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### Lectures on hyperbolic groups and convergence groups

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# Lectures on hyperbolic groups and convergence groups

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## Lecture 1

### Motivation

We start by stating an important theorem in 3-manifold topology.

**Theorem 1** ([28, 24, 21, 25, 11, 6]). *Let  $M$  be a closed, orientable, irreducible 3-manifold. If  $\pi_1 M$  has an infinite cyclic normal subgroup, then  $M$  is Seifert fibered.*

Since Seifert manifolds are easily seen to be geometric in Thurston's sense, this theorem can be viewed as a partial answer to the Geometrization Conjecture. It was known in some circles as the 'Seifert Conjecture' (although I doubt that Seifert ever conjectured this.)

Here is an outline of proof: let  $Z$  be an infinite cyclic normal subgroup of  $\pi_1 M$  and set  $\Gamma := \pi_1 M / Z$ . Let  $\hat{M}$  be the covering space of  $M$  such that  $\pi_1 \hat{M} = Z$ . Then  $\Gamma$  acts on  $\hat{M}$  with quotient space  $M$ .

**Step 1** The 3-manifold  $\hat{M}$  is *tame*, i.e. homeomorphic to  $S^1 \times \mathbf{R}^2$ .

**Step 2** The group  $\Gamma$  is *quasi-isometric* to a complete Riemannian metric on  $\mathbf{R}^2$ . (The definition of quasi-isometric will be given later.)

**Step 3** Any finitely generated group quasi-isometric to a complete Riemannian metric on  $\mathbf{R}^2$  is *virtually  $\mathbf{Z}^2$*  (i.e. has a finite index subgroup isomorphic to  $\mathbf{Z}^2$ ,) or is quasi-isometric to the hyperbolic plane  $\mathbf{H}^2$ .

**Step 4** Any finitely generated group quasi-isometric to  $\mathbf{H}^2$  acts *geometrically* on  $\mathbf{H}^2$ . (This means that the action is isometric, properly discontinuous, and cocompact.)

**Step 5** The fundamental group of  $M$  is isomorphic to the fundamental group of some irreducible Seifert fibered space  $N$ .

**Step 6**  $M$  is homeomorphic to  $N$ , hence Seifert fibered.

The goal of these lectures is to explain Step 4 in some detail, putting it into the more general context of *hyperbolic groups* and *convergence groups*. Before that, we make some brief comments about the other steps, in the chronological order of their proofs.

Step 5 is classical: the class of groups that are virtually  $\mathbf{Z}^2$  or act geometrically on  $\mathbf{H}^2$  has been well-known for some time: they are called ‘planar discontinuous groups’ in [31]. Extensions of  $\mathbf{Z}$  by these groups can be classified using group cohomology [32]. When such a group is torsion free, it is not hard to construct a Seifert fiber space realizing it.

Step 6 is due to Peter Scott [24]: he proves that if  $M, N$  are closed, orientable, irreducible 3-manifolds with isomorphic fundamental groups, and one of them is Seifert, then  $M$  and  $N$  are homeomorphic. Hence at that time the Seifert Conjecture was reduced to proving that  $\Gamma$  is planar discontinuous.

Steps 1–3 are due to Mess [21]. In fact, the conclusion of Step 2 and hypothesis of Step 3 in his original approach is that  $\Gamma$  is quasi-isometric to a *quasihomogeneous* metric on  $\mathbf{R}^2$ . That Step 3 is valid without this additional hypothesis is proved in [19].

Step 1 uses many important results in 3-manifold topology: the Meeks-Simon-Yau theorem about irreducibility of covering spaces, the Scott compact core theorem, the theory of characteristic submanifolds due to Jaco-Shalen and Johannson... Step 2 uses minimal surfaces. Complete proofs of more general results can be found in [20] (see [1] for an introduction). I also refer you to Juan Souto’s lectures in this summer school for an introduction to open 3-manifolds.

Step 3 consists of intricate 2-dimensional arguments. A simpler proof was given in [19].

Step 4 can be further decomposed into 2 substeps: if  $\Gamma$  is quasi-isometric to  $\mathbf{H}^2$ , then Mess observed that  $\Gamma$  acts on the circle  $S^1$  as a *convergence group*. More generally, if  $\Gamma$  is a *hyperbolic group* in the sense of Gromov, then it acts in a natural way on a compact topological space  $\partial\Gamma$  called its *boundary*, and this action is a convergence action. This will be explained in Lectures 1 and 2.

To finish the proof of Theorem 1, one needs:

**Theorem 2 (Convergence Group Theorem [25, 11, 6]).** *Let  $\Gamma$  be a group. If  $\Gamma$  acts as a convergence group on  $S^1$ , then  $\Gamma$  has a finite normal subgroup  $F$  such that  $\Gamma/F$  is Fuchsian.*

Recall that a Fuchsian group is a discrete subgroup of isometries of  $\mathbf{H}^2$ . Hence the conclusion is equivalent to saying that  $\Gamma$  acts properly discontinuously and isometrically on  $\mathbf{H}^2$ . If  $\Gamma$  is quasi-isometric to  $\mathbf{H}^2$ , then one can see that such an action has to be cocompact, hence we get the conclusion of Step 4 as stated above.

In lectures 3 and 4, I will discuss the proof of Theorem 2, following Tukia and Casson-Jungreis.

For other (more group-theoretic) approaches to the Seifert Conjecture and related questions, see e.g. [9, 4, 18].

## Hyperbolicity according to Rips and Gromov

Let  $(X, d)$  be a metric space. A *geodesic segment* in  $X$  is an isometric embedding of a compact interval  $[a, b] \subset \mathbf{R}$  into  $X$ . We say that  $X$  is *geodesic* if for all  $x, y \in X$ , there is a geodesic segment  $c : [a, b] \rightarrow X$  such that  $c(a) = x$  and  $c(b) = y$ .

Basic examples of geodesic spaces are complete Riemannian manifolds, in particular Euclidean space  $\mathbf{E}^n$  and hyperbolic space  $\mathbf{H}^n$ . Note however that our notion of ‘geodesic segment’ is commonly called ‘minimizing geodesic’ in Riemannian geometry. A nonminimizing geodesic (such as a large portion of the equator in the round 2-sphere) is *not* a geodesic segment in our sense.

Other basic examples are *graphs*, i.e. connected 1-dimensional cellular complexes: let  $X$  be a graph. The distance between two vertices  $x, y$  is defined to be the minimal number of edges in an edge-path connecting  $x$  to  $y$ . This defines a distance on the set of vertices of  $X$ . It can be extended to a distance on all of  $X$  by fixing a parametrization of each edge, and counting the length of the pieces of edges in a path connecting the two points.

Next we define hyperbolicity in the sense of Rips: the definition involves looking at *geodesic triangles*. Before this, we give some general definitions in metric spaces: if  $A$  is a subset of  $X$  and  $C \geq 0$ , the  *$C$ -neighborhood* of  $A$  is the set  $N_C(A) := \{x \in X \mid d(x, A) \leq C\}$ . If  $x \in X$ , the  $C$ -neighborhood of  $\{x\}$  is also called the *ball* of radius  $C$  around  $x$ , and denoted by  $B_C(x)$ .

A *geodesic triangle* in  $X$  is a triple  $(\alpha_1, \alpha_2, \alpha_3)$  of geodesic segments such that there exist points  $x_1, x_2, x_3$  such that  $\alpha_i$  connects  $x_{i+1}$  to  $x_{i+2}$  (with indices in  $\mathbf{Z}/3$ .) It is  *$\delta$ -thin* if  $\alpha_i$  lies in the  $\delta$ -neighborhood of  $\alpha_{i+1} \cup \alpha_{i+2}$ .

**Definition.** Let  $\delta \geq 0$ . A geodesic metric space is  $\delta$ -hyperbolic if all its geodesic triangles are  $\delta$ -thin. It is *hyperbolic in the sense of Rips*, or simply *hyperbolic*, if it is  $\delta$ -hyperbolic for some  $\delta$ .

*Examples.*

- $\mathbf{H}^n$  is hyperbolic;  $\mathbf{E}^n$  is not hyperbolic for  $n \geq 2$ . Pinched negatively curved manifolds are hyperbolic, as well as their convex subsets.
- Simplicial trees (i.e. 1-connected graphs) are 0-hyperbolic.
- Any bounded geodesic space is hyperbolic. Any product of a hyperbolic space and a bounded space, e.g.  $S^2 \times \mathbf{H}^2$  is hyperbolic. These examples show that the definition of Rips hyperbolicity captures the geometry of negative curvature only ‘in the large’.

**Exercise 1.** If  $X$  is 0-hyperbolic, then for all  $x, y \in X$ , there is a unique topological arc connecting  $x$  to  $y$ , i.e. topological embedding  $c : [a, b] \rightarrow X$  such that  $c(a) = x$  and  $c(b) = y$ . (Of course, the uniqueness is up to parameterization.)

A metric space is *proper* if all metric balls are compact. Basic examples of proper spaces are complete Riemannian manifolds and locally finite metric graphs.

A group action is called *geometric* if it is isometric, properly discontinuous, and cocompact.

**Definition.** A *hyperbolic group* is a group that acts geometrically on some proper hyperbolic space.

*Examples.*

- Convex cocompact Kleinian groups, in particular surface groups.
- Free groups, because they act geometrically on trees.

*Remark.* Any finitely generated group  $\Gamma$  acts geometrically on some proper geodesic space. The simplest construction is called the Cayley graph: let  $S$  be a finite generating set of  $\Gamma$ . Then the *Cayley graph* is the graph whose vertices are elements of  $\Gamma$ , and there is an edge between  $\gamma_1$  and  $\gamma_2$  if and only if  $\gamma_1^{-1}\gamma_2 \in S$ .

**Exercise 2.** Let  $F_2$  be the free group on two generators. Describe the Cayley graph of  $F_2$  with respect to a generating system of your choice.

## Quasi-isometries; quasi-isometry invariance of hyperbolicity

**Definition.** Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be two metric spaces. We say that a map  $f : X_1 \rightarrow X_2$  is a *quasi-isometric embedding* if there exist  $\lambda \geq 1$  and  $C \geq 0$  such that the inequality

$$\lambda^{-1} d_1(x, x') - C \leq d_2(f(x), f(x')) \leq \lambda d_1(x, x') + C$$

holds for any  $x, x' \in X_1$ .

If in addition, the  $C$ -neighborhood of the image of  $f$  is all of  $X_2$ , we say that  $f$  is a *quasi-isometry*.

We say that  $(X_1, d_1)$  is *quasi-isometric* to  $(X_2, d_2)$  if there exists a quasi-isometry  $f : X_1 \rightarrow X_2$ . Sometimes we just say that  $(X_1, d_1)$  and  $(X_2, d_2)$  are quasi-isometric.

**Exercise 3.** Prove that if  $(X_1, d_1)$  is quasi-isometric to  $(X_2, d_2)$ , then  $(X_2, d_2)$  is quasi-isometric to  $(X_1, d_1)$ .

Prove that if  $(X_1, d_1)$  is quasi-isometric to  $(X_2, d_2)$  and  $(X_2, d_2)$  is quasi-isometric to  $(X_3, d_3)$ , then  $(X_1, d_1)$  is quasi-isometric to  $(X_3, d_3)$ .

**Theorem 3.** *Let  $X$  be a hyperbolic space. Let  $Y$  be a geodesic metric space. If  $X$  and  $Y$  are quasi-isometric, then  $Y$  is hyperbolic.*

The proof of Theorem 3 relies on the following fundamental property of hyperbolic spaces.

A  $(\lambda, C)$ -*quasigeodesic segment* in a space  $X$  is a quasi-isometric embedding of a compact interval  $[a, b]$  in  $X$ .

**Theorem 4 (Morse Lemma on quasigeodesic stability).** *Let  $\delta, \lambda \geq 0$ . There is a number  $D \geq 0$  such that if  $X$  is a  $\delta$ -hyperbolic space, then any  $(\lambda, \delta)$ -quasigeodesic segment lies in the  $D$ -neighborhood of a geodesic segment with the same endpoints.*

**Exercise 4.** Deduce Theorem 3 for spaces from Theorem 4.

(Hint: take a geodesic triangle in  $B$ ; its image by a quasi-isometry is a “quasigeodesic triangle” in  $A$ , which by Theorem 4 is close to a geodesic triangle; then use the hyperbolicity of  $A$  to conclude. You will need to prove that sides of the geodesic triangle lie in the  $D'$ -neighborhood of the sides of the quasi-geodesic triangle for some constant  $D'$ .)

**Exercise 5.** Prove Theorem 4 when  $X$  is a simplicial tree.

## Consequences for hyperbolic groups

Let  $\Gamma$  be a finitely generated group. Let  $S$  be a finite generating set. Without loss of generality, we may assume that  $S$  is *symmetric*, i.e.  $S = S^{-1}$ . We define the *word metric*  $d_S$  on  $\Gamma$  by setting  $d_S(\gamma_1, \gamma_2)$  equal to the least integer  $n \geq 0$  such that  $\gamma_1^{-1}\gamma_2$  can be written as a product of  $n$  elements of  $S$ . Note that this is the metric obtained by viewing  $\Gamma$  as the 0-skeleton of its Cayley graph and restricting the graph metric.

**Exercise 6.** Let  $S_1, S_2$  be two symmetric generating sets. Show that the identity map  $(\Gamma, d_{S_1}) \rightarrow (\Gamma, d_{S_2})$  is a quasi-isometry. (Hence it makes sense to say that a group  $\Gamma$  is quasi-isometric to some metric space  $X$ .)

Show that the inclusion of  $\Gamma$  into its Cayley graph is a quasi-isometry.

The link between groups and spaces is provided by the following fundamental proposition due independently to Efremovič, Švarc and Milnor. (See e.g. [13, 17, 5].)

**Proposition 5.** *Let  $X$  be a proper geodesic metric space. Let  $\Gamma$  be a group acting geometrically on  $X$ . Then  $\Gamma$  is finitely generated and quasi-isometric to  $X$ .*

**Corollary 6.** *If a finitely generated  $\Gamma$  is quasi-isometric to some hyperbolic space  $X$ , then  $\Gamma$  is hyperbolic.*

In particular:

**Corollary 7.** *If  $\Gamma$  is quasi-isometric to  $\mathbf{H}^2$ , then  $\Gamma$  is hyperbolic.*

*Remarks.*

- What we are looking for is a kind of converse to Proposition 5 in the case where  $X = \mathbf{H}^2$ . In general, given a metric space  $X$ , one may ask whether all groups quasi-isometric to  $X$  act geometrically on  $X$ , and if not, what are the groups quasi-isometric to  $X$ . For a survey on this, see Misha Kapovich's notes [16].
- Hyperbolic groups were introduced by Gromov. The standard references are his seminal paper [15] and the books [7, 13, 14]. Hyperbolic groups are also discussed in the more recent book [5].
- There are striking recent applications of  $\delta$ -hyperbolic spaces to low-dimensional geometry. Among them, the most famous is probably the work of Brock, Canary and Minsky on the Ending Lamination Conjecture (see [22] for an introduction).

## Lecture 2

### The boundary of a hyperbolic space

In all of this lecture,  $(X, d)$  is a proper geodesic metric space.

A *geodesic ray* in  $X$  is an isometric embedding of  $[0, +\infty)$  into  $X$ . We say that two geodesic rays  $c, c'$  are *asymptotic* if the function  $t \mapsto d(c(t), c'(t))$  is bounded.

**Definition.** Let  $(X, d)$  be a proper hyperbolic space. Fix a basepoint  $p \in X$ . The *boundary* of  $X$ , denoted by  $\partial_p X$ , is the set of equivalence classes of geodesic rays  $c$  in  $X$  such that  $c(0) = p$ , where two rays are equivalent if they are asymptotic.

Using properness of  $X$  and the Ascoli-Arzelà theorem, it is easy to see that if one chooses a different basepoint  $q$ , then there is a natural bijection  $\partial_p X \rightarrow \partial_q X$ . Hence it is legitimate to drop the basepoint in the notation. We will henceforth denote the boundary of  $X$  simply by  $\partial X$ .

We are going to define a topology on  $X \cup \partial X$ . For this, it is convenient to make some more definitions: a *generalized ray* is a map  $c : [0, +\infty) \rightarrow X$  such that either  $c$  is a geodesic ray, or there exists  $R \geq 0$  such that the restriction of  $c$  to  $[0, R]$  is a geodesic segment, and the restriction of  $c$  to  $[R, +\infty)$  is constant. We consider two generalized rays to be equivalent if either they are asymptotic geodesic rays, or they are eventually constant and equal. Hence we can view the set  $X \cup \partial X$  as a quotient of the set  $\mathcal{R}_p$  of generalized rays starting at  $p$ .

**Definition.** We endow  $\mathcal{R}_p$  with the compact-open topology, i.e. a fundamental system of neighborhoods of a generalized ray  $c$  is given by sets of the form

$$V_\epsilon = \{c' \mid d(c(t), c'(t)) \leq \epsilon \quad \forall t \in [0, \epsilon^{-1}]\}.$$

The space  $\bar{X} = X \cup \partial X$  is given the quotient topology.

Hence a sequence  $c_n \in \mathcal{R}_p$  converges to  $c$  if and only if it converges uniformly on compact subsets.

**Proposition 8.** *The topology on  $\bar{X}$  is independent of the choice of basepoint. It is a compact Hausdorff space. The inclusion  $X \rightarrow \bar{X}$  is an embedding with open image. (Hence  $\partial X$  is compact.)*

**Exercise 7.** Prove that  $\partial \mathbf{H}^n$  is homeomorphic to the  $(n-1)$ -sphere  $S^{n-1}$ .

Let  $T$  be a Cayley graph of the free group on two generators. Prove that  $\partial T$  is a Cantor set.

**Proposition 9 (Quasi-isometry invariance of the boundary).** *Let  $X$  and  $Y$  be proper hyperbolic spaces. If  $X$  and  $Y$  are quasi-isometric, then  $\partial X$  and  $\partial Y$  are homeomorphic.*

*Remark.* One can show that the boundary of a proper hyperbolic space is always metrizable. This will be assumed implicitly in the sequel. In fact one can construct explicitly a family of metrics on  $\partial X$  that induce the topology. This is important, but will not be discussed in these lectures. See [7] or [13].

## From hyperbolic groups to convergence groups

**Definition.** Let  $M$  be an infinite compact metrizable topological space and  $\Gamma$  be a group acting by homeomorphisms on  $M$ .

We say that  $\Gamma$  *acts as a convergence group* if for each sequence  $\{g_n\}$  of distinct elements of  $\Gamma$ , there exist points  $a, b \in M$  and a subsequence  $\{g_{n_k}\}$  such that  $\lim g_{n_k} \cdot x = a$  uniformly on compact subsets not containing  $b$ .

**Proposition 10 ([3, 10, 27]).** *Let  $X$  be a proper hyperbolic space. Let  $\Gamma$  be a group acting properly discontinuously by isometries on  $X$ . Then  $\Gamma$  acts as a convergence group on  $\partial X$ .*

**Corollary 11.** *Let  $\Gamma$  be a group quasi-isometric to  $\mathbf{H}^2$ . Then  $\Gamma$  acts as a convergence group on  $S^1$ .*

## Basics of convergence groups

Next we discuss an important characterization of convergence groups. Denote by  $\Theta(M)$  the set of triples  $(x, y, z) \in M^3$  such that  $x, y, z$  are pairwise distinct, topologized as a subset of  $M^3$ .

**Proposition 12 ([3]).** *Let  $M$  be an infinite compact metrizable topological space and  $\Gamma$  be a group acting by homeomorphisms on  $M$ . Then  $\Gamma$  acts as a convergence group if and only if the induced action of  $\Gamma$  on  $\Theta(M)$  is properly discontinuous.*

**Exercice 8.** Show that  $\Theta(S^1)$  is homeomorphic to two copies of  $S^1 \times \mathbf{R}^2$ . (Hint: construct a map from  $\Theta(S^1)$  to the unit tangent bundle of  $\mathbf{H}^2$ .)

Let  $\Gamma$  act as a convergence group on  $M$ . An element  $\gamma \in \Gamma$  is *elliptic* (resp. *parabolic*, resp. *hyperbolic*) if it has finite order (resp. has a unique fixed point, resp. has exactly two fixed points).

**Proposition 13.** *Any element of  $\Gamma$  is either elliptic, parabolic, or hyperbolic.*

*Remarks.*

- We have seen that hyperbolic groups are convergence groups. It is a natural question to characterize hyperbolic groups among convergence groups. Such a characterization has been given by Brian Bowditch [2]. This theorem has been extended by Asli Yaman [29] to a characterization of *relatively hyperbolic groups*. For more on this, see François Dahmani’s thesis [8].
- Convergence groups on spheres were introduced by Gehring and Martin [12]. Their original definition is more general: the convergence groups considered in these lectures are what they call *discrete* convergence groups.

## Lecture 3

In this lecture, I will explain the main ideas of the proof of the following ‘half’ of the Convergence Group Theorem.

**Theorem 14 ([25]).** *Let  $\Gamma$  be a group acting as a convergence group on  $S^1$ . Assume that  $\Gamma$  is torsion free, and acts preserving orientation and without parabolic elements. Then  $\Gamma$  has a finite normal subgroup  $F$  such that  $\Gamma/F$  is Fuchsian.*

An *axis* is a pair  $(a, b)$  of distinct points of  $S^1$ . We say that two axes  $(a_1, b_1), (a_2, b_2)$  *cross* if  $a_2$  and  $b_2$  belong to different components of  $S^1 \setminus \{a_1, b_1\}$ . An axis  $A$  is *simple* if for every  $g \in \Gamma$ ,  $gA$  does not cross  $A$ . An axis is *hyperbolic* if it consists of the fixed points of a hyperbolic element of  $\Gamma$ .

Here is the key lemma:

**Lemma 15.** *Let  $\Gamma$  be as in Theorem 14. Then there is a simple hyperbolic axis.*

*Remarks.*

- The idea of finding a simple axis goes back to the work of Nielsen [23], and was further developed by Zieschang [30] in connection with the Nielsen Realization Problem for mapping classes of homeomorphisms of surfaces.
- The paper [25] is rather long and technical. The author has given a nice outline in [26].

## Lecture 4

This lecture will be devoted to the other ‘half’ of the Convergence Group Theorem.

**Theorem 16.** *Let  $\Gamma$  be a group acting as a convergence group on  $S^1$ . If the action is orientation-preserving and if  $\Gamma$  has a torsion element of order at least 3, then  $\Gamma$  has a finite normal subgroup  $F$  such that  $\Gamma/F$  is Fuchsian.*

Of course, Theorems 14 and 16 do not imply the Convergence Group Theorem. However, the methods discussed in the previous lecture extend to cover all cases not covered by Theorem 16 (see [25]) and even some more. Roughly, the proof of Theorem 14 extends without difficulties if there are orientation-reversing elements or elements of order 2. If there are parabolics, one proves the existence of a simple *regular axis* (which may not consist in the fixed points of a hyperbolic element).

I will follow the Casson-Jungreis approach [6]. For a completely different proof, see Gabai’s paper [11].

If time permits, I will discuss a proof of the Morse Lemma using ultra-limits, following [17].

## References

- [1] M. Boileau, S. Maillot, and J. Porti. *Three-dimensional orbifolds and their geometric structures*, volume 15 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2003.
- [2] B. H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.
- [3] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.
- [4] B. H. Bowditch. Planar groups and the Seifert conjecture. *J. Reine Angew. Math.*, 576:11–62, 2004.
- [5] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

- [6] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118:441–456, 1994.
- [7] M. Coornaert, T. Delzant, and A. Papadopoulos. *Géométrie et théorie des groupes*, volume 1441 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], With an English summary.
- [8] F. Dahmani. *Les groupes relativement hyperboliques et leurs bords*. Prépublication de l’Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 2003/13. Université Louis Pasteur Département de Mathématique Institut de Recherche Mathématique Avancée, Strasbourg, 2003. Thèse, l’Université Louis Pasteur (Strasbourg I), Strasbourg, 2003.
- [9] M. J. Dunwoody and E. L. Swenson. The algebraic torus theorem. *Invent. Math.*, 140(3):605–637, 2000.
- [10] E. M. Freeden. Negatively curved groups have the convergence property. I. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(2):333–348, 1995.
- [11] D. Gabai. Convergence groups are Fuchsian groups. *Annals of Math.*, 136:447–510, 1992.
- [12] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)*, 55(2):331–358, 1987.
- [13] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d’après Mikhael Gromov*. Birkhäuser Boston Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.
- [14] É. Ghys, A. Haefliger, and A. Verjovsky, editors. *Group theory from a geometrical viewpoint*, River Edge, NJ, 1991. World Scientific Publishing Co. Inc.
- [15] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer, New York, 1987.
- [16] M. Kapovich. Lecture notes on geometric group theory. available at <http://www.math.ucdavis.edu/%7Ekapovich/eprints.html>.
- [17] M. Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.

- [18] M. Kapovich and B. Kleiner. Coarse Alexander duality and duality groups. Preprint 1999.
- [19] S. Maillot. Quasi-isometries of groups, graphs and surfaces. *Comment. Math. Helv.*, 76(1):29–60, 2001.
- [20] S. Maillot. Open 3-manifolds whose fundamental groups have infinite center, and a torus theorem for 3-orbifolds. *Trans. Amer. Math. Soc.*, 355(11):4595–4638 (electronic), 2003.
- [21] G. Mess. The Seifert conjecture and groups which are coarse quasiisometric to planes. Preprint.
- [22] Y. N. Minsky. Combinatorial and geometrical aspects of hyperbolic 3-manifolds. In *Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001)*, volume 299 of *London Math. Soc. Lecture Note Ser.*, pages 3–40. Cambridge Univ. Press, Cambridge, 2003.
- [23] J. Nielsen. *Jakob Nielsen: collected mathematical papers*. Contemporary Mathematicians. Birkhäuser Boston Inc., Boston, MA, 1986. Edited and with a preface by Vagn Lundsgaard Hansen.
- [24] P. Scott. There are no fake Seifert fibered spaces with infinite  $\pi_1$ . *Annals of Math.*, 117:35–70, 1983.
- [25] P. Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.
- [26] P. Tukia. Homeomorphic conjugates of Fuchsian groups: an outline. In *Complex analysis, Joensuu 1987*, volume 1351 of *Lecture Notes in Math.*, pages 344–353. Springer, Berlin, 1988.
- [27] P. Tukia. Convergence groups and Gromov’s metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.
- [28] F. Waldhausen. Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten. *Topology*, 6:505–517, 1967.
- [29] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.
- [30] H. Zieschang. *Finite groups of mapping classes of surfaces*, volume 875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

- [31] H. Zieschang, E. Vogt, and H.-D. Coldewey. *Surfaces and planar discontinuous groups*, volume 835 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. Translated from the German by John Stillwell.
- [32] H. Zieschang and B. Zimmermann. Über Erweiterungen von  $\mathbf{Z}$  und  $Z_2 * Z_2$  durch nichteuklidische kristallographische Gruppen. *Math. Ann.*, 259(1):29–51, 1982.