \\ &= \begin{pmatrix} 1 + d_n c_n & d_n^2 \\ -c_n^2 & 1 - d_n c_n \end{pmatrix}, \end{aligned}$$

$$c_{n+1} = -c_n^2, \quad d_{n+1} = 1 - d_n c_n$$

Let $\varepsilon := |c| = |c_1|$; suppose $0 < \varepsilon < 1$.
(then $|c_n| \leq \varepsilon$)

$$d_2 \leq 1 + |d_1|\varepsilon$$

$$d_3 \leq 1 + |d_2|\varepsilon \leq 1 + \varepsilon + |d_1|\varepsilon^2$$

!

$\Rightarrow \{d_n\}_{n \in \mathbb{N}}$ are bounded

$\Rightarrow V_n, n \in \mathbb{N}$ are bounded, so some subsequence V_{n_i} converges and

$V_{n_i}^{-1}V_{n_{i+1}} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, contradicting the discreteness of G .

Lemma Let $f, g \in PSL(2, \mathbb{R})$. Let g be hyperbolic and assume that f and g have exactly one fixed point in common, say ∞ . Then $\langle f, g \rangle$ is not discrete. 34

Pf. Let $g(z) = \alpha z$, with $|\alpha| > 1$ (otherwise consider g^{-1}), fixing 0 and ∞ . Let $f(z) = az + b$, fixing ∞ but not 0 , so $b \neq 0$. Then

$$g^{-n} f g^n(z) = az + \alpha^{-n} b \xrightarrow{n \rightarrow \infty} az,$$

so α is not properly discontinuous.

Pf of the Prop.

Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $ad - bc = 1$.

If $c = 0$ then, by the second Lemma, V is parabolic fixing ∞ and Σ .

If $c \neq 0$ then, by the first Lemma, $|c| \geq 1$.

$$V(x+iy) = x^* + iy^* = \frac{a(x+iy) + b}{c(x+iy) + d}$$

$$= \frac{(ax+b+iy)(cx+d-iy)}{|cx+d|^2 + |c|^2 y^2}$$

$$= \frac{-\dots + iy}{|cx+d|^2 + |c|^2 y^2}$$

$$\Rightarrow y^* \leq \frac{y}{|c|^2 y^2} \leq \frac{1}{y}; \text{ so } y^* < 1 \text{ if } y > 1.$$

3. Hyperbolic 3-space and hyperbolic 3-manifolds

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3.1 Hyperbolic 3-space

$$\mathbb{H}^3 = \mathbb{R}_+^3, \text{ metric } ds^2 = \frac{1}{x_3^2} (dx_1^2 + dx_2^2 + dx_3^2)$$

volume $\frac{dx_1 dx_2 dx_3}{x_3^3}$

$$\begin{aligned}\partial \mathbb{R}_+^3 &= \overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\} \\ &= \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \\ &\cong \mathbb{CP}^1 \cong S^2 \quad \text{"Riemann sphere" } ^4\end{aligned}$$

$\text{PSL}(2, \mathbb{C})$ acts on $\overline{\mathbb{C}}$; this action extends to \mathbb{H}^3 to give the or. pres. isometry group $\text{Iso}_+(\mathbb{H}^3)$ ("Poincaré-extension"):

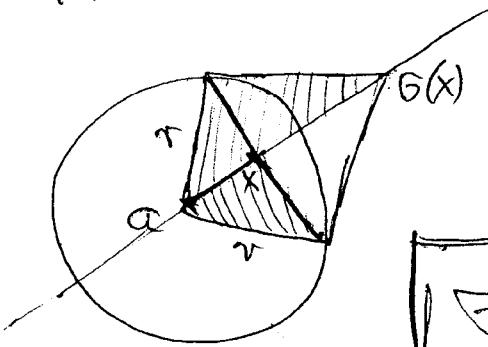
every element of $\text{PSL}(2, \mathbb{C})$, acting on $\overline{\mathbb{R}^2}$, is a composition of reflections in circles $S(a, r)$ and lines in \mathbb{R}^2 :

$$S(x) = a + s(x-a)$$

$$\text{s.t. } |S(x)-a||x-a| = r^2, \text{ or}$$

$$\frac{|S(x)-a|}{r} = \frac{r}{|x-a|}, \text{ so}$$

$$\boxed{S(x) = a + \frac{r^2}{|x-a|^2} (x-a)}$$



each such reflection extends
canonically to $\overline{\mathbb{R}^3}$ and \mathbb{R}_+^3
as the reflection in the sphere
orthogonal to \mathbb{R}^2 in $S(a, r)$

(one has to show that this is
well-defined)

alternative definition using quaternions:

$$\mathbb{R}_+^3 = \{z + t\bar{i} : z \in \mathbb{C}, t \in \mathbb{R}_+\}$$

$$g \in PSL(2, \mathbb{C}), \quad g(z) = \frac{az+b}{cz+d}$$

$$g(z + t\bar{i}) = \frac{a(z + t\bar{i}) + b}{c(z + t\bar{i}) + d};$$

then $g(\mathbb{R}_+^3) = \mathbb{R}_+^3$, and g
is an isometry of \mathbb{H}^3 .

(see [Bordoni])

disk model of \mathbb{H}^3

$$\mathbb{H}^3 = \overset{\circ}{B^3}, \quad \partial \overset{\circ}{B^3} = S^2 \cong \overline{\mathbb{C}}$$

metric $ds^2 = \frac{4(dx_1^2 + dx_2^2 + dx_3^2)}{(1 - v^2)^2}$

$$v = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Remark. Let \tilde{G} be the reflection in the sphere $S(a, r) \subset \mathbb{R}^2$, and \tilde{G} its extension to \mathbb{R}^3 :

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$$\begin{aligned}
 |\tilde{G}(y) - \tilde{G}(x)| &= r^2 \sqrt{\frac{(y-a)}{|y-a|^2} - \frac{(x-a)}{|x-a|^2}} \\
 &= r^2 \sqrt{\frac{x-a}{|x-a|^2} - \frac{x-a}{|x-a|^2} \frac{y-a}{|y-a|^2} - \frac{x-a}{|x-a|^2}}^{\frac{1}{2}} \\
 &= r^2 \left(\frac{1}{|y-a|^2} - \frac{2 \langle x-a, y-a \rangle}{|x-a|^2 |y-a|^2} + \frac{1}{|x-a|^2} \right)^{1/2} \\
 &= r^2 \left(\frac{\langle (y-a) - (x-a), (y-a) - (x-a) \rangle}{|x-a|^2 |y-a|^2} \right)^{1/2} \\
 &= r^2 \frac{|y-x|}{|x-a||y-a|}
 \end{aligned}$$

$$\tilde{G}(x) = a + r^2 \frac{(x-a)}{|x-a|^2}$$

$$\underbrace{\tilde{G}(x)_3}_{\text{third coordinate}} = r^2 \frac{x_3}{|x-a|^2}$$

$$\begin{aligned}
 \frac{|\tilde{G}(y) - \tilde{G}(x)|^2}{\tilde{G}(y)_3 \tilde{G}(x)_3} &= \frac{r^4 |y-x|^2}{(|x-a|^2 |y-a|^2) \cdot r^2 \frac{x_3}{|x-a|^2} r^2 \frac{y_3}{|y-a|^2}} \\
 &= \frac{|y-x|^2}{y_3 x_3},
 \end{aligned}$$

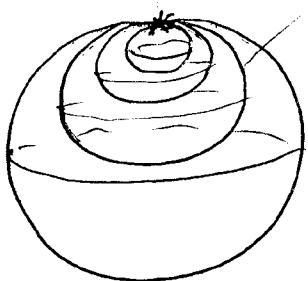
Since $\frac{|y-x|^2}{y_3 x_3}$ is invariant under \tilde{G} , and

$$\frac{|d\tilde{\sigma}(x)|}{\tilde{\sigma}(x)_3} = \frac{|dx|}{x_3}$$

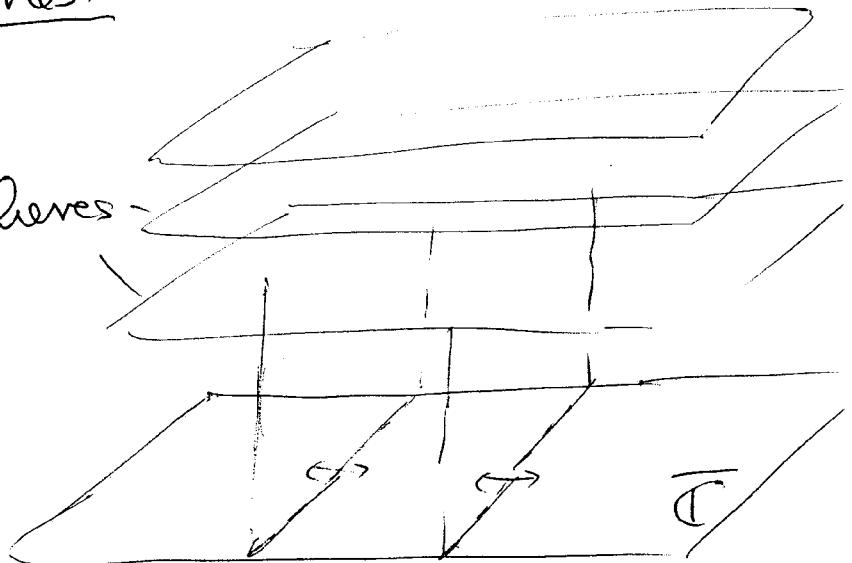
then $ds = \frac{|dx|}{x_3}$ is invariant
 under \tilde{G} , and $\tilde{\sigma}$ is an boundary
 of $H^3 = \mathbb{R}_+^3$

types of isometries:

parabolic



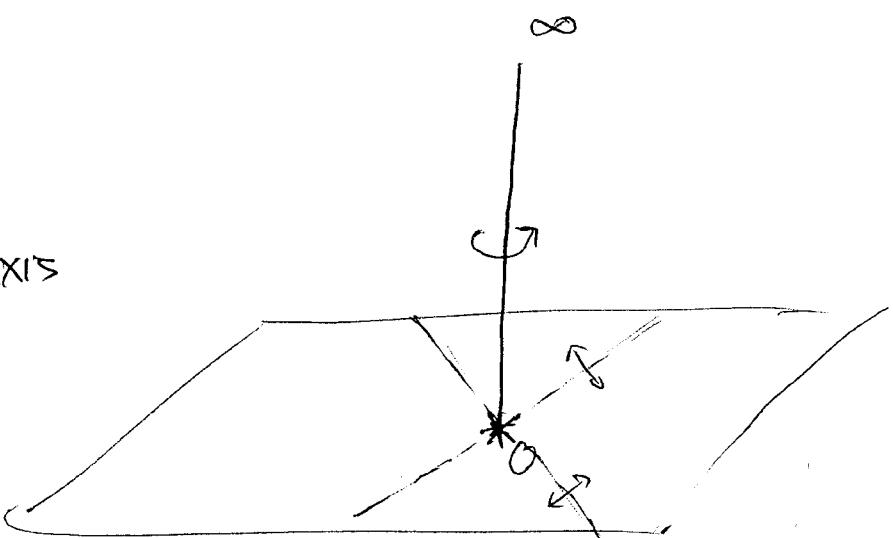
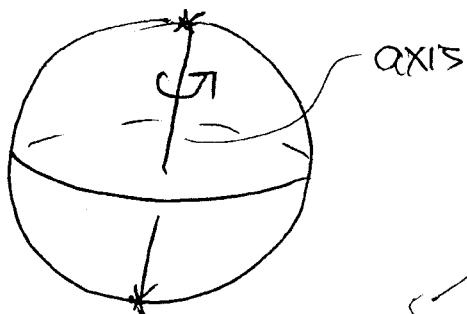
horospheres -



$z \rightarrow z + a$, $a \in \mathbb{C}$
translations

fixing one point,
in \mathbb{C}

elliptic

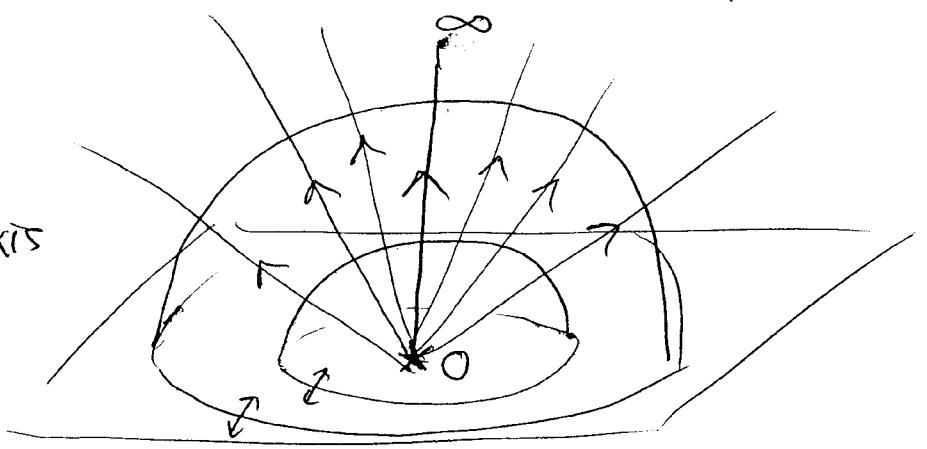


fixes an
axis of
rotation

hyperbolic



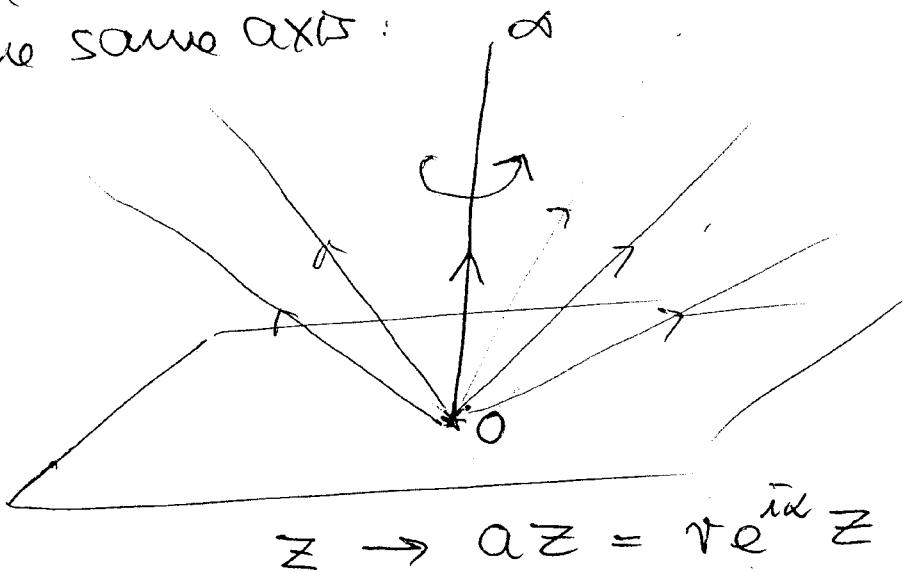
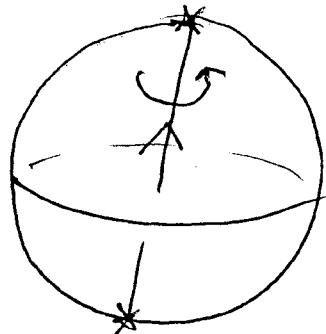
\rightarrow one, infinite



$z \rightarrow rz$, $r \in \mathbb{R}$.

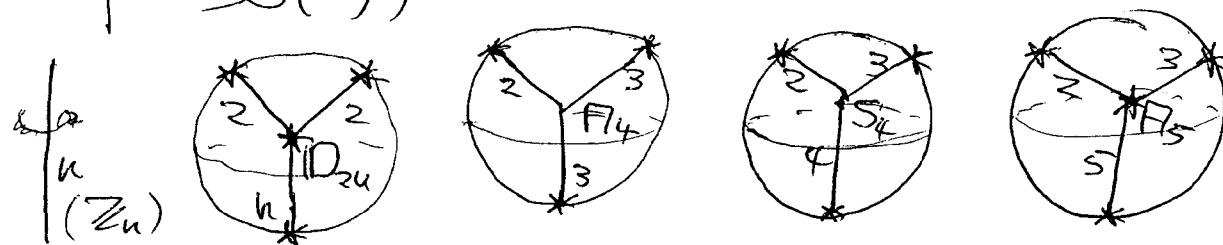
loxodromes

composition of
hyperbolic and
elliptic with the same axis:



- Remarks.
- Every horosphere has Euclidean geometry; the subgroup of $\text{PSL}(2, \mathbb{C})$ mapping a horosphere to itself and fixing $\infty \in \overline{\mathbb{R}_+^3}$ consists of elements of the form $z \rightarrow az + b$, $|a| = 1$ and is the group $\text{Iso}_+(\mathbb{R}^2) = \mathbb{R}^2 \times \text{SO}_2$ of translations and rotations;
 - The subgroup fixing ∞ is the group of similarities of $\mathbb{R}^2 = \partial \overline{\mathbb{R}_+^3} - \{\infty\}$ ($z \rightarrow az + b$, $a, b \in \mathbb{C}$).
 - The subgroup of $\text{PSL}(2, \mathbb{C})$ fixing $o \in B^3$ is $\text{SO}(3)$ (rotations around axes in \mathbb{R}^3 through o)

iii) Locally, the singular set of
 a hyperbolic, euclidean, spherical
 3-orbifold B is one of the following
 types (corresponding to the finite subgroups
 of $\mathrm{SO}(3)$):



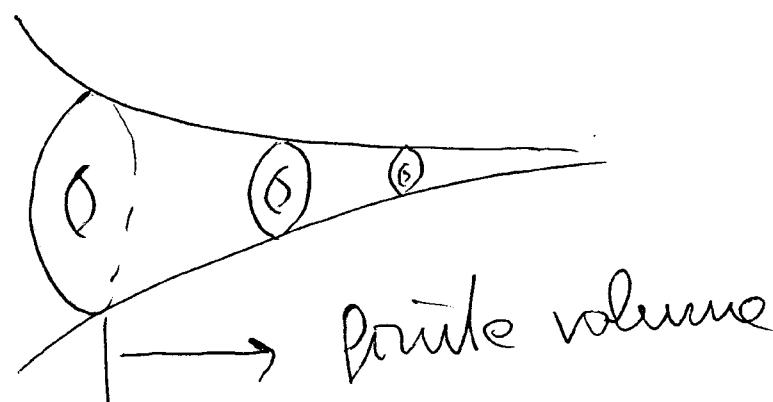
Globally, it is a 3-valent graph
 (some components may be knots (circles),

Using the representation of isometries
 of \mathbb{R}^3_+ by quaternions, the proof
 in the 2-dim case generalizes and gives:

Prop. If $G \subset \mathrm{PSL}(2, \mathbb{C})$ is discrete and
 contains the parabolic element $z \rightarrow z+1$
 then the ^{sub}group P of G fixing ∞
 consists of parabolic and elliptic
 elements, and the hyperbolic 3-orbifold
 \mathbb{H}^3/G contains a subset (as an "end")
 Σ/P , $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 : x_3 > 1\}$,
 (horoball)

P a discrete subgroup of $\mathrm{Iso}_+(\mathbb{R}^2)$.

If G is torsionfree, then $P \cong \mathbb{Z} \times \mathbb{Z}$,
 if in addition \mathbb{H}^3/G has finite volume
 then $P \cong \mathbb{Z} \times \mathbb{Z}$ and Σ/P is a
torus cusp (\cong torus $\times (0, \infty)$)



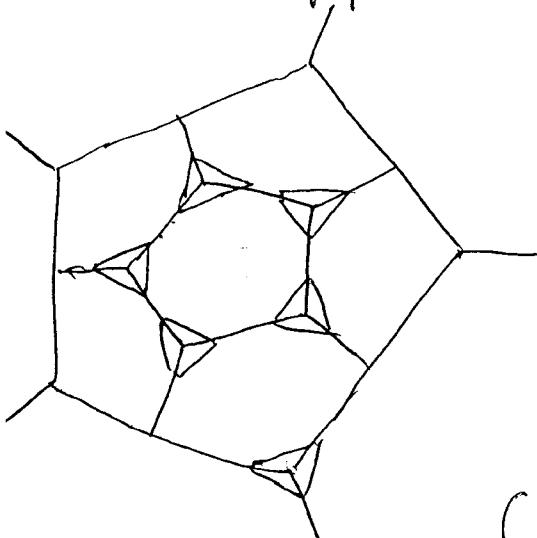
Prop. Let $M = \mathbb{H}^3/G$ be an or. hyperbolic 3-mfd. If M has a subgroup $P \cong \mathbb{Z} \times \mathbb{Z}$ then P is conjugate to a subgroup of the fundamental group of a torus cusp (or of a boundary torus when compactifying each torus-cusp by a torus at infinity): P is "peripheral"

If M is closed (compact) then $G \subseteq \pi_1 M$ has no parabolic elements, and in particular no subgroups $\mathbb{Z} \times \mathbb{Z}$ (this follows also from the fact that parabolic elements move points arbitrarily small distances):

M is atroroidal

Examples

- i) Identify opposite faces of a regular dodecahedron D , by pushing each face to the opposite side and twisting by $2\pi/10$, to get space M . The edges of D get identified in sets of 3.



Consider the link of a vertex, after identification:

This gives a closed surface (by identifying the small triangles in the figure along their sides, according to the identification of faces of D). The identification space M is a manifold iff all these links give the 2-sphere S^2 (in a vertex, M is the cone over its link). It is easy to see that all links are S^2 iff $\chi(M) = 0$ (using that the Euler characteristic of a closed 3-manifold is zero).

The dihedral angles of a Euclidean dodecahedron are $116,565^\circ$. There is a regular spherical dodecahedron with dihedral angles 180° (whose boundary is a geodesic or great sphere S^2 in S^3). Shoving its vertices uniformly to its center, the

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dodecahedron becomes small and has almost euclidean angles, so in between there is a spherical dodecahedron with angles $2\pi/3 = 120^\circ$. Identifying opposite faces by spherical isometries now, the result is a spherical 3-manifold M (the structure behaves well along the edges, and also in the vertices: at a vertex, the little triangles close up nicely to give a covering of S^2 so its link has to be S^2 (which has no non-trivial coverings)).

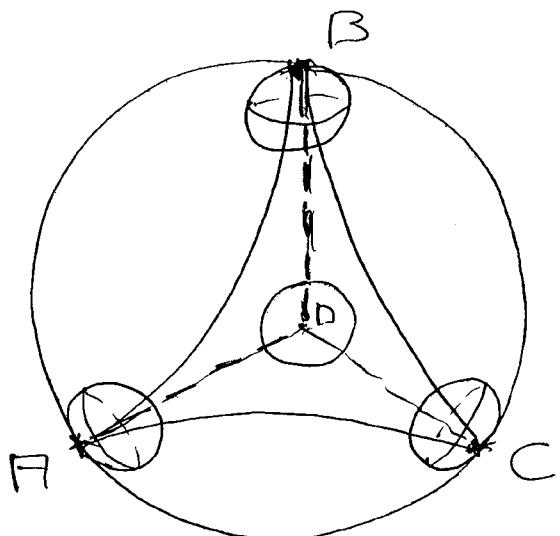
M is the Poincaré (homology) 3-sphere

- ii) Now identify opposite faces of a dodecahedron after a twist of $2\pi \cdot \frac{3}{10}$, then edges get identified in sets of 5.
- The dihedral angle of an ideal (all vertices in ∂H^2) regular hyperbolic dodecahedron are 60° (the angles of an equilateral triangle: put one vertex of the dodecahedron at $\infty \in \partial \mathbb{R}^3_+$).
- Contracting vertices uniformly to $0 \in \partial B^3$, there is a hyperbolic regular dodecahedron with angles $2\pi/5 = 72^\circ$ ($60^\circ < 72^\circ < 116^\circ$,

Identifying opposite faces of this $2\pi/5$ -dodecahedron by hyp. isometries (loxodromic transformations) one obtains a closed hyperbolic 3-mfd, the Seifert-Weber dodecahedral space.

iii) a non-orientable hyp. 3-mfd

Let \mathfrak{F} be a regular ideal hyperbolic tetrahedron in H^3 , all dihedral angles are 60° . Identify the faces by hyperbolic isometries in the following way:



$$x : BAD \rightarrow BDC$$

(rotation around axis BD followed by reflection in plane through B orthogonal to line DC)

$$y : FDC \rightarrow FCB$$

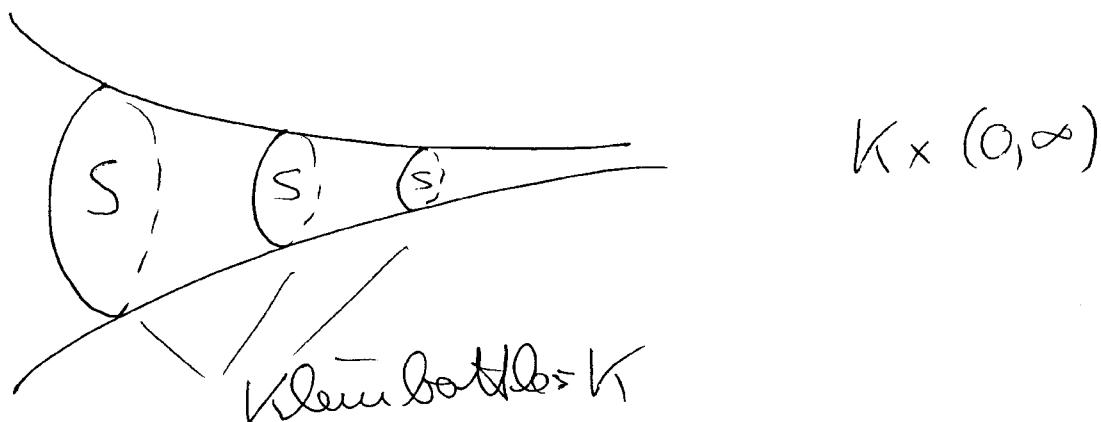
(similarly)

then all 6 edges of \mathfrak{F} get identified:

$$FA \xrightarrow{x} FC \xrightarrow{x} FB \xrightarrow{x} DB \xrightarrow{x} CB \xrightarrow{x} DC \xrightarrow{x} FD$$

the resulting space M is a complete
non-orientable hyp. 3-mfd. of finite
volume, with one Klein-Bottle cusp

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(complete because the system of 4
horospheres in the figure is invariant
under the identifying transformations);
also $y^2x^2y^{-1}x^{-1} = id$.

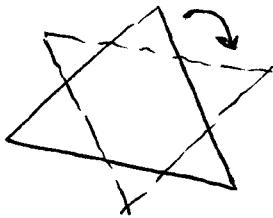
In fact, $\pi_1 M \cong \langle x, y \mid y^2x^2y^{-1}x^{-1} = 1 \rangle$

(by "Poincaré's theorem on fundamental
polyhedra": a presentation of the universal
covering group of M).

M is the Gieseking-mfd. the
orientable 2-fold covering of M is the
complement of the figure-8 knot 
so also this has a complete
hyp. structure (finite volume, one torus-cusp)
(e.g. compute fundamental groups and show that
they are isomorphic;
by a result of Waldhausen for Haken 3-mfds,
they are homeomorphic)

46 a

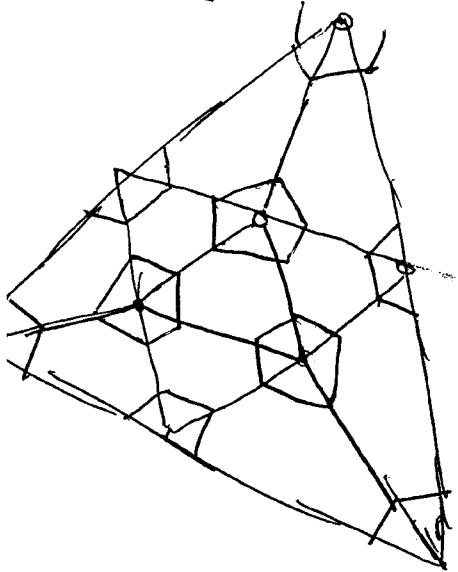
iv) Identify opposite faces (triangles) of a regular icosahe^dron, by pushing faces to the opposite side and rotating by an angle of $2\pi/6$. All vertices get identified, and the edges get identified in sets of 5. However the Euler-characteristic of the resulting space is $\neq 0$, and locally a nbhd. of the unique vertex after identification is a cone over a surface of genus 4.



Starting with a small almost euclidean regular icosahe^dron around $0 \in \mathbb{B}^3$ (of dihedral angles $> 2\pi/3$) and pushing vertices out on diameters of \mathbb{B}^3 , one obtains first a hyperbolic regular icosahe^dron with dihedral angles $2\pi/3$, then a regular ideal icosahe^dron of dihedral angles $3\pi/5$, and then a regular icosahe^dron of angles $2\pi/5$ (where vertices are outside $\mathbb{B}^3 = \overline{\mathbb{H}^3}$). This $2\pi/5$ -icosahe^dron can be truncated by planes orthogonal to its faces, obtaining a truncated regular $2\pi/5$ -icosahe^dron, with a

46 b

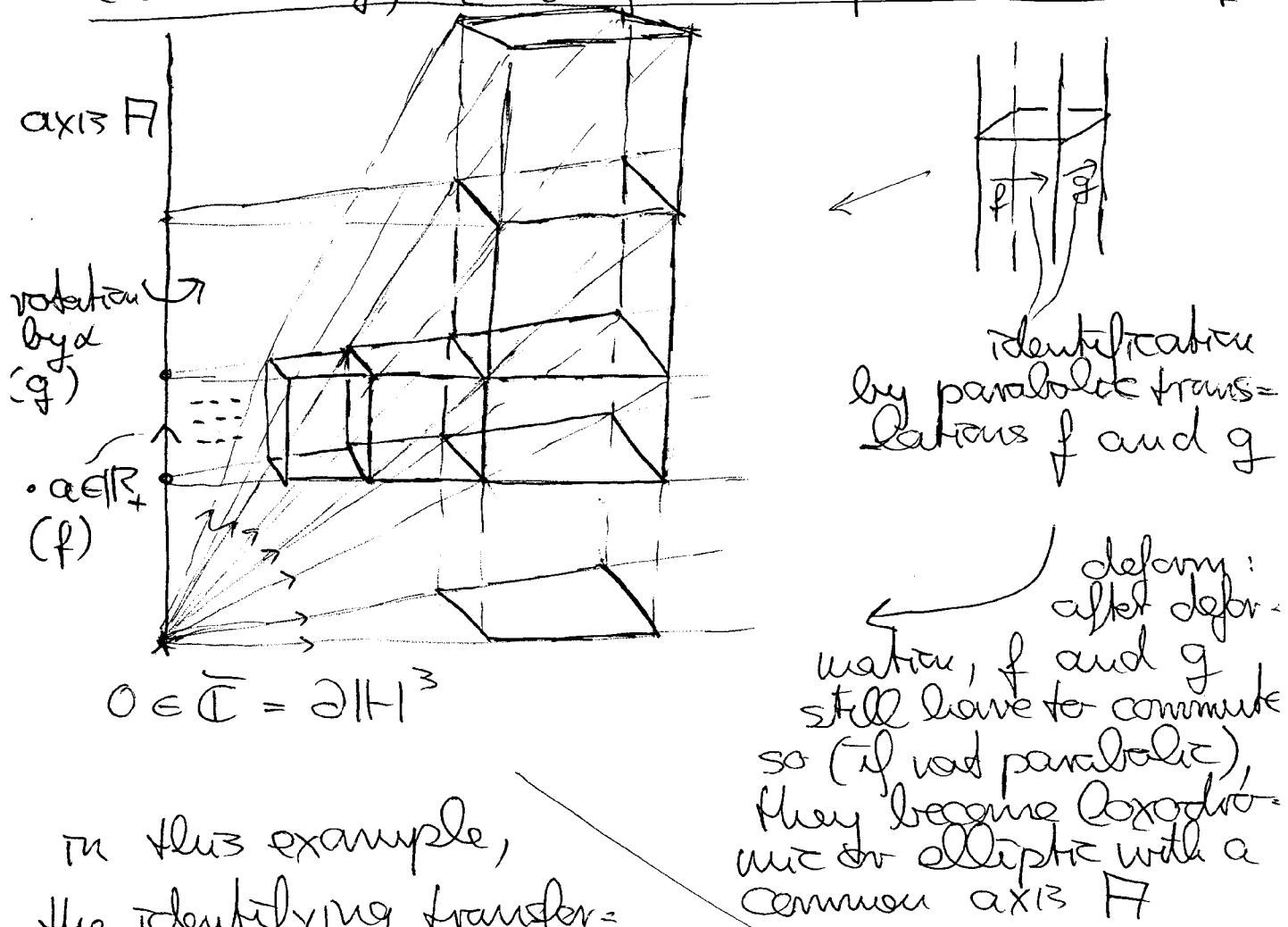
tessellation by 12 regular pentagons lying on the truncating planes, and 20 hexagons coming from the original faces (this is the tessellation of a soccer-ball).



Identifying the 20 hexagons as before by hyperbolic isometries, one obtains a compact hyperbolic 3-manifold whose boundary is a totally geodesic surface of genus 4, with an isometric F_5 -action.

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v) Deforming the parabolic covering group
 (holonomy) $\mathbb{Z} \times \mathbb{Z}$ of a complete torus cusp



In this example, the identifying transformations became hyperbolic (f: multiplication by $a \in R_+$) and elliptic (g: rotation by angle α around axis A)

The developing image (image of the developing map) is R_+^3 -axis A;
 the completion of the deformed cusp is obtained by adding the circle A/f ,
 the result is the interior of a solid torus

If $\alpha = 2\pi/n$, the completed cusp has the structure of a hyperbolic orbifold, with singular set the central curve of the solid torus (of branching index n); in general, this gives a hyperbolic cone-manifold, with angle α around the central curve.

In general, small deformations of the parabolic holonomy give 2 loxodromic elements f and g with the same axis A .

The developing image is again $\mathbb{R}_+^3 - A$, but in order to get a completion which is a manifold, the induced action of $\langle f, g \rangle$ on A has to be a group \mathbb{Z} of translations of A (generated by some product $f^r g^s$), and hence some product $f^{p_1} g_1^{q_1} (p_1 q_1 = 1)$ has to be a rotation around the axis A .

If this rotation has angle α , the completion is a hyperbolic cone manifold; and a hyp. mfd. if $\alpha = \pm 2\pi$ (and $f^{p_1} g_1^{q_1} = 1$, lift to the universal covering of the deformed non-complete cusp (the universal covering of $\mathbb{R}_+^3 - A \cong \mathbb{R}^2 \times S^1$) where rotations around A become translations of \mathbb{R} by the rotation-angle). In any case,

the completion is the manifold obtained by (p/q) -surgery on the original complete torus-cusp.

Surgery or Dehn-filling

Let M be a 3-manifold which has a torus T as a boundary component; Let f and g be generators of the fundamental group $\pi_1 T \cong \mathbb{Z} \times \mathbb{Z}$. Let $V = D^2 \times S^1$ be a solid torus; Identify ∂V and T by a homeomorphism which sends a meridian m of V (bounding a disk in V) to a curve representing fpg^{-1} , $(p/q) = 1$. The resulting 3-mfd. is obtained by (p/q) -surgery (or Dehn filling) on the boundary component T of M (fixed f and g ; it depends only on (p/q) or p/q).

Consider (p/q) as an element of $S^2 = \mathbb{R}^2 \cup \{\infty\} = \overline{\mathbb{R}^2}$, and call the elements of S^2 generalized Dehn filling coefficients.

Let M be a compact 3-manifold whose boundary ∂M consists of tori T_1, \dots, T_k ; suppose that the interior of M has a complete hyperbolic structure of finite volume, with k torus-cusps.

Theorem (Thurston's hyperbolic surgery theorem):
 There exists a neighbourhood U of (∞, \dots, ∞) in $S^2 \times \dots \times S^2$ (k copies) such that the complete hyp. structure of $\text{int}(M) = \overset{\circ}{M}$ has a space of hyperbolic deformations (non-complete except for (∞, \dots, ∞)) parametrized by U such that:

If, for a point $x \in M$, the generalized Dehn filling coefficient for the i 'th torus is of the form $\beta(p_i q)$, $p_i q \in \mathbb{Z}$, $(p_i q) = 1$, $\beta \in \mathbb{R}_+$, then the completion of the cusp is the hyperbolic cone-manifold obtained by $(p_i q)$ -surgery on T_i , and the cone-angle of the central curve of the added solid torus V_i is $2\pi/\beta$;

If $\beta = n \in \mathbb{Z}_+$, then it is a hyperbolic orbifold, and if $\beta = 1$ a hyperbolic manifold.

Corollary. Excluding finitely many surgery coefficients $(p_i q_i)$ for every cusp, the manifold obtained by $((p_1 q_1), \dots, (p_k q_k))$ - surgery on M is a complete hyperbolic 3-manifold $M((p_1 q_1), \dots, p_k q_k)$ where $p_i q_i \in \mathbb{Z}$, $(p_i q_i) = 1$ (some of the surgery coefficients may be ∞ which means that no surgery is performed on the cusp). If $(p_i q_i) \rightarrow \infty$ then the volumes of $M((p_1 q_1), \dots, (p_k q_k))$ converge from below to the volume of $M((p_1 q_1), \dots, \infty, \dots, (p_k q_k))$.

2 Margulis Lemma and the finite volume case

Margulis Lemma. For any dimension k there is a distance $\varepsilon > 0$ such that any discrete subgroup of $\text{Iso}(H^k)$ generated by elements that move some point x by less than ε contains a normal abelian subgroup of finite index.

Let $M = H^3/G$ be a hyperbolic 3-manifold, and $\varepsilon > 0$ as above. We consider the decomposition

$$M = M_{(0, \varepsilon]} \cup M_{(\varepsilon, \infty)}$$

where $M_{(0, \varepsilon]}$ consists of all points in M through which passes a nontrivial (in $\pi_1 M$) closed path of length $\leq \varepsilon$ (whereas for a point $x \in M_{(\varepsilon, \infty)}$ every such path has length $\geq \varepsilon$):

the Margulis or thin-thick-decomposition

Corollary There is an $\varepsilon > 0$ such that

for any oriented hyperbolic 3-mfd

$M = \mathbb{H}^3/G$, each component of $M_{(0, \varepsilon]}$ is either

- a horoball modulo \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}$

- $B_r(g)/\mathbb{Z}$ where g is a geodesic in \mathbb{H}^3 , $r \geq 0$,

where $B_r(g) := \{x \in \mathbb{H}^3 : d(x, g) \leq r\}$.

If a discrete torsionfree abelian subgroup of $\text{Iso}_+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ is \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}$.

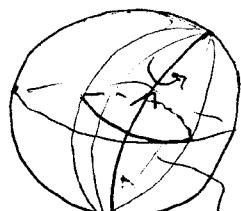
Let $x \in M_{(0, \varepsilon]}$ and $\tilde{x} \in \mathbb{H}^3$

project to x . There is some covering transformation $\gamma \in G$ which moves \tilde{x} a distance $\leq \varepsilon$. If γ is loxodromic,

let g be its axis. Then $\tilde{M}_{(0, \varepsilon]}$

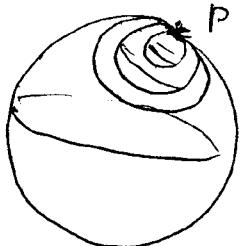
(the preimage of $M_{(0, \varepsilon)}$ in \mathbb{H}^3)

contains $B_r(g)$, for some $r \geq 0$.



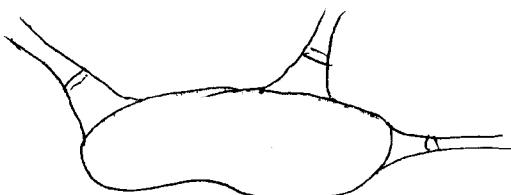
g $B_r(g)$

If γ is parabolic with fixed point $p \in \partial H^3$, $\tilde{M}_{(0,\varepsilon)}$ contains a horoball at p containing x . 54



Whenever two of these $B_r(g)$ and/or horoballs are not disjoint, they correspond to two covering transformations γ_1 and γ_2 , which move some point x a distance $\leq \varepsilon$, so γ_1 and γ_2 commute (by the Margulis Lemma) and have the same axis or parabolic fixed point; then the corresponding horoballs or solid cylinders $B_r(g)$ are concentric.

Prop. F7 hyperbolic 3-mfd $M = H^3/G$ of finite volume is the union of a compact submanifold ~~the~~ bounded by tori and a finite collection of torus-cusps (horoballs modulo $\mathbb{Z} \times \mathbb{Z}$ -actions).



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Pf. $M_{[\varepsilon, \infty)}$ is closed; if it is covered by finitely many closed ε -balls in M then $M_{[\varepsilon, \infty)}$ is compact. Otherwise, there is an infinite sequence of points in $M_{[\varepsilon, \infty)}$ with pairwise distance at least ε , and hence an infinite sequence of disjoint $\frac{\varepsilon}{2}$ -balls in M . By definition of $M_{[\varepsilon, \infty)}$, these $\varepsilon/2$ -balls are isometric to $\varepsilon/2$ -balls in H^3 , so the volume of M would be infinite.

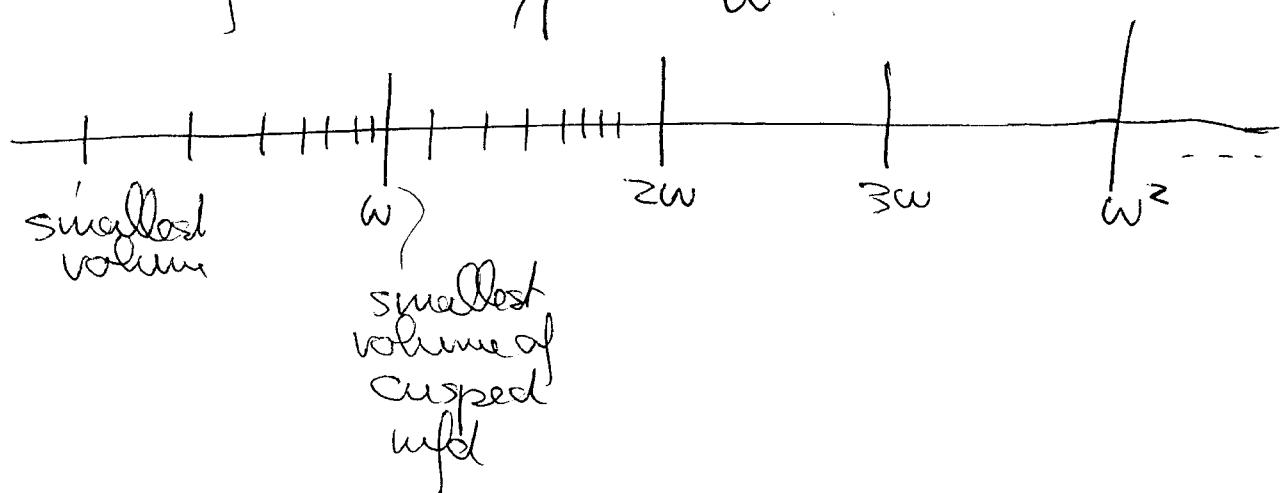
Hence $M_{[\varepsilon, \infty)}$ is compact, and has finitely many tori as boundary components. By filling in the compact components $B_r(g)/\mathbb{Z}$ of $M_{[0, \varepsilon]}$ one obtains the Proposition.

Cor. A finite volume or. hyp. 3-mfd B the interior of a compact 3-mfd bounded by (finitely many) tori.

Corollary of Margulis Lemma

For each dimension n , there is a positive lower bound for the volumes of hyperbolic n -manifolds

- Remarks
- The ^{ov} hyperbolic 3-orbifold of smallest volume is of type $(0; 2, 3, 7)$, of volume $(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}) 2\pi$ = $\frac{2\pi}{42}$; the ^{ov} hyperbolic 2-mfd of smallest volume 2π are $(0; \infty, \infty, \infty)$ (2-sphere with 3 punctures) and $(1; \infty)$ (torus with one puncture).
 - The volumes of hyperbolic 3-manifolds are ^{well ordered} of order type w^n .



3.3 Geometrization conjecture,
hyperbolic case.

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The interior of a compact or. 3-mfd M with infinite fundamental group is hyperbolic iff M is irreducible, atoroidal (every subgroup $\mathbb{Z} \times \mathbb{Z}$ of $\pi_1 M$ is peripheral, i.e. conjugate to a subgroup of a boundary torus), and not the twisted I -bundle over the Klein-bottle $T^2 \times I/\tau$

(the involution τ is a reflection on I and the covering involution of the Klein bottle on the torus T^2)

Remarks. i) If M is Haken, in particular if $\partial M \neq \emptyset$, this is Thurston's hyperbolization theorem.
 ii) the hyp. structure has finite volume iff ∂M consists of tori, and M is not $D^2 \times S^1$ (the solid torus), $T^2 \times I$ or $T^2 \times I/\tau$.

So the following case remains open:

Conjecture. Let M be a closed or irreducible 3-mfd such that $\pi_1 M$ is infinite and has no subgroups $\mathbb{Z} \times \mathbb{Z}$. Then M is hyperbolic (of finite volume)

The other open case of the general geometrization conjecture for 3-mfds is the following:

Conjecture. Let M be a closed or irreducible (prime) 3-mfd with $\pi_1 M$ finite. Then $M = S^3/G$ is spherical (the case $\pi_1 M = 1$ is the Poincaré conjecture)

General geometrization conjecture
The interior of every compact 3-mfd has a canonical decomposition along spheres and tori, into pieces which have a geometric structure (geometric 3-mfds)

3.4 Moscow rigidity

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Moscow rigidity theorem

Let $M_1 = \mathbb{H}^n/G_1$ and $M_2 = \mathbb{H}^n/G_2$ be or. hyperbolic n-mds of finite volume, $n \geq 3$. Then every isomorphism between $\pi_1 M_1$ and $\pi_1 M_2$ is induced by an isometry between M_1 and M_2 ; equivalently : every isomorphism α between G_1 and G_2 is induced by conjugation with an isometry g of \mathbb{H}^n ($\alpha(g_1) = g g_1 g^{-1}, \forall g_1 \in G_1$) (remarque for groups with torsion)

Corollary Let $M = \mathbb{H}^n/G$ be a hyperbolic n-md. of finite volume. Then the group of isometries $\text{Iso}(M)$ is finite and isomorphic to the outer automorphism group

Out $\pi_1 M = \text{Aut } \pi_1 M / \text{Inn } \pi_1 M$ of $\pi_1 M$ (automorphisms modulo inner automorphisms)

equivalently, G has finite index
in its normalizer

$$N = N_{\text{Iso}(H^n)} G = \{k \in \text{Iso}(H^n) : kGk^{-1} = G\}$$

in $\text{Iso}(H^n)$, and $N/G \cong \text{Aut } G$.

Pf. If isometries $k_1, k_2 \in N$ induce
by conjugation the same automorphism
of G then $k_1 k_2^{-1}$ reduces the identity,
i.e. commutes with every element of G .
This is possible only if $k_1 k_2^{-1} = \text{id}$,
that is $k_1 = k_2$.

Also N is discrete and properly
discontinuous (otherwise, if

$k_n \rightarrow \text{id}$, $k_n \in N$, then

$k_n g k_n^{-1} \rightarrow \text{id}$, $\forall g \in G$, so
 G would not be discrete). Then

$$\text{volume}(H^n/G) = [N : G] \text{ volume}(H^n/N)$$

so $[N : G]$ is finite

It follows that $\text{Aut } H^n/M$ can be "realized"
by isometries of M : this is a version
of the Nielsen realization problem for
free abelian groups of finite volume in dim. $n \geq 3$.

In dimension 2: (see [ZVC])

Thm. Let $M_1 = \mathbb{H}^2/G_1$ and $M_2 = \mathbb{H}^2/G_2$ be closed or. surfaces. Then every isomorphism between $\pi_1 M_1$ and $\pi_1 M_2$ is induced by a homeomorphism between M_1 and M_2 ;

more generally: every isomorphism of cocompact Fuchsian groups G_1 and G_2 is induced by conjugation with a homeomorphism of \mathbb{H}^2 .

Remark. The isometry group of a closed ^{hyp.} surface F_g of genus $g \geq 2$ is finite (proof as above) but $\text{Out}(\pi_1 F_g)$ is infinite;
 (in fact, most hyp. surfaces have trivial isometry group)

Structure of Gromov's proof of Mostow rigidity

Let $M_1 = \mathbb{H}^n/G_1$ and $M_2 = \mathbb{H}^n/G_2$ be closed hyperbolic manifolds and $\alpha: \pi_1 M_1 \rightarrow \pi_1 M_2$ an isomorphism. Then α is induced by a homotopy equivalence $f_0: M_1 \rightarrow M_2$. Let $f: \mathbb{H}^n \rightarrow \mathbb{H}^n$ be a lift of f_0 , $f \circ g_1 = \alpha(g_1) \circ f$, $\forall g_1 \in G_1$ ($\alpha: G_1 \cong \pi_1 M_1 \xrightarrow{\cong} G_2 \cong \pi_1 M_2$).

Prop. 1 $f: \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a pseudo-isometry: there are constants c_1, c_2 such that

$$c_1^{-1} d(x, y) - c_2 \leq d(f(x), f(y)) \leq c_1 d(x, y)$$

distance in \mathbb{H}^n "Lipschitz condition"

$\forall x, y \in \mathbb{H}^n$ (using compactness)

Prop 2. f extends to a continuous map $\bar{f}: \overline{\mathbb{H}^n} \rightarrow \overline{\mathbb{H}^n}$, injective on $\partial \mathbb{H}^n$ (in fact, a homeomorphism), and

$$\bar{f} \circ g_1 = \alpha(g_1) \circ \bar{f}, \quad \forall g_1 \in G_1.$$

The crucial step is now the following:

Prop 3. \bar{f} maps the vertices of a regular ideal n -simplex to the vertices of a regular ideal n -simplex.

(regular means that every permutation of the vertices is induced by an isometry of ∂H^n : maximal symmetry)

Fixing n of the $n+1$ vertices of a regular ideal n -simplex, there are exactly 2 regular ideal n -simplices having these n vertices (related by the reflection in the hyperplane generated by the n vertices):

This is true for $n > 2$ (for $n=2$, any 2 ideal 2-simplices are isometric)

Now, fixing a regular ideal n -simplex Δ , there is a hyperbolic isometry (Möbius transformation) I of ∂H^n which acts on its vertices exactly as \bar{f} . This implies that $\bar{f} \partial H^n = I | \partial H^n$, and

$$I \circ g_1 = \alpha(g_1) \circ I,$$

$$I \circ g_1 I^{-1} = \alpha(g_1), \quad \forall g_1 \in G_1 :$$

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Consider the group generated by the reflections in the faces of Δ ; the images of the vertices of Δ under this group action are dense in ∂H^4 , and \mathcal{F} and \mathcal{I} coincide on these images (by Prop. 3), so they are equal on ∂H . So Prop. 3 implies Mostow's rigidity theorem.

For the proof of Prop. 3, one uses the Gromov-invariant and the following

Prop. 4. An n -simplex in H^4 has maximal volume if and only if it is regular and ideal.

Gromov-invariant: For any singular k -chain $c = \sum a_i G_i \in S_k(X; \mathbb{R})$ in a topological space X , let

$$\|c\| := \sum |a_i| \quad (a_i \in \mathbb{R})$$

(simplicial norm of c). For a homology class $\alpha \in H_k(X; \mathbb{R})$, let

$$\|\alpha\| := \inf \left\{ \|c\| : c \text{ is a } k\text{-cycle representing } \alpha \right\}$$

(simplicial norm of α)

The Gromov-invariant of a closed connected orientable n -manifold M is the simplicial norm $\|M\|$ of a fundamental class (generator) of M in $H_n(M; \mathbb{R})$.

Prop. 5. (Gromov's theorem)

Let M be a closed orientable hyperbolic n -manifold, $n > 1$, and let v_n be the volume a regular ideal n -simplex in \mathbb{H}^n . Then $\|M\| = \text{Vol}(M)/v_n$.

Corollary. If M_1 and M_2 are homotopy-equivalent, closed orientable hyperbolic n -manifolds, $n > 1$, then $\text{Vol}(M_1) = \text{Vol}(M_2)$

(follows from: if $f: X \rightarrow Y$ is continuous and $\alpha \in H_n(X; \mathbb{R})$, then $\|f_*(\alpha)\| \leq \|\alpha\|$)

For the proof of Prop. 3, one obtains a contradiction to the Corollary assuming that the images under $\overline{f}: \overline{\mathbb{H}}^n \rightarrow \overline{\mathbb{H}}^n$ of the vertices of a regular ideal n -simplex are not the vertices of such a simplex (i.e. of a simplex of smaller volume).

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4. Seifert fiber spaces and their geometries

If closed orientable 3-mfd M is a Seifert fiber space if it has a fibration by circles S^1 which is locally trivial except around finitely many exceptional or singular fibers which have fibered neighborhoods of the following type:

take a solid cylinder $D^2 \times I$ trivially fibered by intervals I and identify top and bottom after rotating by



$$\frac{2\pi\delta}{\alpha}, \text{ for integers } \alpha \geq 1, (\alpha, \delta) = 1$$

This gives a solid torus Σ fibered by circles: in general, α fibers close up to give a circle fiber, only the central fiber $0 \times I$ closes at once and gives an exceptional fiber of multiplicity α . On the torus ∂V , let m be a meridian/bounding a disk in V , f a fiber and s a section of the fibration of ∂V ; then, $m \in H_1(\partial V) \cong \mathbb{Z} \times \mathbb{Z}$,

$$m = s^\alpha f^\beta$$

$$m = s^{\alpha} f^{\beta}$$

(because m intersects each fiber α times); changing s (by s^{α}), β changes mod α . We say that the central fiber of V is exceptional of type (α, β) . (oriented)

Let M be a closed or Seifert fiber space with exceptional fibers of types (α_i, β_i) , $i = 1, \dots, n$. Identifying each fiber to a point, we get a closed surface B (the basis of the fibration), of some genus g and with n singular points of multiplicities $\alpha_1, \dots, \alpha_n$, and a projection

$$p: M \longrightarrow B = (g; \alpha_1, \dots, \alpha_n)$$

a "2-orbifold"; we assume that B is orientable

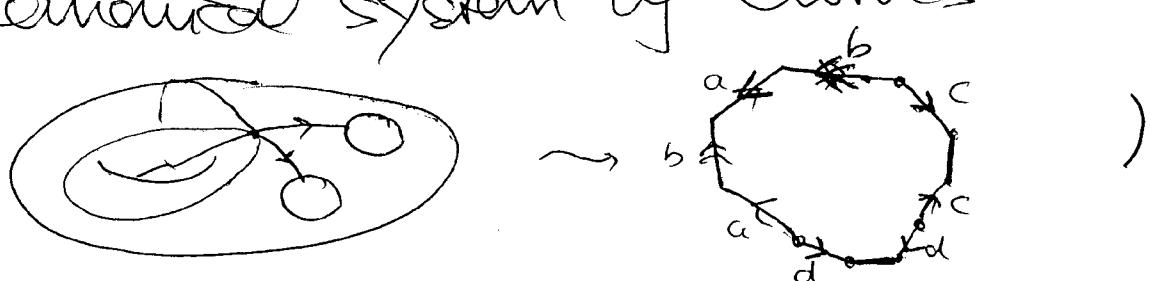
$$\chi(B) := 2 - 2g - \sum_{i=1}^n \left(1 - \frac{1}{\alpha_i}\right)$$

(orbifold Euler-characteristic),

B is a spherical, euclidean, hyperbolic or a "bad orbifold" of type $(0; \alpha)$, $(0; \alpha_1, \alpha_2)$ with $\alpha_1 \neq \alpha_2$.

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Remove the interiors of disjoint regular fibers V_i (solid tori) of the exceptional fibers (and also of one regular fiber if there are no except. fibers) to get a Seifert fiber space \bar{M} with boundary and no except. fibers and a projection $p: \bar{M} \rightarrow \bar{B}$ where \bar{B} is a surface with boundary. Then there exists a global section $s: \bar{B} \rightarrow \bar{M}$ (which may be arbitrarily prescribed over all but one boundary component of \bar{B}), and is then determined on this last boundary component: cut \bar{B} into a disk along a curved system of curves:



In particular, $\bar{M} \cong \bar{B} \times S^1$. Now the section s determines sections s_i of the fibrations of ∂V_i , and hence pairs (α_i, β_i) , with $w_i = s_i^{\alpha_i} f^{\beta_i}$.

Let $e := - \sum_{i=1}^n \beta_i / \alpha_i \in \mathbb{Q}$:
 $w_1, \dots, w_n \in \mathcal{S}_i$. Then $\dots, \alpha_i \in M$

The Euler number e is an invariant of 65
 M , up to fiber- and or.-pres. homeomorphisms

in $H_1(\bar{M})$, there is an equation $s_1 + \dots + s_n = 0$,

if a different section s' is chosen, then

$$\text{also } s'_1 + \dots + s'_n = 0, \text{ and } s'_i = s_i f_i^{u_i}$$

(written additively and multiplicatively),

now it is easy to check that e does not
 depend on the section s ($\sum_{i=1}^n u_i = 0$)

Up to fiber- and or.-pres. homeomo-
 plism, M is determined by the data

$$(g; e; (\alpha_1 \beta_1), \dots, (\alpha_n \beta_n)),$$

where the β_i can be arbitrarily
 normalized mod α_i .

Remarks. i) If M has no exceptional
 fibers then $M \cong \text{surface} \times S^1$ iff $e = 0$

ii) The Euler number behaves multipli-
 catively under finite fiber-pres. coverings

$$p: M' \rightarrow M:$$

$$e(M') = (a/b) e(M)$$

(see [Scott])

degree on
 the bases

degree of the covering on
 regular fibers

iii) Using Van Kampen's theorem,

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$$\pi_1 M = \langle a_1, b_1, \dots, a_g, b_g, s_1, \dots, s_n, f \mid \\ \prod [a_i, b_i] \prod s_i = f^{\beta}, \\ s_i^{\alpha} f^{\beta} = 1, \\ [a_i, f] = [b_i, f] = [s_i, f] = 1 \rangle$$

$(= s_0^{-1})$

(cut out also a whld. V_0 of a regular fiber, with $m_0 = s_0 f^\beta$), so there is an exact sequence

$$1 \rightarrow \langle f \rangle \rightarrow \pi_1 M \rightarrow G \rightarrow 1$$

where G is isomorphic to the orbifold fundamental group of spherical, euclidean, hyperbolic or bad 2-orbifold.

If the basis B of the fibration is a bad 2-orbifold $(O; \alpha_1, (0; \alpha_1, \alpha_2))$ ($\alpha_1 \neq \alpha_2$) or a spherical orbifold $(O; \alpha, \alpha)$ then M is a lens space

(union along the boundary of 2 solid tori V_1, V_2 , so of Heegaard genus 1): These are spherical 2-mkls

Suppose B is a good orbifold;⁶⁷
 then G has a normal torsionfree
 subgroup of finite index, and hence
 M has a finite regular covering
 by a Seifert fiber space M' without
 exceptional fibers; in particular,
 $e(M) = e(M') = 0$

Prop. M has a finite covering by
 $(\text{surface} \times S^1)$ iff $e(M) = 0$.

example Let $f: F \rightarrow F$ be a
 periodic homeomorphism of a closed
 surface F ; then the mapping torus
 $M = F \times I / (x, 1) = (f(x), 1)$ is
 Seifert fibered and finitely covered
 by $F \times S^1$.

Examples : Seifert fiber spaces (we consider only closed 3-mfds)

I) spherical base orbifold, $e = 0$
 $(X = \mathbb{S}^2)$

$$\text{e.g. } M = S^2 \times S^1$$

$X = \boxed{S^2 \times \mathbb{R}}$, with product metric

$$G = \text{Iso}(X) = O(3) \times (\mathbb{R} \rtimes \mathbb{Z}_2)$$

preserves both fibrations of X

translations

reflections

$$G_0 = SO(3) \times \mathbb{R} \subseteq \text{Iso}_+(X) \subseteq G$$

a (X, G_0) -mfld is of the form

$$(S^2 \times I)/_{(x, 0)} = (g(x), 1)$$

where g is a periodic euclidean boundary
of S^2 (a rotation of order α)

These are $S^2 \times S^1$ or of type

$(g=0; e=0; (\alpha, \beta), (\alpha, -\beta))$ ($\cong S^2 \times S^1$, but
with basis $(0, \alpha, \alpha)$ not fiber-prns.).

the spherical 2-orbifolds $(0, 2, 2, n), (0, 2, 3, 3),$
 $(0, 2, 3, 4)$ and $(0, 2, 3, 5)$ do not occur as
basis because, e.g., $\frac{\pm 1}{2} + \frac{\pm 1}{3} + \frac{m}{n} \neq 0$

other (X, G) -mfds:

$$\mathbb{RP}^2 \times S^1, \quad S^2 \times S^1 \text{ (twisted)}, \quad \underbrace{\mathbb{RP}^3 \# \mathbb{RP}^3}_{\text{orientable}} \textcircled{*}$$

II) hyperbolic base orbifold, $e = 0$
 $(\chi(B) < 0)$

e.g. $M = F_g \times S^1$

hyp. surface of genus $g > 1$

$$X = \boxed{\mathbb{H}^2 \times \mathbb{R}}, \text{ with product metric}$$

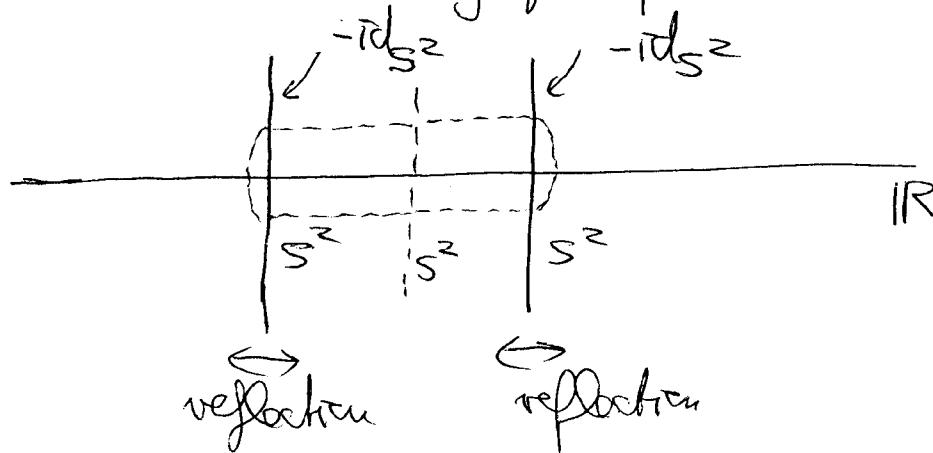
$$G = \text{Iso}(X) = \text{Iso}(\mathbb{H}^2) \times \text{Iso}(\mathbb{R})$$

$$G_0 = \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \subseteq \text{Iso}_+(X) \subseteq G$$

the closed (X, G_0) -mfds are

$$(F_g \times I)/_{(x, 0)} = (g(x), 1), \quad g: F_g \rightarrow F_g \text{ periodic isometry}$$

* universal covering group on $S^2 \times \mathbb{R}$:



fibers over
 \mathbb{RP}^2 ;
 union
 along the
 boundary of
 2 I-bundles
 over \mathbb{RP}^2

III) euclidean base orbifold, $e = 0$ 70
 $(\chi(B) = 0)$

e.g. $M = (\text{2-torus}) \times S^1 \cong \text{3-torus}$

$$X = \boxed{\mathbb{R}^3}, G = \text{Iso}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes O(3)$$

$$G_+ = \overline{\text{Iso}_+(\mathbb{R}^3)} = \mathbb{R}^3 \rtimes SO(3)$$

translations rotations

ex. one $(\text{torus} \times I) / (x, 0) = (g(x), 0)$

where g is a periodic isometry of the torus with fixed points; the possible orders of g are

- | | | |
|---|---------------------------|--|
| 2 | : basis $(0; 2, 2, 2, 2)$ | 
has another fibration over the Klein bottle |
| 3 | : basis $(0; 3, 3, 3)$ | |
| 4 | : basis $(0; 2, 4, 4)$ | |
| 6 | : basis $(0; 2, 3, 6)$ | |
| 1 | : basis is 2-torus | |

There is only one other orientable flat (=euclidean) 3-mfd ("Hantzsche-Wendt mfd") up to affine diffeomorphisms, there are 6 or. flat 3-mfds and 4 non-or. ones; the classification can be obtained using the following result (see [Scott], based on a famous result of Bieberbach)

Prop. Let Γ be a non-cyclic discrete group of isometries of \mathbb{R}^3 acting freely. Then Γ leaves invariant some family of parallel straight lines in \mathbb{R}^3 , and \mathbb{R}^3/Γ is S^2 fibered by circles which are the images of these lines.

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A crystallographic group is a discrete compact group of isometries of \mathbb{R}^n .

Thm. (Bieberbach) A crystallographic group Γ of \mathbb{R}^n has a normal subgroup \mathbb{Z}^n of finite index acting by translations. Up to conjugation with affine diffeomorphisms, there are only finitely many crystallographic groups in each dimension;

in particular: up to affine diffeomorphisms, in each dimension, there are only finitely many flat (closed) manifolds, and each is finitely by an n -torus.

all euclidean or flat 3-mfd's have
 $(\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1, \text{Iso}(\mathbb{R}^2) \times \text{Iso}(\mathbb{R}^1))$ -geometry

III) spherical and bad base orbifolds,^{7a}
 $e \neq 0$

ex. the unit tangent bundle $T_1 S^2$ of S^2
 (fibered by unit tangent vectors in
 each point of S^2):

$$T_1 S^2 \cong \underbrace{SO(3)}_{\text{acting on } S^2} \cong \mathbb{RP}^3 \cong S^3 / \{\pm \text{id}\}$$

and sharply framely
 on $T_1 S^2$ (by differentials)

so $T_1 S^2$ has spherical geometry
 $(\boxed{S^3}, O(4))$, and $e \neq 0$

(there is no vector field on S^2 which
 never vanishes, so $T_1 S^2 \not\cong S^2 \times S^1$;
 or consider fundamental groups)

the stabilizers of the action of $SO(3)$
 on S^2 are conjugate to $SO(2) \subset SO(3)$
 $(SO(2) \cong S^1)$, so $SO(3)/SO(2) \cong S^2$

Let Γ be a finite subgroup of
 $SO(3)$ (one of the polyhedral groups)

$$\begin{array}{ccc}
 SO(3) & \longrightarrow & SO(3)/SO(2) \cong S^2 \\
 \downarrow & & \downarrow \\
 \boxed{\Gamma \backslash SO(3)} & \longrightarrow & \Gamma \backslash SO(3)/SO(2) \cong \Gamma \backslash S^2
 \end{array}$$

Seifert fiber space, with basis $\Gamma \backslash S^2$
 (which is one of the spherical 2-orbifolds)

Consider $S^3 \longrightarrow SO(3) \cong RP^3$:

thus is a 2-fold covering of Lie groups,
 $S^3 = \text{unit quaternions}$

$$\begin{aligned}
 &= \{x_1 + x_2 i + x_3 j + x_4 k : x_i \in \mathbb{R}, \sum x_i^2 = 1\} \\
 &= \{z_1 + z_2 j : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}
 \end{aligned}$$

$$1 \rightarrow \mathbb{Z}_2 \xrightarrow{\quad \text{+13} \quad} S^3 \xrightarrow{\qquad \cup \qquad} SO(3) \rightarrow 1$$

$$1 \rightarrow \mathbb{Z}_2 \xrightarrow{\quad \text{+13} \quad} \Gamma^* \xrightarrow{\qquad \cup \qquad} \Gamma \rightarrow 1$$

a binary polyhedral group

2) S^3 act on itself by left multiplication,
 hence by ~~isometries~~; then

$$\Gamma^* \backslash S^3 \cong \Gamma \backslash SO(3)$$

is a spherical Seifert fiber space,

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with basis S^2 , $(0, \alpha, \alpha)$, $(0, z, z, \alpha)$,
 $(0, z, 3, 3)$, $(0, 2, 3, 4)$ or $(0, 2, 3, 5)$

other examples: the lens spaces are spherical and exactly the Seifert fiber spaces which fiber over the bad orbifolds $(0, \alpha)$, and the orbifolds $(0; \alpha_1, \alpha_2)$ or S^2

Hopf fibration:

$$S^3 \longrightarrow S^2 \cong \overline{\mathbb{C}}$$

$$q = \frac{z_1 + z_2 \bar{i}}{\sqrt{2}} \longrightarrow z_1/z_2$$

$$S^3 \longrightarrow SO(3) \quad \begin{matrix} \text{2-fold} \\ \text{covering of Lie} \\ \text{groups:} \end{matrix}$$

consider $S^3 \longrightarrow SO(4)$ (acting on S^3)

$$q \longmapsto (x \in S^3 \rightarrow q x q^{-1} \in S^3)$$

the kernel is the center $\{\pm 1\}$ of S^3 ,
 the image fixes $1 \in S^3$, acts on
 $\mathbb{R}^3 = \{x_1 i + x_2 j + x_3 k\}$ and is isomorphic
 to $SO(3)$

$$S^3 \times S^3 \longrightarrow SO(4)$$

$$(q_1, q_2) \longrightarrow / (x \in S^3 \rightarrow q_1 x q_2^{-1} \in S^3)$$

The kernel is $\{(1,1), (-1,-1)\}$,
the image is $SO(4)$, hence

$$SO(4) \cong (S^3 \times S^3)/\mathbb{Z}_2 \cong \underbrace{S^3 \times S^3}_{\mathbb{Z}_2}$$

central product:
direct product with identified centers

$$\begin{array}{ccc} S^3 \times S^3 & \longrightarrow & SO(4) \\ \downarrow / \{\pm 1\} \times \{\pm 1\} & & \downarrow / \{\pm \text{id}_{S^3}\} \\ & \longrightarrow & SO(3) \times SO(3) \end{array}$$

This is used to classify the finite subgroups of $SO(4)$, and the finite subgroups acting freely on S^3 : these give all Seifert fiber spaces with bad and spherical basis and $e \neq 0$

IV) hyperbolic base orbifold, $e \neq 0$ 76

ex. the unit tangent bundle $T_1 F_g$
 of a hyp. surface $F_g = H^2/\Gamma$, $g > 1$,
 $\Gamma \subset PSL(2, \mathbb{R})$

$e \neq 0$ because F_g has no vector field
 which never vanishes,

there is the covering

$$T_1 H^2 \xrightarrow{\pi} T_1 F_g$$

$PSL(2, \mathbb{R})$ acts on H^2 with stabilizers
 $SO(2) \cong S^1$, so $PSL(2, \mathbb{R})/SO(2) \cong H^2$;
 it acts on $T_1 H^2$ simply transitively, so
 $T_1 H^2 \cong PSL(2, \mathbb{R})$

$$\downarrow \quad \downarrow \\ H^2 \cong PSL(2, \mathbb{R})/SO(2)$$

$T_1 H^2 \subset T_1 H^2$ has a Riemannian metric
 such that $PSL(2, \mathbb{R})$ acts by isometries;
 alternatively, consider a left-invariant
 metric on $PSL(2, \mathbb{R})$ (acting on
 itself from the left)

Let Γ be a Fuchsian group

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$$\text{PSL}(2, \mathbb{R}) \longrightarrow \text{PSL}(2, \mathbb{R})/\text{SO}(2) \cong \mathbb{H}^2$$

$$\underbrace{\Gamma \backslash \text{PSL}(2, \mathbb{R})}_{\text{Seifert fiber}} \longrightarrow \Gamma \backslash \text{PSL}(2, \mathbb{R})/\text{SO}(2) \cong \Gamma \backslash \mathbb{H}^2$$

Seifert fiber space with basis $\Gamma \backslash \mathbb{H}^2$ which is a hyperbolic \mathbb{Z} -orbifold, and $e \neq 0$.

Topologically, $\Gamma \backslash \mathbb{H}^2 \cong \mathbb{H}^2 \times S^1$, so

$$\pi_1(\text{PSL}(2, \mathbb{R})) \cong \pi_1(\Gamma \backslash \mathbb{H}^2) \cong \mathbb{Z}, \text{ let}$$

$$\widetilde{\text{SL}}(2, \mathbb{R}) \longrightarrow \text{SL}(2, \mathbb{R}) \xrightarrow{\text{z-fold}} \text{PSL}(2, \mathbb{R})$$

Be the universal covering of $\text{PSL}(2, \mathbb{R})$ which is a simply connected Lie group,

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{SL}}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R}) \rightarrow 1$$

$$\quad \downarrow \qquad \qquad \qquad \downarrow$$

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma^* \longrightarrow \Gamma \rightarrow 1,$$

$$\text{and } \Gamma^* \backslash \widetilde{\text{SL}}(2, \mathbb{R}) \cong \Gamma \backslash \text{PSL}(2, \mathbb{R}),$$

$\widetilde{\text{SL}}(2, \mathbb{R})$ with a left invariant metric,

so the geometry is

$$(\widetilde{SL}(2, \mathbb{R}), G = \text{Iso}(\widetilde{\Sigma})),$$

Note that $\widetilde{SL}(2, \mathbb{R})$ is a line bundle over H^2 , but with twisted geometry

for the component of the identity
of G , there is an exact sequence

$$1 \rightarrow \mathbb{R} \longrightarrow G_0 \longrightarrow PSL(2, \mathbb{R}) \rightarrow 1$$

lift of the S^1 -action on T^*H^2
which rotates all fibers the same
angle

all Seifert fiber spaces over hyperbolic
2-orbifolds and with $e \neq 0$ belong
to this geometry

III) euclidean base orbifold, $e \neq 0$ 79

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus;
consider the torus-bundle over S^1

$$M = (T \times I) / (x, 0) = (g(x), 1)$$

where $g : T \rightarrow T$ is induced by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \text{ acting on } \mathbb{R}^2$$

(a torus bundle with "parabolic monodromy")

$$T_1 M = \langle f, a, g \mid [f, a] = 1, gfg^{-1} = f^n, gag^{-1} = f^na \rangle$$

$$= \langle g, a, f \mid gag^{-1}a^{-1} = f^n, [a, f] = [g, f] = 1 \rangle$$

M is also a Seifert fiber space over the torus, with $e = n$.

$$\text{Let } \mathrm{Nil} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

(the Heisenberg group)

$$1 \rightarrow \mathbb{R} \rightarrow \mathrm{Nil} \rightarrow \mathbb{R}^2 \rightarrow 0$$

matrices with
 $x = y = 0$: the center of Nil

Nil is a nilpotent Lie group which
is a line bundle over \mathbb{R}^2 ($\cong \mathbb{R}^3$) 8c

the commutator of

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence $\pi_1 M$ admits an injection
into Nil, and

$$\Gamma \backslash \text{Nil} \cong M$$

(image of $\pi_1 M$)

take a left-invariant metric on the
Lie-group Nil to get the geometry

$$(\text{Nil}, G = \text{Iso}(\text{Nil}))$$

all Seifert fiber spaces over a euclidean
basis and with $e \neq 0$ are of this
geometry, and all torus-bundles
over S^1 with parabolic monodromy;

geometrically, Nil is a twisted
line-bundle over \mathbb{R}^2

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the torus-bundles over S^1 with
 "elliptic monodromy" are euclidean mfd,
 those reward the torus-bundles over S^1
 with "hyperbolic monodromy" (2 real
 eigenvalues different from ± 1):
 These are not Seifert fibered and
 belong to the Sol - geometry
 (a solvable 3-dim. Lie group)

$$1 \rightarrow \mathbb{R}^2 \rightarrow \text{Sol} \rightarrow \mathbb{R} \rightarrow 0$$

where $t \in \mathbb{R}$ acts on \mathbb{R}^2 by:

$$t(x, y) t^{-1} = (e^t x, e^{-t} y)$$

$$= \underbrace{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}}_{\text{conjugate}}(x, y)$$

of hyp. matrix in $SL(2, \mathbb{Z})$

scheme of geometries for Seifert
 fiber spaces:

	$X > 0$	$X = 0$	$X < 0$
$e = 0$	$S^2 \times \mathbb{R}$	\mathbb{R}^3	$H^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{SL}(2, \mathbb{R})$

Geometric 3-manifolds

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Theorem. Let M be a closed orientable geometric 3-manifold. Then the universal covering X of M is one of the following 8 simply connected 3-dim. geometries (so M is a quotient of X by a group of isometries of the respective geometry):

euclidean geometry $\boxed{\mathbb{R}^3}$, spherical geometry $\boxed{\mathbb{S}^3}$,
hyperbolic geometry $\boxed{\mathbb{H}^3}$
(with or.-pres. point stabilizers $SO(3)$) .

the product geometries $\boxed{\mathbb{S}^2 \times \mathbb{R}}$ and $\boxed{\mathbb{H}^2 \times \mathbb{R}}$
(with point stabilizers $SO(2)$);

the following three Lie-groups with a left-invariant metric:

$\boxed{\widetilde{SL}(2, \mathbb{R})}$, $\boxed{\text{Nil}}$ (point stabilizers $SO(2)$)
 $\boxed{\text{Sol}}$ (trivial point stabilizers).

Scheme of proof.

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Let $(X, G := \text{Iso}(X))$ be as in the theorem;

for $x \in X$, let G_x be the component of the identity in the stabilizer of x in G .

G_x operates as a group of isometries of the tangent space $T_x X$, so G_x is a connected Lie-subgroup of $\text{SO}(3)$.

There are 3 cases:

- i) $G_x = \text{SO}(3)$; then X has constant (sectional) curvature and is equal to \mathbb{R}^3 , S^3 or H^3
- ii) $G_x = \text{SO}(2) \cong S^1$ (rotations around an axis in $T_x X$). We have one distinguished direction of rotation in each tangent space $T_x X$, and then also an orthogonal plane field E . There are 2 subcases:
 - a) E is integrable, i.e. there are 2-dim. submanifolds with tangent space E . This leads to the product geometries $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$.

- b) E is non-integrable; this leads to the twisted geometries $\widetilde{SL}(2, \mathbb{R})$ and Nil (twisted line bundles over H^2 or \mathbb{R}^2)
- iii) $G_x = \text{trivial group}$. Then we are in the case $X = \text{Sol}$.
-

Theorem. Let M be a closed 3-mfd.

- i) M has a geometric structure modelled on Sol iff M is finitely covered by a torus-bundle over S^1 with hyperbolic monodromy; in particular M itself is either a bundle over S^1 with fiber the torus or the Klein bottle, or the union of 2 twisted \mathbb{I} -bundles over the torus or the Klein-Bottle.
- ii) M has a geometric structure modelled on one of $S^3, \mathbb{R}^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \widetilde{SL}(2, \mathbb{R})$ or Nil iff M is a Seifert fiber space, and the geometry of M is determined by X and e according to the above scheme.

Lectures on geometric 3-manifolds

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