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Tameness and Marden's conjecture

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① Open manifolds

M 3-srf, orientable, $\partial M = \emptyset$, non-compact, irreducible

Def M is tame or almost compact or missing boundary manifold
if $\exists \bar{M}$ compact manifold with $M = \text{int}(\bar{M})$

Question: Which conditions ensure that M is tame??

Lemma:

M tame \Leftrightarrow

- (1) $\pi_1(M)$ is finitely generated
- (2) $M \setminus K$ is tame $\forall K \subset M$ compact submanifold.

Proof. ■

Theorem (Tucker)

M is tame $\Leftrightarrow \forall K \subset M$ compact submf then $\pi_1(M \setminus K)$ is f.g. ■

Why is the stress on f.g. and not on finitely presented??

Theorem (Scott)

If $\pi_1(M)$ is f.g. then it is f.p. as well. ■

Theorem (Scott)

If $\pi_1(M)$ is f.g. then $\exists C \subset M$ compact submf. such that $C \hookrightarrow M$ is a homotopy equivalence. ■

Def The submanifold C is said to be a compact core or Scott core of M .

Lemma: Every component of $M \setminus C$ is connected. ■

Components of $M \setminus C$ are called ends of M .

Observation: M is tame \Leftrightarrow every end is tame ■

② Failures

Some attempts to give topological conditions ensuring tameness?

1. Attempt: M contractible

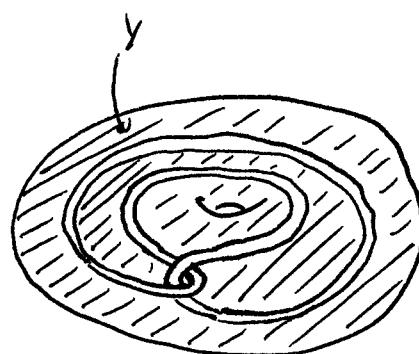
$$M_0 = D^2 \times S^1$$

$$M_1 = M_0 \cup Y \cong D^2 \times S^1$$

$$M_2 = M_1 \cup Y \cong D^2 \times S^1$$

$$\vdots$$

$$M_m = M_{m-1} \cup Y \cong D^2 \times S^1$$



$$M = \bigcup_n M_n = \varinjlim_n M_n \quad (\text{Whitehead manifold})$$

Claim: M is irreducible and contractible.

Claim: M is not tame.

Proof: (1) $\pi_1(M \setminus M_0)$ is not f.g.

(2) Haken finiteness

•

Remark: The same construction can be made with many different knots or with combination of knots.

⇒ There are uncountably many homeomorphism types of non tame contractible 3-manif.

• Other constructions.

① Using higher genus handlebodies

② Using handlebodies of higher and higher genus

③ „Connected sums“

A positive result:

Theorem (Whitehead)

$$M = \bigcup B_i, B_i \subset B_{i+1}, B_i \text{ ball } \mathbb{H}^3 \Rightarrow M = \mathbb{R}^3$$

Different formulation of Whitehead theorem:

„ M contractible & M a union of nested compact cores $\Rightarrow M$ is tame“

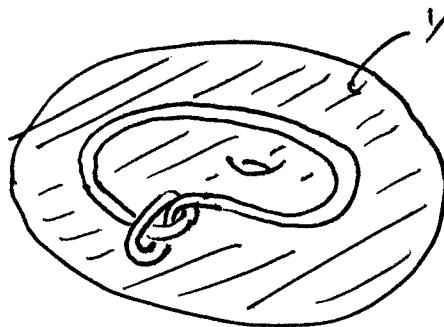
2. Attempt $M = \text{union of nested compact cores.}$

$$M_0 = D^2 \times S^1$$

$$M_1 = M_0 \cup Y \cong D^2 \times S^1$$

:

$$M_{n+1} = M_n \cup Y \cong D^2 \times S^1$$



$$M = \bigcup_n M_n$$

Claim M irreducible, $\pi_1(M) \cong \mathbb{Z}$, $M_i \hookrightarrow M$ compact core, M non-tame.

Remark: Again there are lots of counter-examples.

O.K., forget about Whitehead manifolds and similar things.

3. Attempt $\tilde{M} = \text{Universal cover of } M \cong \mathbb{R}^3$.

Claim If M is in 2. Attempt then $\tilde{M} = \mathbb{R}^3$.

Proof Using the fact that M contains a properly embedded essential disk one proves that $\tilde{M} = \text{union of nested balls.}$

③ Harden's Conjecture

Harden's tameness conjecture

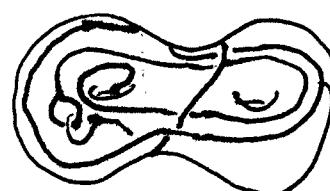
M hyperbolic and $\pi_1(M)$ f.g. $\Rightarrow M$ tame.

Recall that M hyperbolic $\Leftrightarrow \exists \Gamma \subset PSL_2 \mathbb{C} = \text{hom}_\mathbb{R} H^3$ torsion free with $M \cong H^3/\Gamma$

$\Leftrightarrow M$ admits a complete metric with constant sectional curvature $K = -1$.

4th failure M admits a Riemannian metric which is hyperbolic outside of a compact set.

Construction of counterexample using Thurston's hyperbolization thm.



History of Marden's conjecture

- Marden proved it if M is geometrically finite.
- Thurston in some cases (necessarily to prove the hyperbolization then)
- Bonahon if $\pi_1(M)$ is freely indecomposable.
- Canary proved lots of things
- Oshika, Minsky, Kleiner, Brock, Bromberg, Evans, ... proved some cases.
- Christmas 2004: Proof of the Tameness conjecture by Agol, Calegari-Gabai
- Other proofs by Soma, Choi.

Goal of these lectures: Prove Tameness conjecture.

Proof: Combination of Agol, Calegari-Gabai & Soma.

Standing assumption

M is a hyperbolic 3-manifold such that there is $\varepsilon > 0$ with $\kappa_{ij}M \geq \varepsilon \quad \forall p \in M$ (\Leftrightarrow If $\gamma \subset M$ is a closed geodesic then $C_M(\gamma) \geq 2\varepsilon$).

Remark about the assumption:

- If M has parabolics then one can either avoid them using a trick due to Agol or work 10 times more, but things are not really different.
- If M has no parabolics ~~then~~ but there are arbitrarily short geodesics then the proof remains word-by-word the same but one needs a little bit more of notation.

④ The geometrically finite case

$M = \mathbb{H}^3/\Gamma$ is geometrically finite if $CH(\Lambda_\Gamma)/\Gamma$ is compact. Here $\Lambda_\Gamma = \text{limit set}$ of Γ and $CH(\Lambda_\Gamma)$ is convex-hull of Λ_Γ in \mathbb{H}^3 .

$$\text{let } K = N_r(CH(\Lambda_\Gamma)/\Gamma) = \{x \in M \mid d(x, CH(\Lambda_\Gamma)/\Gamma) \leq r\}$$

Prop

$K \subset M$ is a compact cone, is convex and has C^1 -boundary.

Proof

$$K = N_r(CH(\Lambda_\Gamma)/\Gamma).$$

General facts about convex subsets of $\mathbb{H}^3 \Rightarrow$ claim holds. ■

Then (Marden)

If M is geometrically finite $\Rightarrow M$ is tame.

Proof

Using the convex-projection one gets that $M \cap K = \partial K \times \mathbb{R}$

Corollary (Thurston)

$M \cong \text{int } \bar{M}$, $\mathbb{Z}^2 \not\subset \pi_1(M)$, $\chi(\bar{M}) \neq 0$ and $N \rightarrow M$ covering with $\pi_1(N)$ f.g. $\Rightarrow M$ is tame.

Proof

① $\chi(\bar{M}) = \frac{1}{2} \chi(\partial \bar{M}) \neq 0 \Rightarrow$ some component of ~~not~~ $\partial \bar{M}$ is not a torus.

② Thurston's hyperbolisation theorem $\Rightarrow M$ admits geometrically finite hyperbolic metric with infinite volume.

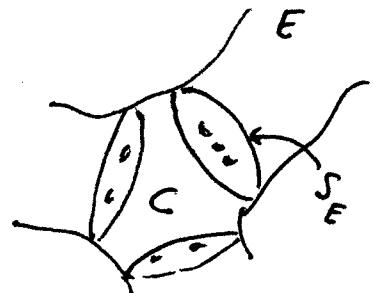
③ $\pi_1(N)$ is geometrically finite. ■

⑤ Reduction to the really interesting cases

$C \subset M$ compact core, $E = M \setminus C$ end of M .

(Recall that it suffices to show that each end of M is tame.)

Let $S_E \subset \partial C$ be the component of ∂C facing E



Prop (Bonahon)

$\exists C_E \subset C$ such that

$$\textcircled{1} S_E \subset C_E$$

$\textcircled{2} C_E$ is homeomorphic to either trivial interval bundle, handlebody or compression body

$$\textcircled{3} \pi_1(C_E) \hookrightarrow \pi_1(C) \cong \pi_1(M)$$

$\textcircled{4}$ ~~covered by~~ The end E lifts homeomorphically to the cover M_ϵ of M determined by $\pi_1(C_E)$.

Proof

Take a regular neighborhood of S_E and compress until boundary $\setminus S_E$ is incompressible.

Corollary

The tameness conjecture holds iff it holds for those hyperbolic 3-mfs whose compact core is homeomorphic to one of the following: trivial interval bundle over a closed surface, compression body or handlebody.

Another standing assumption

M hyperbolic 3-mf, $\text{inj}(M) \geq \varepsilon > 0$ and M admits a compact core C homeomorphic to a handlebody

Remark The case of the handlebody is the most difficult.

⑥ Tameness Criterium

Then

Assume that there is a sequence (S_i) of connected surfaces in M with the following properties:

① $\chi(S_i) = \chi(\partial C)$

② $0 \neq [S_i] \in H_2(M \setminus C, \mathbb{Z})$

③ $\forall K \subset M$ compact $\exists i_K$ with $S_i \cap K = \emptyset \quad \forall i \geq i_K$.

Then M is tame.

Corollary

If M admits a nested exhaustion by compact cores then M is tame.

Proof

If $C \subset C' \subset M$ are compact cores then $\partial C'$ is incompressible outside of C , it separates C of ∞ and $\chi(\partial C') = \chi(\partial C)$. ■

Proof of the tameness criterium:

1. Step Get embedded surfaces.

Then (Gabai) If S is a immersed surface in a 3-manif N then there is an embedded surface S' with no component homeomorphic to \mathbb{S}^1, RP^2 , with $\chi(S') \geq \chi(S)$ and with $[S] = [S'] \in H_2(N, \mathbb{Z})$. ■

Applying Gabai's theorem to $S_i \subset N_\varepsilon(S_i)$ we get sequence (S'_i) of embedded surfaces in M which exit every compact set, with $0 \neq [S'_i] \in H_2(M, \mathbb{Z})$ and with $\chi(S'_i) \geq \chi(S_i)$.

Lemma

~~If $S \subset M$ is nonseparable~~

Lemma

If $\Sigma \subset M \setminus C$ is incompressible then $\Sigma = \partial C'$ where $C' \subset M$ is a handlebody with $C \subset C'$ and $\chi(C') \leq \chi(C)$.

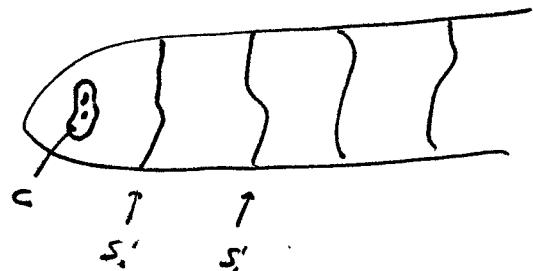
Lemma implies that the surfaces S_i' are incompressible.

Theorem (Waldhausen)

S, S' homotopic incompressible embedded surfaces with $S \cap S' = \emptyset$
 $\Rightarrow S, S'$ bound trivial interval bundle. \blacksquare

~~Goal~~

Goal: Use Waldhausen's Theorem.



2. Step Get the surfaces to be incompressible.

Def A link $L \subset M$ is algebraically disk burting if $\vee \pi_1(M) = A * B$
 $(A, B \neq \text{Id}) \exists r \in L$ with $r \notin A, B$.

Lemma

If y_1, \dots, y_m are generators of $\pi_1(M)$ then $L = \langle y_1, y_2, \dots, y_m, \gamma_1, \gamma_2, \dots, \gamma_n \rangle$ is a disk burting link.

Proof

See what happens in $H_1(M)$. \blacksquare

Taking geodesic representative of $\langle y_1, y_2, \dots, y_m, \gamma_1, \gamma_2, \dots, \gamma_n \rangle$ one gets a geodesic link L which is algebraically disk-burting.

Remark If the obtained geodesics are not ~~seaked~~ disjoint one can perturb the metric so that they are.

~~longing~~

~~the link to make~~

Lemma

The surfaces S_i' are incompressible in $M \setminus L$. \blacksquare

3. Step $\exists D$ such that S_i' is homotopic in $M \setminus L$ to a surface X_i' with the following properties $\forall i$

- 1) $\text{diam } X_i' \leq D$
- 2) $X_i' \cap N_1(L) \neq \emptyset$.

Assuming the 3. Step we conclude the proof of the tameness criterium.

Let $U \subset M$ be a compact irreducible sub-manifold of M which contains $\mathbb{B}_{2D+1}(L)$, and set $Z_i = X_1 \cup \dots \cup X_i$.

Then (Freedman - Hass - Scott)

If an incompressible surface X is homotopic to an embedded surface then there is an embedded surface S homotopic to X and contained in $N_1(S)$.

The surface Z_i is incompressible and homotopic to the embedded surface $S_1 \cup \dots \cup S_i$ in $M \setminus L$. \Rightarrow There is for all i an embedded surface $\tilde{Z}_i \subset U \setminus L \subset M \setminus L$ which is homotopic to $S_1 \cup \dots \cup S_i$.

Haken finiteness \Rightarrow Infinitely many of the S_i' are homotopic.
Waldhausen \Rightarrow The end of M is tame.

It remains to prove the claim made in 3. Step. This is the only point in the proof where some geometry is used.

(7) Simplicial hyperbolic surfaces

Def M hyperbolic manifold, S closed surface, $f: S \rightarrow M$ is a simplicial hyperbolic surface if the following conditions hold:

- There is a triangulation of S such that $\forall \Delta$ 2-cell the image $f(\Delta)$ is a hyperbolic triangle.
- The sum of the angles at every vertex is $\geq 2\pi$.

If $f: S \rightarrow M$ is a simplicial hyperbolic surface we consider S always with the pulled-back metric.

Fact: If $f: S \rightarrow M$ is a simplicial hyperbolic surface then ~~the~~ the universal cover \tilde{S} of S is a CAT(-1) space.

Proposition

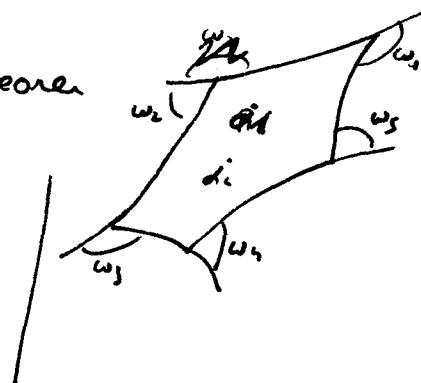
$f: S \rightarrow M$ simplicial hyperbolic surface

- ① any two points in \tilde{S} are joined by a unique geodesic segment.
- ② $\text{Vol}(S) \leq 2\pi |\chi(S)|$
- ③ If $\text{inj}_x(S) \geq \varepsilon \Rightarrow \text{Vol}(B_x(\varepsilon)) \geq \pi \varepsilon^2$

Proof

Based on Gauß-Bonnet theorem

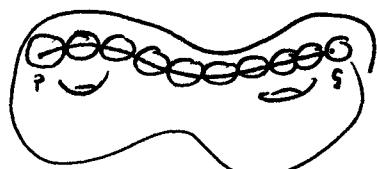
$$\begin{aligned} \text{Vol}(\Delta) &= -2\pi + \sum_i (\pi - \omega_i) \\ &\leftarrow -\sum_i (\alpha_i - 2\pi) \end{aligned}$$



Bounded diameter lemma

If $f: S \rightarrow M$ is a simplicial hyperbolic surface ~~equipped with~~ $\text{inj}(S) \geq \varepsilon \Rightarrow \text{diam}(S) \leq \frac{2}{\varepsilon^2} |\chi(S)|$

Proof



$$\# \text{Balls} \geq \frac{\text{diam}}{\varepsilon}$$

$$2\pi |\chi(S)| \geq \text{vol}(S) \geq \pi \varepsilon^2 \# \text{Balls}$$

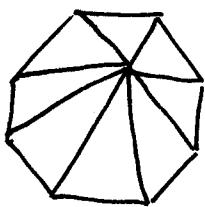
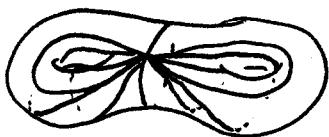
How does one get simplicial hyperbolic surfaces?

Prop

If N is a hyperbolic 3-manifold without cusps and $f: S \rightarrow N$ is such that $f_*: \pi_1(S) \hookrightarrow \pi_1(N)$ then f is homotopic to a simplicial hyperbolic surface $f': S \rightarrow N$.

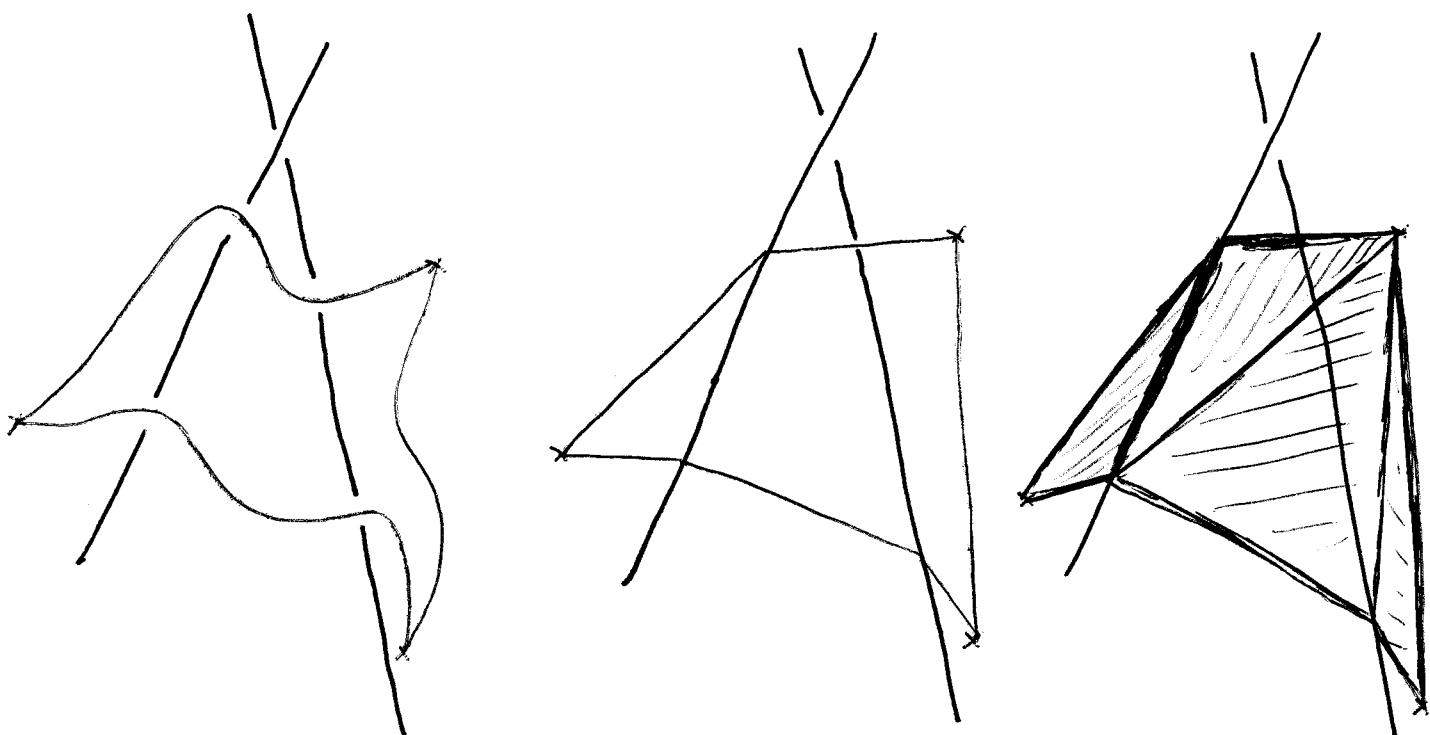
Proof

Construct appropriate triangulation and pull straight.



This proposition is of no use for us since the surfaces S' we are interested in are compressible.

Idea Use the link L as a "banner" = shrink-wrapping.



Theorem (Soma)

M hyperbolic 3-manifolds without parabolic, $\Delta \subset M$ geodesic link, $f: S \rightarrow M \setminus \Delta$ 2-incompressible rel. to Δ . Then there is a homotopy $F: \Sigma \times [0,1] \rightarrow M$ s.t.

- i) $F(x, 0) = f(x)$
- ii) $F(\Sigma \times \{0,1\}) \cap \Delta = \emptyset$
- iii) $F(\Sigma \times \{1\})$ is a simplicial hyperbolic surface.

Moreover, if S is compressible then F can be chosen s.t. $F(\Sigma \times \{1\}) \cap \Delta \neq \emptyset$.

Proof

Schmitz-wrap \bullet

Remark Formally, the proof is based on the fact that the completion of the universal cover of $M \setminus \Delta$ is a CAT(-1) metric space.

End of the 3. Step of the proof of the tameness criterion

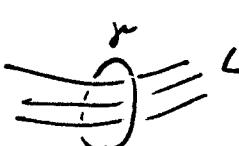
By step 1 + step 2 we have surfaces $S_i' \subset M$ which go to infinity, are doubly incompressible in $M \setminus L$, compressible in M and $X(S_i') = X(\partial C)$. By Soma's theorem the surface S_i' is homotopic in $M \setminus L$ to a surface S_i'' with

$$\text{diam } S_i'' \leq 1 + \frac{2}{\varepsilon^2} |X(S)| \stackrel{\text{def}}{=} D$$

where $\varepsilon = \min \{ \text{inj}(M), l_M(r) \}$ where r

bounds an essential

singular disk in M which meets L at least 3 times }



The surface S_i'' is obtained taking stopping Soma's homotopy short before it ends. \bullet

⑧ What does one get from tameness?

Theorem (Canary)

If $E \subset M \setminus C$ is a end of a hyperbolic 3-manifold then either E is geometrically finite or there is a sequence S_i of surfaces of negative curvature of bounded genus which exit the end.

Proof (following Calegari - Babai)

- ① If E is not geometrically finite then there is a sequence γ_i of closed geodesics such that $\forall K \subset E$ compact $\exists i_K$ with $\gamma_i \cap (E \setminus K) \neq \emptyset$.
- ② Take surfaces S_i homotopic in $M \setminus (\cup \gamma_i)$ to the missing boundary and stretch-wrap them to get X_i .

The Bounded diameter lemma implies that $X_i \rightarrow \infty$.

Corollary (Shapiro's measure theorem) [Shapiro, Thurston, Canary] [Thurston, Canary]

If H^3/Γ is tame then either $\Lambda_\Gamma = \mathbb{S}^2$ or Lebesgue(Λ_Γ) = 0.

Corollary (Covering Theorem) [Thurston, Canary]

If M is tame ~~and~~, $E \subset M$ is not geometrically finite and $M \rightarrow N$ is a Riemannian cover then either the covering is finite-to-one when restricted to E or N is compact and finitely covered by a hyperbolic manifold fibering over the circle.

Corollary of the corollary.

If N is a closed hyperbolic manifold and $\Gamma \subset \pi_1(N)$ is f.g. then either it commensurable to a virtual fiber or H^3/Γ is convex-cocompact and hence a quasi-convex subgroup of $\pi_1(N)$.

⑨ Proof of the tameness conjecture

Theorem (Agol, Calegari-Gabai)

Every hyperbolic 3-manifold will f.g. π_1 is tame.

Recall our assumptions:

- $\pi_1(M)$ is free
- $\text{inj}(M) \geq \varepsilon > 0$

Let $C \subset M$ be a compact core, L an algebraically disk-bounding geodesic link and γ_i a sequence of geodesics in M s.t. $\forall K \subset M$ compact there is i with $\gamma_i \notin K$; set $L_i = L \cup \gamma_i$.

Given i let $U_i \subset M$ be a compact submanifold s.t. its component of $M \setminus U_i$ is bounded (there are no silly holes), $C \subset U_i$ and U_i contains the track of a homotopy of L_i into C .

Lemma (Myers)

Up to changing the compact core C to a new one C_i one can assume that ∂U_i is incompressible in $M \setminus L_i$.

Proof

Enlarging U_i one can assume that ∂U_i is incompressible in $M \setminus U_i$; assume that this is the case and that there is a disk $\Delta \subset U_i \setminus L_i$ (say, separating). Consider the associated decomposition $\pi_1(U_i) = A + B$ and the homomorphism

$$\pi_1(C) \rightarrow \pi_1(U_i)$$

The assumption that L is algebraically disk-bounding implies that $\pi_1(C)$ is mapped into one of the factors. In particular one can homotope C to a compact core C_i which does not meet the disk. The homotopy of L_i into C_i can also be moved out of the disk.

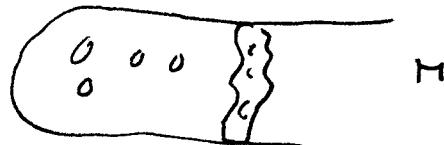
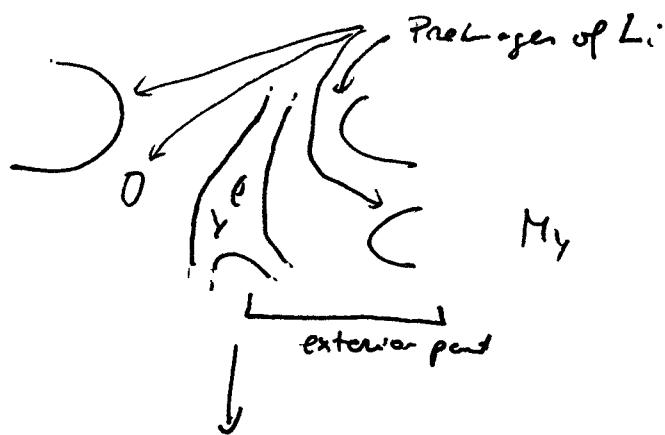
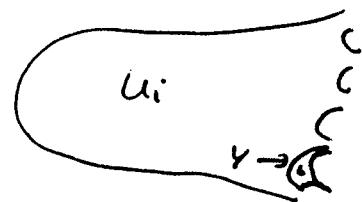
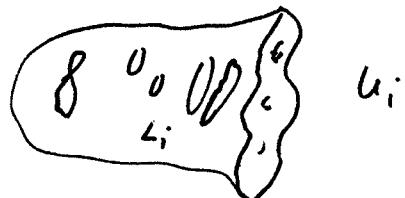
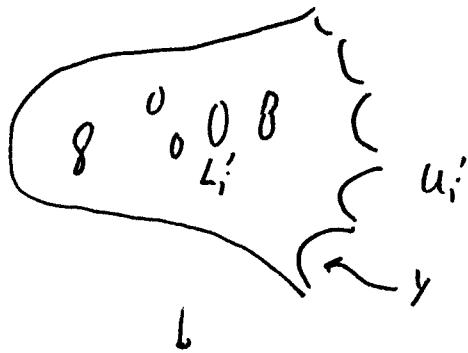
Let $U'_i \rightarrow U_i$ be the cover corresponding to $\pi_1(C_i)$ and let L'_i be lift of L_i .

By Thurston's theorem U'_i is tame and hence homeomorphic to the interior of G_i .

Choose a surface $S_i \subset U'_i$ which is homotopic in $U'_i \setminus L'_i$ to the missing boundary. One would like to shrink-wrap it ~~but~~ in U'_i but this is not possible since U'_i is mostly like a CAT(1) space.

Idea Complete U'_i

If Y is a component of $\partial U'_i$ then it corresponds to a cover of ∂U_i . Take the cover of $M \setminus L_i$ corresponding to Y [this is possible since $\pi_1(Y) \subset \pi_1(\partial U_i) \hookrightarrow \pi_1(M \setminus L_i)$] In this cover one finds the surface Y again and hence, chopping along Y , one finds a piece to be glued into Y .



exterior part of M_Y

Doing this for every component γ of ∂U_i ~~one gets~~ and completing one gets a CAT(-1) space V_i which contain U_i , s.t. $\pi_1(U_i \setminus L'_i) \hookrightarrow \pi_1(V_i \setminus L'_i)$ and such that there is ~~an~~ a branched cover $V_i \rightarrow M$.

In V_i we can shrink-wrap S'_i to a simplicial hyperbolic surface X_i which, in V_i , separates homologically the link L'_i from ∞ .

By construction the projection Σ_i of X_i to M is a simplicial hyperbolic surface as well which also separates L_i from ∞ (homologically).

We have the desired surfaces X_i needed to apply the tameness criterium. This concludes the proof of the tameness conjecture. \square



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