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Introduction to Ricci flow

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These are informal notes which are meant to be a support for the lectures of the "Summer school on geometry and topology of 3-manifolds", to be held at ICTP, Trieste, 6-10 June 2005. We have attempted here to summarize the results from a vast area of research; this might have led to somce inaccuracies in the exposition. The interested reader is encouraged to look at the references in the bibliography for a comprehensive treatment of the subject.

1 The Ricci flow

Let $\mathcal{M}^n$ be an n-dimensional riemannian manifold with a metric g_0. The *Ricci flow* of $(\mathcal{M}^n, g_0)$ is a time-dependent family $g(t)$ (with $t \geq 0$) of metrics on $\mathcal{M}^n$ satisfying $g(0) = g_0$ and evolving according to the equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)} \tag{1.1}$$

where $Ric_{g(t)}$ is the Ricci curvature tensor associated with the metric $g(t)$. As we will see later, the Ricci flow is a parabolic system of partial differential equations which has a solution at least in some finite time interval $t \in [0, T[$. The choice of sign in the right-hand side is essential for this property, since otherwise in general the flow would not have a solution for positive times.

The Ricci flow was introduced by R. Hamilton in [6]. To explain the interest of the flow, let us recall the main result of that paper.

Theorem 1.1 *Any closed three-dimensional riemannian manifold with positive Ricci curvature is diffeomorphic to a quotient of the sphere S^3 under a finite group of isometries.*

To prove this result, Hamilton considered the evolution of the metric under the Ricci flow and showed that it converges to a metric of constant positive sectional curvature. More precisely, there is a finite time $T > 0$ at which the flow becomes singular and the manifold "shrinks to a point" (that is, the metric tends to zero and the curvature becomes unbounded everywhere); however, by choosing an appropriate rescaling factor $\rho(t)$, the normalized metric $\rho(t)g(t)$ converges, as $t \to T$, to a metric of positive constant sectional curvature. On the other hand, it is known that a manifold with such a metric must be S^3 or one of its quotients.

A remarkable feature of Hamilton's proof was the use of techniques from the theory of parabolic PDEs, like the maximum principle, to solve a geometric problem. This approach was partly inspired by a previous paper by Eels and Sampson [4] where they used the heat equation to find harmonic maps between two given riemannian manifolds.

In the following years, many other results were obtained by a similar technique. On one hand, there was a detailed study of the Ricci flow in dimension 2, showing that every metric on a closed surface converges under Ricci flow to a metric of constant curvature, thus providing an alternative approach to the uniformization theorem. On the other hand, convergence results to quotients of the sphere in dimension greater than three were obtained under suitable conditions on the initial metric (see [1, 9] and the references therein).

After some time, Hamilton started pursuing the more ambitious goal of proving the *Thurston geometrization conjecture* using Ricci flow. As it is well known, this conjecture provides a complete classification of the closed three-dimensional manifolds, and includes in particular the

Poincaré conjecture: Every closed simply connected three-dimensional manifold is homeomorphic to the sphere S^3.

Hamilton's program consisted in showing that the Ricci flow on a general closed three-manifold converges to one of the canonical structures described by Thurston. However, a more subtle technique is needed with respect to the results described before. Namely, the Ricci flow on a general three manifold may develop singularities before the metric has converged to one of the desired limits. Hamilton's idea in this case is to define a flow with surgeries: the Ricci flow is stopped shortly before the singular time, the regions with large curvature are removed by a surgery and replaced by more regular ones, and the flow is restarted. To define this procedure rigorously it is necessary to have a detailed knowledge of the possible structure of singularities.

In spite of many relevant results, some crucial parts of Hamilton's program remained unsolved until recently. Between 2002 and 2003 G. Perelman posted on the web three papers which introduce several new ideas for the analysis of Ricci flow and give a proof of Thurston conjecture [13, 14, 15]. The details of the proof are still being checked by the experts in the field; however, the main ideas of the papers are by now widely understood.

In these lectures we will present the basic properties of Ricci flow and the main results about the analysis of singularities which are used in the proof of the geometrization conjecture. The Ricci flow with surgeries and the proof of the conjecture will be the object of the lectures by Bessieres and Besson next week.

2 Examples

Some explicit solutions of the Ricci flow are presented in [9, §2] and in [2, §2]. Easy examples are given by the spaces with constant curvature, which evolve by homotheties.

A sphere shrinks to a point in finite time, while a hyperbolic space of constant negative curvature expands more and more as $t \to \infty$, while the curvature tends to zero.

Other interesting examples are the so–called *solitons*. A steady Ricci soliton is a manifold $\mathcal{M}^n$ (not necessarily closed) with a metric $\tilde{g}$ which is a constant solution of the Ricci flow up to a diffeomorphism. By this we mean that there exists a family of diffeomorphism ϕ_t of $\mathcal{M}^n$ such that, if we set $g(t) = \phi_t^*(\tilde{g})$ (the pull-back metric under the diffeomorphism) then $g(t)$ solves the Ricci flow. If the diffeomorphisms are generated by the gradient of a function f, then the solution is called a gradient soliton; this is equivalent to requiring that the metric satisfies $\mathrm{Ric}_{\tilde{g}} + D^2 f = 0$. In general, a metric $\tilde{g}$ satisfying $\mathrm{Ric}_{\tilde{g}} + D^2 f = \rho\tilde{g}$ for some constant ρ is called a expanding (resp. shrinking) gradient Ricci soliton if $\rho < 0$ (resp. if $\rho > 0$). In this case, the Ricci flow is given by composing diffeomorphism and homotheties.

An explicit example of steady gradient soliton in dimension 2 is the so-called *cigar*, which is $\mathbb{R}^2$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

(see [2, §2.2] for the details).

Another case where the Ricci flow has a special structure is the one of a homogeneous metric (like Thurston's three-dimensional model geometries). Here the Ricci flow reduces to a system of ordinary differential equations (see [2, §1.4-1.7] and the references therein).

Particularly important for our purposes are the examples of "intuitive solutions" in [9, §3]. Consider the case of a manifold (of dimension at least three) consisting approximately of two big spheres S^n joined by a thin tube (or "neck") $S^{n-1} \times [a, b]$. One expects that the tube shrinks at least in some part before the two spheres do the same. Such a behaviour is called *neckpinch*. A solution with these properties has been constructed rigorously in a recent paper (see [2, §2.5]). Hamilton's conjecture (confirmed by Perelman's results) was that, intuitively speaking, the neckpinch is the only possible singular behaviour in the three-dimensional Ricci flow, in addition to the shrinking behaviour described after Theorem 1.1. When a neckpinch singularity forms, it is possible to remove the singular part by a surgery procedure such that the possible change of topology of the manifold is controlled. In this notes we will not deal with the surgery construction, but we will describe instead parts of the argument which gives the classification of the possible singular behaviours.

3 Short time existence

When written in coordinates, the Ricci flow is a parabolic system of partial differential equations for the components of the metric. There exists a standard theory giving short time existence of solutions for systems which are strictly parabolic; however, the Ricci flow does not completely fit into this framework since for this system the parabolicity is not strict. Nevertheless, using the special structure of the equations, Hamilton was able to prove short time existence for the Ricci flow, as shown by the next result [6].

Theorem 3.1 *The Ricci flow on any closed manifold has a unique solution in a time interval* $[0, t_0)$ *for some* $t_0 > 0$*.*

Shortly afterwards, De Turck gave a simpler proof of this result, based on an equivalent formulation of the flow where the parabolicity becomes strict. For more details about these matters, one can consult [6, §4-6], [9, §6], [2, §3.1-3.4]. Let us point out that, while the flow admits a solution for positive times, it is in general not be solvable for negative times (this is a typical feature of parabolic problems).

As we have already seen, the Ricci flow in some cases exists only in a finite time interval. The next theorem (proved in [9, §8]) shows that, if this is the case, then the curvature necessarily becomes unbounded.

Theorem 3.2 *Each solution of the Ricci flow on a compact manifold can be extended to a maximal time interval* $[0, T)$*, with* $T \leq +\infty$*. If* T *is finite, then necessarily*

$$\limsup_{t \to T} M(t) = +\infty,$$

where $M(t)$ *is the maximum of the norm of the Riemann curvature tensor at time* t*.*

We describe the above behaviour by saying that the flow *becomes singular* at time T.

4 Evolution of curvature, preservation of positivity

As the metric on a manifold evolves by Ricci flow, the Riemann curvature tensor also evolves and satisfies an equation which can be computed explicitly and has the form

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + Q(\mathrm{Rm}). \tag{4.2}$$

Here $\Delta = \Delta_{g(t)}$ is the Laplace operator associated to the evolving metric $g(t)$, while $Q(\mathrm{Rm})$ is a tensor which is a quadratic function of Rm and whose expression can be found in [9, §4]. Many important results about Ricci flow can be obtained starting from this equation and using the maximum principle. In addition to the usual maximum principle for scalar functions evolving by parabolic equations (see e.g. [5, §7.1.4]), we can apply in this case also the following maximum principle for tensors by Hamilton [7, Theorem 4.3].

Theorem 4.1 *Let* $(\mathcal{M}^n, g(t))$ *be a riemannian manifold evolving by Ricci flow and let* F *be a time dependent tensor on* $\mathcal{M}^n$ *which evolves by the system*

$$\frac{\partial F}{\partial t} = \Delta F + \Phi(F) \tag{4.3}$$

for some function Φ *from the tensor bundle into itself. Let* Z *be a closed subset of the tensor bundle which is invariant under parallel translation and such that its intersection with each fiber is convex. If* Z *is invariant in each fiber under the ordinary differential system* $dZ/dt = \Phi(Z)$*, then* Z *is also invariant for system* (4.3)*.*

Using the maximum principle one can obtain the following invariance results for the positivity of curvature under Ricci flow (see [2, §6], [9, §5]). Here and in the following R denotes the scalar curvature. In addition we call "curvature operator" the Riemann curvature tensor interpreted as a symmetric bilinear operator on the space of two-forms (see [9, §4] or [2, §6.3] for details).

Theorem 4.2 *Let $g(t)$ be a solution to the Ricci flow on a closed manifold $\mathcal{M}^n$.*

(i) *The minimum of R is nondecreasing under the flow. In particular, positive scalar curvature is preserved under the flow (that is, if the initial metric $g(0)$ has positive scalar curvature, then so does $g(t)$ for all $t > 0$).*

(ii) *If $n = 3$ then positive Ricci curvature and positive sectional curvature are preserved under the flow. (NB this property may fail in dimension greater than 3).*

(iii) *Positive curvature operator is preserved under the flow.*

Using the maximum principle one can prove further estimates which become interesting near the singular time. An important example is given by the next result.

Theorem 4.3 *Let $g(t)$ be a solution of the Ricci flow on a closed three-manifold $\mathcal{M}^3$ and let R_0 be the minimum of the scalar curvature at time 0. Then there exists a function $\phi : [R_0, +\infty) \to (0, \infty)$ such that $\phi(r)/r \to 0$ as $r \to +\infty$ and such that any sectional curvature K of the solution at any time satisfies*

$$K \geq -\phi(R). \tag{4.4}$$

The above result is often called the *Hamilton-Ivey pinching estimate* and is proved in [9, Th. 24.4]. Intuitively speaking, the theorem says that when the scalar curvature becomes large (that is, when the singular time is approached) the negative sectional curvatures, if there are any, become negligible compared to the other ones. Thus, even if the sign of the curvature at the initial time is completely arbitrary, the asymptotic profile near the singularity necessarily has nonnegative curvature. This property will be stated in a more precise way when we will introduce the rescaling of a solution near a singularity.

5 Differential Harnack inequality

The classical Harnack inequality for elliptic equations is an estimate controlling the oscillation of positive solutions. Similar estimates hold for parabolic equations. In [12] P. Li and S.-T. Yau introduced an alternative approach to Harnack inequalities, showing that in certain cases they can be obtained from suitable estimates involving derivatives, which are called *differential Harnack inequalities.* Hamilton subsequently developed extensively this approach for varius geometric evolution equations. In particular, for the Ricci flow he obtained the following result [8].

Theorem 5.1 *Let $(\mathcal{M}^n, g(t))$ be a solution to the Ricci flow, defined for $t \in [0,T)$, which is either closed or complete with bounded curvature, and has nonnegative curvature operator. Then, for any vector V and any time $t \in [0,T)$ we have*

$$\frac{\partial R}{\partial t} + \frac{1}{t}R + 2\langle \nabla R, V\rangle + 2\mathrm{Ric}(V,V) \geq 0. \tag{5.5}$$

The above result is called "trace differential Harnack inequality" because it is obtained taking the trace of a more general tensor inequality. Integrating along a suitable path in space time one obtains the following result.

Corollary 5.2 *Under the same hypotheses, given any $P_1, P_2 \in \mathcal{M}^n$ and $0 < t_1 < t_2$, we have*

$$R(P_2,t_2) \geq \frac{t_1}{t_2}\, R(P_1,t_1)\, e^{-\frac{d^2}{2(t_2-t_1)}}$$

where d is the distance between P_1 and P_2 at time t_1.

The above estimate shows that the scalar curvature at the later time t_2 controls the curvature at time t_1 and is similar to the classical Harnack estimate satisfied by the solutions of linear parabolic equations. (see e.g. [5, §7.1.4b]).

6 Analysis of singularities

To study the behaviour of solutions of the Ricci flow when the curvature is unbounded one can use rescaling procedures which are common also for other kinds of PDEs. We will describe the technique in an informal way because the rigorous statements are rather technical (see [9, §16]).

Let us first observe that the Ricci flow is invariant under parabolic rescalings, that is, if we dilate a solution by a factor $\lambda > 0$ in space and λ^2 in time, we obtain another solution of the flow, which will have the norm of the curvature $|\mathrm{Rm}|$ reduced by a factor λ^2. Suppose now that we have a solution $(\mathcal{M}^n, g(t))$ of the Ricci flow which becomes singular as $t \to T$. We can consider a sequence of rescalings with larger and larger factors near the singular time and then take a limit which describes, intutively speaking, the singular profile of the original solution. More precisely, let us take a sequence of points $P_j \in \mathcal{M}^n$ and times t_j such that $t_j \uparrow T$ and in addition

$$|\mathrm{Rm}(P,t)| \leq C|\mathrm{Rm}(P_j,t_j)| \qquad \forall P \in \mathcal{M}^n,\ t \in [0,t_j]$$

for some constant $C \geq 1$ independent of j. For any $j \geq 1$ we now rescale our flow by a factor λ_j, where $\lambda_j = \sqrt{|\mathrm{Rm}(P_j,t_j)|}$. In addition, we take P_j to be the origin of the rescaled flow and we translate the time so that t_j becomes zero. Then the j-th flow is defined for $t \in [-\lambda_j^2 t_j, (T-t_j)\lambda_j^2]$. Observe that the initial endpoint of the time interval tends to $-\infty$ at $j \to \infty$; the final endpoint is positive, and it can be proved that it stays bounded away from zero for all j. By construction, each rescaled flow satisfies $|\mathrm{Rm}| \leq C$ everywhere at all times $t \leq 0$. It is possible to show that this curvature bound ensures the existence of a converging subsequence, provided the rescaled flows also satisfy an injectivity radius bound.

Theorem 6.1 *Let $(\mathcal{M}^n, g(t))$ a solution of the Ricci flow which becomes singular as $t \to T$, and let us consider a family of rescaled flows defined as above. Suppose in addition that the injectivity radius of our manifold satisfies the estimate*

$$\mathrm{inj}(P,t) \geq \frac{c}{\sqrt{\max_{\mathcal{M}^n} |\mathrm{Rm}|(\cdot,t)}}, \qquad \forall P \in \mathcal{M}^n,\ t \in [0,T)$$

for some $c > 0$. Then a subsequence of the rescaled flows converges uniformly on compact sets to a limit $(\hat{\mathcal{M}}^n, \hat{g}(t))$, which is a solution to the Ricci flow and is defined in an interval of the form $(-\infty, T^*)$, with $T^* > 0$ (possibly infinite). If $n = 3$ then the limit flow has nonnegative sectional curvature at every point and satisfies the improved differential Harnack estimate

$$\frac{\partial R}{\partial t} + 2\langle DR, V\rangle + 2\mathrm{Ric}(V,V) \geq 0. \tag{6.6}$$

For the proof of the first part of this statement, see [9, §16]. The assertion concerning the sectional curvature can be obtained from Theorem 4.3; in fact, the right-hand side of (4.4) disapperars in the rescaling procedure due to the sublinearity of ϕ. Observe also that in three dimensions positive sectional curvature is equivalent to positive curvature operator. Thus the limit flow satisfies the Harnack inequality (5.5) where the R/t term can be replaced by $R/(t - t_0)$ with t_0 arbitrarily small since the solution is defined in $(-\infty, T^*)$. Thus, letting $t_0 \to -\infty$, this term vanishes and we obtain the improved inequality (6.6).

In [9, §16] one can also find a precise definition of the "convergence on compact sets" mentioned in the statement. In particular, even if the rescaled flows are all compact, their diameter can go to $+\infty$, and thus the limit flow can be noncompact. The typical example is the neckpinch of Section 2, where the limit flow is an infinite cylinder $S^2 \times \mathbb{R}$.

A solution defined in $(-\infty, T^*)$ is called an *ancient solution*; if $T^* = +\infty$, it is called an *eternal solution*. Such solutions of the flow are special because, as we have mentioned before, the Ricci flow in general is not solvable backwards in time.

Hamilton then proved the following classification results of the possible structure of the limit flow in dimension 3.

Theorem 6.2 *Let $g(t)$ be a solution of the Ricci flow on a closed three-manifold $\mathcal{M}^3$. Suppose that the flow becomes singular as $t \to T$ and that we have an injectivity radius estimate of the form*

$$\mathrm{inj}(P,t) \geq \frac{c}{\sqrt{\max_{\mathcal{M}^3} |\mathrm{Rm}|(\cdot,t)}}, \qquad \forall P \in \mathcal{M}^3,\ t \in [0,T)$$

for some $c > 0$. Then it is possible to choose the sequence (P_j, t_j) in the above construction in such a way that the limit flow is one of the following (or a quotient under a finite group of isometries)

(i) *the shrinking sphere S^3, or*

(ii) *the shrinking cylinder $S^2 \times \mathbb{R}$, or*

(iii) $\Sigma \times \mathbb{R}$, *where* Σ *is the "cigar" soliton introduced in Section 2.*

The above theorem is stated at the end of the paper [9] and the proof uses all the properties of the limit flow which we have mentioned before. Although such a result already gave strong restrictions on the possible structure of the singularities, there remained two unsatisfactory aspects. One was the lack of a general argument which could provide the injectivity radius estimate needed in the theorem. The second problem regarded case (iii): if such a limit really occurs then it represents a fatal obstruction to Hamilton's program, because there is no clear way to do surgery on a singularity which exhibits such a profile. Actually, Hamilton conjectured that case (iii) cannot occur, but did not succeed in proving this. We will see in the next sections how Perelman's new ideas have solved both of these difficulties.

7 Perelman's monotonicity formula

In [13, §3] Perelman introduced the following functional. Let $\mathcal{M}^n$ be a closed manifold. Given a metric g on $\mathcal{M}^n$, a function $f : \mathcal{M}^n \to \mathbb{R}$ and a positive number τ, consider

$$\mathcal{W}(g, f, \tau) = \int_{\mathcal{M}} [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f} dV_g.$$

Define also, for fixed g and τ,

$$\mu(g, \tau) = \inf \left\{ \mathcal{W}(g, f, \tau) \ : \ f \text{ such that } \int_{\mathcal{M}} (4\pi\tau)^{-n/2} e^{-f} dV_g = 1 \right\}.$$

Then the following monotonicity result holds.

Theorem 7.1 *If $g(t)$ is a solution of the Ricci flow for $t \in [t_0, t_1]$ on a closed manifold $\mathcal{M}^n$, and if $\tau(t) = \bar{t} - t$ for some $\bar{t} > t_1$ then the quantity $\mu(g(t), \tau(t))$ is nondecreasing in t for $t \in [t_0, t_1]$.*

The above result has an important application to the analysis of singularities. Let us introduce the notion of local collapsing.

Definition 7.2 *Let $(\mathcal{M}^n, g(t))$ be a solution of the Ricci flow for $t \in [0, T)$, with T finite. We say that the solution is* locally collapsing *at time T if there exists a sequence of times $t_k \uparrow T$, of points $P_k \in \mathcal{M}$ and of radii $r_k > 0$ such that $\{r_k\}$ is bounded and such that, if we denote by B_k the ball of center P_k and radius r_k with respect to the metric $g(t_k)$, we have that $|\mathrm{Rm}|(P, t_k) \leq r_k^{-2}$ for all $P \in B_k$ and that $\mathrm{Vol}\,(B_k)/r_k^n \to 0$ as $k \to \infty$.*

It turns out that the monotonicity of $\mathcal{W}$ prevents the collapsing behaviour described above. In fact, as a consequence of Theorem 7.1, Perelman obtains the following crucial result [13, §4].

Theorem 7.3 *If $g(t)$ is a solution of the Ricci flow for $t \in [0, T)$ on a closed manifold $\mathcal{M}^n$, then $(\mathcal{M}^n, g(t))$ is not locally collapsing at time T.*

To prove this result, Perelman shows that if the flow is collapsing at time T, then $\mu(g(t_k), r_k^2) \to -\infty$, by plugging suitable functions f in the functional $\mathcal{W}$. On the other hand, by Theorem 7.1, $\mu(g(t_k), r_k^2) \geq \mu(g(0), t_k + r_k^2)$, which cannot be arbitrarily small, and this gives a contradiction.

It is shown that the collapsing behaviour is related to the smallness of the injectivity radius at the points (P_k, t_k). In particular, if the solution is not locally collapsing, then it also satisfies the injectivity radius estimate required in Theorem 6.2. Thus, Perelman's result ensures that the injectivity radius estimate is always satisfied.

Theorem 7.3 also allows to exclude that the cigar $\Sigma \times \mathbb{R}$ is obtained as limit of rescaled flows. In fact, one can check that the metric on the cigar Σ is locally collapsing. Since the collapsing property is invariant under rescaling, $\Sigma \times \mathbb{R}$ cannot occur as the limit of the rescalings of a noncollapsed solution.

Thus, Perelman's monotonicity formula is a powerful tool for the analysis of singularities of the Ricci flow. More detailed results about the singularities of the flow in dimension three are obtained in [13, §11,12]. Once the singularities are analyzed, the following steps of Hamilton's program for the proof of the geometrization conjecture are the surgery construction and the analysis of the nonsingular solutions. These steps have been done in [14], using also the results from [10, 11]. A shorter argument to obtain the Poincaré conjecture without proving the full Thurston conjecture is given in [15] (see also [3]).

Notes on the bibliography:

Most of the relevant papers on Ricci flow which have appeared before 2002 have been collected in the volume "Collected papers on Ricci flow" (edited by H.D. Cao, B. Chow, S.C. Chu, S.T. Yau), International Press, 2003.

Some notes and commentary about Perelman's papers can be found on the web page of B. Kleiner at "www.math.lsa.umich.edu/research/ricciflow/perelman.html". Particularly interesting are the notes by B. Kleiner and J. Lott that can be found there.

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