The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1662/1

Summer School and Conference on Geometry and Topology of 3-manifolds

(6 -24 June 2005)

Introduction to 3-manifold topology

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ICTP, Trieste, June 6-11, 2005

We will be speaking about 3-manifolds. In general, when dealing with n-manifolds, one should specify which category is employed, namely one of the following:

TOP A manifold is a topological space covered by charts, *i.e.* open subsets homeomorphic to $\mathbb{R}^n$;

DIFF A manifold is a topological space covered by C^∞-compatible charts, *i.e.* open subsets U_i with homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ such that the changes of charts $\varphi_i \circ \varphi_j^{-1}$, suitably restricted, are C^∞ diffeomorphisms;

PL A manifold is a topological space covered by closed subsets P_i with homeomorphisms $g_i : P_i \to \Delta_n$, the n-simplex, so that each $g_i \circ g_j^{-1}$, when applicable, is a linear homeomorphism of a face of Δ_n onto another face of Δ_n. (Whence the name: Piecewise Linear). In addition, after subdividing the P_i's, the union of all the P_i's containing any given point should be homeomorphic to an n-disc with the given point at the centre.

It is a very deep but by now classical fact that the three viewpoints are equivalent for $n \leqslant 3$, namely:

- Every maximal topological atlas (*i.e.*, system of charts), contains maximal differentiable atlases, and two homeomorphic differentiable manifolds are diffeomorphic;

- Every TOP manifold is homeomorphic to a PL manifold, and two homeomorphic PL manifolds are actually PL equivalent.

For this reason, from now on we will freely switch from one viewpoint to the other. Our 3-manifolds will always be connected and compact, and they will often have a boundary, which requires a slight extension of the above definitions. We will be frequently considering submanifolds (1-dimensional, *i.e.* knots or links, or 2-dimensional, *i.e.* surfaces), and we will always assume that they are tamely embedded (or immersed), which makes it possible to interchange the TOP, DIFF, and PL viewpoints also when dealing with them.

For the sake of simplicity, all our manifolds will be orientable, which means, in the DIFF category, that there exists an atlas such that the Jacobian matrices of the changes of charts always have positive determinant.

1 The loop and sphere theorems

Stated informally, the loop theorem says that in a 3-manifold a homotopically trivial circle is actually topologically trivial, while the sphere thereom says that if all embedded spheres are trivial then so are the immersed or singular spheres. To give more precise statements, we begin by recalling that if Σ is a surface and M is a 3-manifold, a map $f : \Sigma \to M$ is called *proper* if $f^{-1}(\partial M) = \partial\Sigma$. The most general result about loops is as follows:

Theorem 1.1. *Let $B \subset \partial M$ be a compact surface (possibly with boundary), and let N be a normal subgroup of $\pi_1(B)$. Let $f : D^2 \to M$ be a proper map such that $f(S^1) \subset B$ and $f(S^1) \notin N$. Then there exists an embedding having the same properties.*

Concerning the statement just given, note that $f(S^1)$ is only well-defined up to conjugation as an element of $\pi_1(B)$, but the assumption $f(S^1) \notin N$ makes sense anyway, because N is normal. This theorem implies quite easily the following more classical statements:

Theorem 1.2 (Loop Theorem). *If $\gamma \subset \partial M$ is an embedded loop which bounds a disc in M, then γ also bounds a properly embedded disc.*

Theorem 1.3 (Dehn's Lemma). *Given $f : D^2 \to M$ such that $f(S^1) \subset \partial M$ and $f(S^1)$ is non-trivial in $\pi_1(\partial M)$, then there exists an embedding having the same properties.*

Corollary 1.4. *Given $f : D^2 \to M$ such that f is an embedding if restricted to some neighbourhood of S^1 and $f^{-1}(f(S^1)) = S^1$, then there exists an embedding $g : D^2 \to M$ such that $g(S^1) = f(S^1)$.*

Corollary 1.5. *If K is a knot in S^3 then K is the trivial knot if and only if $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.*

To state the last corollary, we introduce another notion also needed below. A (possibly disconnected) surface Σ properly embedded in M and having all the components different from the sphere is called *incompressible* if, whenever there exists a 2-disc D in M such that $D \cap \Sigma = \partial D$, actually ∂D bounds a disc also in Σ. It is an easy exercise to prove that a surface is incompressible if and only if all its components are.

Corollary 1.6. *A two-sided surface $\Sigma \subset M$ is incompressible if and only if the embedding $\Sigma \hookrightarrow M$ is injective at the level of fundamental groups.*

We conclude with the statement of the second fundamental result after which the section is named:

Theorem 1.7 (Sphere Theorem). *If $\pi_2(M)$ is non-trivial then there is in M an embedded sphere representing a non-trivial element of $\pi_2(M)$.*

2 Spherical splitting

For the sake of simplicity in this section we only consider manifolds whose boundary does not contain components homeomorphic to S^2. This does not seriously reduce the generality, because from an arbitrary M we can construct a unique $\widehat{M}$ without spheres in the boundary by attaching a ball to the sphere components of ∂M, and we can reconstruct M from $\widehat{M}$ by making the appropriate number of punctures.

A natural binary operation on 3-manifolds is defined as follows. Given M_1 and M_2, we remove from M_i the interior of a closed 3-disc D_i embedded in the interior of M_i, and we construct a new manifold by identifying ∂D_1 with ∂D_2. The result is called *connected sum* of M_1 and M_2, and denoted by $M_1 \# M_2$. Since the 2-sphere has two isotopy classes of automorphisms, at most two manifolds can arise as the connected sum of M_1 and M_2, but the operation is actually well-defined if we insist that M_1, M_2, and $M_1 \# M_2$ should be oriented, and that $M_i \setminus D_i$ should be induced the same orientation from M_i and from $M_1 \# M_2$, for $i = 1, 2$.

It is an easy exercise to show that $\#$ is a commutative and associative operation. The Alexander theorem implies that S^3 is the identity element for the operation of connected sum. In analogy with the case of integer numbers with the operation of product, we define a 3-manifold to be *prime* if whenever it is expressed as a connected sum, one of the summands has to be S^3. A strictly related notion is the following: M is called *irreducible* if every S^2 embedded in M bounds an embedded D^3. Of course every irreducible is prime, and the converse is almost true:

Proposition 2.1. *The only non-irreducible prime 3-manifold is $S^2 \times S^1$.*

We now come to the main statement:

Theorem 2.2. *Any 3-manifold M can be expressed in a unique fashion as a connected sum of primes $M_1, \ldots, M_k$.*

There is a subtlety about the uniqueness part of the previous statement which is worth pointing out. One easily sees that the realization of M as $M_1 \# \ldots \# M_k$ can be obtained by simultaneously cutting M along a family Σ of $k-1$ disjoint embedded spheres, and then filling by balls the resulting spherical boundary components. What the theorem says is that k and the resulting $M_1, \ldots, M_k$ are uniquely determined by M. It does *not* say that Σ is determined up to isotopy.

3 Seifert manifolds

An oriented 3-manifold M is called a *Seifert* manifold if it is expressed as a disjoint union of circles (the *fibres*) in such a way that each fibre not contained in the boundary has a neighbourhood U which is a union of fibres and, for some coprime integers α, ν with $\alpha \geqslant 1$, one has that U is fibre-preserving homeomorphic to $(D^2 \times [0,1]) \big/ \varphi_{\alpha,\nu}$, where $\varphi_{\alpha,\nu} : D^2 \times \{0\} \to D^2 \times \{1\}$ acts as $(z, 0) \mapsto (\exp(2\nu\pi i/\alpha) \cdot z, 1)$, with fibres given by the projections of the arcs $\{z\} \times [0,1]$. The fibres of M are required to give rise to an ordinary fibration in the neighbourhood of ∂M (in particular, ∂M must be a union of fibres).

The space $(D^2 \times [0,1]) \big/ \varphi_{\alpha,\nu}$ just described will be called the (α,ν)-*standardly fibred solid torus*. Since all the fibres except the core (the projection of $\{0\} \times [0,1]$) are homotopic to α times the core within the fibred torus, α is a well-defined invariant of the core. If $\alpha > 1$ we will say that the core is a *singular* fibre. One easily sees that singular fibres are isolated, whence finite in number. The next result describes to what extent ν is an invariant of the fibre.

Proposition 3.1.

- *A fibre- and orientation-preserving homeomorphism between two standardly fibred solid tori with parameters (α, ν_1) and (α, ν_2) exists if and only $\nu_1 \equiv \nu_2$ modulo α.*
- *Reversing orientation, a standardly fibred solid torus with parameters (α, ν) becomes one with parameters $(\alpha, -\nu)$.*

The parameters α and ν, which are well-defined if one puts the restriction $0 \leqslant \nu < \alpha$, are called the *orbital invariants* of a fibre. We now define as *base space* of a Seifert fibration the space B obtained from M by collapsing each fibre to a point. A neighbourhood in B of a point coming from a non-singular fibre is homeomorphic to a disc (or to a half-disc, if the fibre is on the boundary), while for a point coming from a singular fibre of orbital invariants (α, ν) such a neighbourhood is obtained from a wedge of disc of angle $2\pi/\alpha$ by identifying the sides. Since this space is again a disc, we deduce that B is a surface (or, better to say, a 2-orbifold, as another course will explain). Note that B may well be non-orientable, even if M is orientable.

4 Fibred classification of Seifert manifolds

Our next aim is to state the fibred classification of Seifert manifolds. To this end we make a small digression, introducing the notion of Dehn filling, which has crucial importance in 3-dimensional topology and geometry. We begin with the following:

Proposition 4.1.

- *An automorphism of the torus $S^1 \times S^1$ is uniquely determined up to isotopy by its action on the first homology group $H_1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$;*
- *An automorphism of the torus $S^1 \times S^1$ extends to the solid torus $D^2 \times S^1$ if and only if it maps the homology class of the meridian $S^1 \times \{*\}$ to plus or minus itself.*

Suppose now that a 3-manifold M has a boundary component T homeomorphic to the torus, and that a basis λ, μ of $H_1(T; \mathbb{Z})$ has been fixed. Given coprime integers a, b, according to the previous proposition one can uniquely define a new manifold $M \cup_\varphi (D^2 \times S^1)$, where $\varphi : S^1 \times S^1 \to T$ is a homeomorphism such that $\varphi_*(S^1 \times \{*\}) = \pm(a \cdot \lambda + b \cdot \mu)$. This manifold is called the *Dehn filling* of M along $\pm(a \cdot \lambda + b \cdot \mu)$. The condition that a

and b should be coprime is equivalent to the fact that $\pm(a \cdot \lambda + b \cdot \mu)$ admits a simple closed curve as a representative. The isotopy class of such a curve, which is determined by a and b, will be called a *slope* on T.

Back to the fibred classification of Seifert manifolds, let us slightly change our viewpoint, switching from the orbital invariants of fibres to certain *filling* invariants, which we now define. Let us remove from M a fibred neighbourhood of all singular fibres or, in case $\partial M = \emptyset$ and there are no singular fibres, a fibred neighbourhood of a regular fibre. The result is a manifold N with a genuine fibration into circles, whose base space C is obtained by removing some open discs from B. By construction, C has non-empty boundary, so it has the homotopy type of a graph. This easily implies that the fibration $N \to C$ admits a global section s. Let us denote now by $T_1, \ldots, T_k$ the boundary tori of N arising from the drilling of the fibres of M. We choose a homology basis λ_i, μ_i on T_i in such a way that λ_i is represented by an (oriented) fibre, μ_i is contained in the section s (and oriented), and the basis is positive. Pasting back the i-th drilled solid torus then corresponds to a Dehn filling along some slope $\pm(\alpha_i \cdot \mu_i + \beta_i \cdot \lambda_i)$.

Proposition 4.2. *$\alpha_i \neq 0$, whence we can assume $\alpha_i > 0$. With this choice, the orbital invariants of the i-th fibre are (α_i, ν_i), where $\nu_i \cdot \beta_i \equiv 1$ modulo α_i.*

This result shows that we can define the *filling invariants* of the i-th fibre to be (α_i, β_i), where α_i is well-defined and β_i is well-defined modulo α_i. Note $\alpha_i = 1$ means that the fibre is non-singular, which by construction can only occur if there are no singular fibres and no boundary at all. In particular, we have:

Corollary 4.3. *Either $\alpha_i > 1$ for all i or ($k = 1$ and B is closed).*

One can actually reverse the construction just described, *i.e.* one can build a Seifert manifold starting from an arbitrary surface B and certain pairs of coprime integers $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ with $\alpha_i > 0$ and either $\alpha_i > 1$ for all i or ($k = 1$ and B is closed). To do so, one makes k punctures to B, getting a surface C, and one takes the circle-fibration $N \to C$ with orientable total space, *i.e.* the product fibration if B is orientable, and the twisted circle-bundle if B is not. One then chooses a section $s : C \to N$ and proceeds as above, selecting homology bases and doing Dehn filling.

Theorem 4.4.
- *The Seifert manifold $(B; (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$ is well-defined,* i.e. *its construction is independent of the choice of the section of the fibration $N \to C$;*

- *Every Seifert manifold has the form* $(B;(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k))$*;*
- *Two Seifert manifolds*

$$(B;(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k)) \qquad (B';(\alpha_1',\beta_1'),\ldots,(\alpha_{k'}',\beta_{k'}'))$$

  with non-empty boundary are fibre- and orientation-preservingly homeomorphic if and only if $B'=B$*,* $k'=k$*, and, up to reordering,* $\alpha_i'=\alpha_i$ *and* $\beta_i'\equiv\beta_i$ *modulo* α_i *for all* i*;*
- *Two closed Seifert manifolds*

$$(B;(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k)) \qquad (B';(\alpha_1',\beta_1'),\ldots,(\alpha_{k'}',\beta_{k'}'))$$

  are fibre- and orientation-preservingly homeomorphic if and only if $B'=B$*,* $k'=k$*, up to reordering* $\alpha_i'=\alpha_i$ *and* $\beta_i'\equiv\beta_i$ *modulo* α_i *for all* i*, and*

$$\sum_{i=1}^{k}\frac{\beta_i}{\alpha_i}=\sum_{i=1}^{k'}\frac{\beta_i'}{\alpha_i'}.$$

For a closed $M=(B;(\alpha_1,\beta_1),\ldots,(\alpha_k,\beta_k))$ the rational number $\sum\frac{\beta_i}{\alpha_i}$ is called the *Euler number*. For a genuine fibration, *i.e* a Seifert fibration without singular fibres $(B;(1,n))$, the Euler number is n and it coincides with the classical Euler number of the fibration, defined as the obstruction to the existence of a section.

5 Topological classification of Seifert manifolds

We now turn to the topological classification of Seifert manifolds, *i.e.* up to orientation-preserving homeomorphism, neglecting the fibred structure. It turns out that a key ingredient for such a classification is the presence of "essential" fibred annuli (which occurs if and only if there is some boundary) or of fibred tori. For this reason there is a "small" set of closed Seifert manifolds for which the statement of the classification result is somewhat involved, and we will refrain from giving it.

We begin with the case of non-empty boundary, and describe some exceptions. Of course all the standardly fibred solid tori $(D^2; (\alpha, \beta))$ are topologically the same, and there are infinitely many of them up to fibred homeomorphism. On the manifold $S^1 \times S^1 \times D^1$ one can define infinitely many non-isotopic Seifert structures, each depending on the choice of a slope on the torus $S^1 \times S^1$, but of course all these structures are isomorphic to each other. The next exception is more elaborate to explain. Consider the Klein bottle K and the orientable interval-bundle over K, denoted by $K \tilde{\times} D^1$. We can realize such a manifold as the quotient of the cube $D^1 \times D^1 \times D^1$ under the identifications generated by

$$(-1, y, z) \sim (+1, y, z) \qquad (x, -1, z) \sim (-x, +1, -z) \qquad \forall x, y, z \in D^1.$$

Taking as fibres the images of the intervals $D^1 \times \{*\} \times \{*\}$ gives $K \tilde{\times} D^1$ the structure of a Seifert manifold over the Möbius band without singular fibres. On the other hand, if one takes as fibres the images of the intervals $\{*\} \times D^1 \times \{*\}$ one sees that $K \tilde{\times} D^1$ also has the Seifert structure $(D^2; (2,1), (2,1))$.

Theorem 5.1. *With the exceptions just described, any Seifert manifold with non-empty boundary has a unique Seifert fibration up to isotopy.*

Turning to the closed case, let us now say that a Seifert manifold is *large* if it contains an incompressible fibred torus. The next result shows that most Seifert manifolds are large:

Proposition 5.2. *A closed Seifert manifold is large unless one of the following holds:*

- *it is fibred over the sphere with at most three exceptional fibres;*
- *it is fibred over the projective plane with at most one exceptional fibre.*

As in the bounded case, let us begin by describing some exceptions to the uniqueness of the Seifert structure. The manifold $S^1 \times S^1 \times S^1$ admits infinitely many non-isotopic but isomorphic Seifert structures. Doubling the manifold $K \tilde{\times} S^1$ described above one gets the twisted circle bundle over the Klein bottle, which, as one sees using arguments similar to those employed above, can also be given the Seifert structures $(S^2; (2,1), (2,1), (2,-1), (2,-1))$ and $(\mathbb{P}^2; (2,1), (2,-1))$.

Theorem 5.3. *With the exceptions just described, any large closed Seifert manifold has a unique Seifert fibration up to isotopy.*

As already announced, we will not deal with closed Seifert manifolds which are not large. We only mention that the techniques to be employed in this case are completely different from those based on incompressible surfaces used for large manifolds, and that there are many more repetitions. As an example, the following holds:

Proposition 5.4.

- *The Seifert fibred manifold* $(S^2; (p_1, q_1), (p_2, q_2))$ *is homeomorphic to the lens space* $L_{p,q}$ *where* $p = p_1 \cdot q_2 + q_1 \cdot p_2$ *and* $q = p_1 \cdot s_2 + q_1 \cdot r_2$*, where* r_2, s_2 *are integers such that* $p_2 \cdot s_2 - q_2 \cdot r_2 = 1$*;*
- *A lens space* $L_{p,q}$ *is homeomorphic to another lens space* $L_{p',q'}$ *if and only if* $p' = p$ *and* $q' \cdot q \equiv \pm 1$ *modulo* p*.*

In the first assertion of the previous statement, note that p is well-defined because the Euler number is, and that q is well-defined modulo p. Note also that switching indices in the definition of q gives the inverse of q modulo p, whence the same lens space by the second assertion.

6 The JSJ decomposition

Besides cutting along "incompressible spheres" (*i.e.* separating spheres not bounding balls), which cannot be done in a canonical way but leads to the unique splitting into primes, one can try to simplify a given 3-manifold by cutting along incompressible tori. However, one sees that Seifert manifolds can be cut along tori in many different ways, so one avoids these cuts altogether. The result of these cuts, which turns out to be canonical, is the so-called JSJ (Jaco-Shalen and Johansson) decomposition of M, which we will only state in a simplified geometric fashion.

We define a manifold M to be *atoroidal* is every incompressible torus T embedded in M is parallel to a boundary component of M (*i.e.*, together with a boundary component it bounds an embedded product manifold $T \times [0, 1]$).

Theorem 6.1. *If* M *is irreducible then there exists in* M *and is unique up to isotopy a finite family* $\mathcal{T}$ *of incompressible tori satisfying the following:*

- *Each component of* $M \setminus \mathcal{T}$ *is either atoroidal or Seifert;*
- $\mathcal{T}$ *is minimal with respect to the previous property.*

7 Normal surfaces

One key ingredient in the proof of both Theorems 2.2 and 6.1 is the theory of normal surfaces. Since this theory has independent interest, in particular in connection with the algorithmic theory of 3-manifolds, we sketch its main features here. For the sake of simplicity we only refer to the case of closed manifolds, and we only describe normal surfaces with respect to triangulations, omitting the (often more useful) theory of normal surfaces with respect to handle decompositions.

Let $\mathcal{T}$ be a triangulation of M, *i.e.* a realization of M as a gluing of tetrahedra along codimension-1 faces. (In the original PL definition of triangulation one should require the tetrahedra to be embedded in M and to intersect pairwise in at most one common face, but this restriction is not necessary here).

To give the definition, we call *triangle* in a tetrahedron Δ the intersection of Δ with a plane separating one vertex from the other three, and *square* the intersection of Δ with a plane separating two vertices from the other two. A surface Σ embedded in M is called *normal* with respect to $\mathcal{T}$ if it meets each tetrahedron of $\mathcal{T}$ in a union of triangles and squares.

The first of the results we will now state requires an additional definition. A collection Σ of disjoint spheres embedded in M is called *essential* if no component of $M \setminus \Sigma$ is a punctured disc.

Proposition 7.1. *Any essential system of spheres can be replaced by one having the same number of components and being in normal position with respect to $\mathcal{T}$.*

Proposition 7.2. *Suppose that M is irreducible. Consider a possibly disconnected surface Σ properly embedded in M and having all the components different from the sphere. Suppose that Σ is incompressible. Then Σ can be isotoped to a surface in normal position with respect to $\mathcal{T}$.*

Theorem 7.3. *There exists an integer ν depending on M and $\mathcal{T}$ such that, if Σ is a normal surface with respect to $\mathcal{T}$ having more than ν components, then two components of Σ are parallel.*