HEEGAARD SPLITTINGS, PANTS DECOMPOSITIONS, AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS

JEFFREY BROCK

BROWN UNIVERSITY

Joint work with

Juan Souto
U. Toulouse,
U. Chicago,
and U. Michigan

The Goal

In this talk we will tie together:

- Heegaard splittings
- Pants decompositions of surfaces
- Volumes of hyperbolic 3-manifolds

Or attempt to answer the question:

is there life after Perelman?

CERTAINLY YES

Existence and uniqueness of hyperbolic structures on manifolds give little information about their structure.

Motivating Question:

Given a combinatorial description of a 3-manifold that admits a hyperbolic structure what can be said about the geometry of that structure?

Effective Rigidity

For M a closed or finite volume hyperbolic manifold, Mostow rigidity guarantees uniqueness of the hyperbolic structure. More effective information could include:

- The volume of M,
- The length spectrum for closed geodesics in M, or
- A bi-Lipschitz model for M.

A Sample Case

The 3-manifold M *fibers over the circle with fiber S and monodromy $\psi \in$* Diff$^+(S)$ if

$$M \cong M_\psi = S \times [0,1]/(x,0) \sim (\psi(x),1).$$

Theorem. (Thurston) *The 3-manifold M_ψ is hyperbolic if and only if ψ is pseudo-Anosov.*

Definition. *ψ is pseudo-Anosov if no power of it preserves the isotopy class of any essential simple closed curve on S.*

A SAMPLE THEOREM

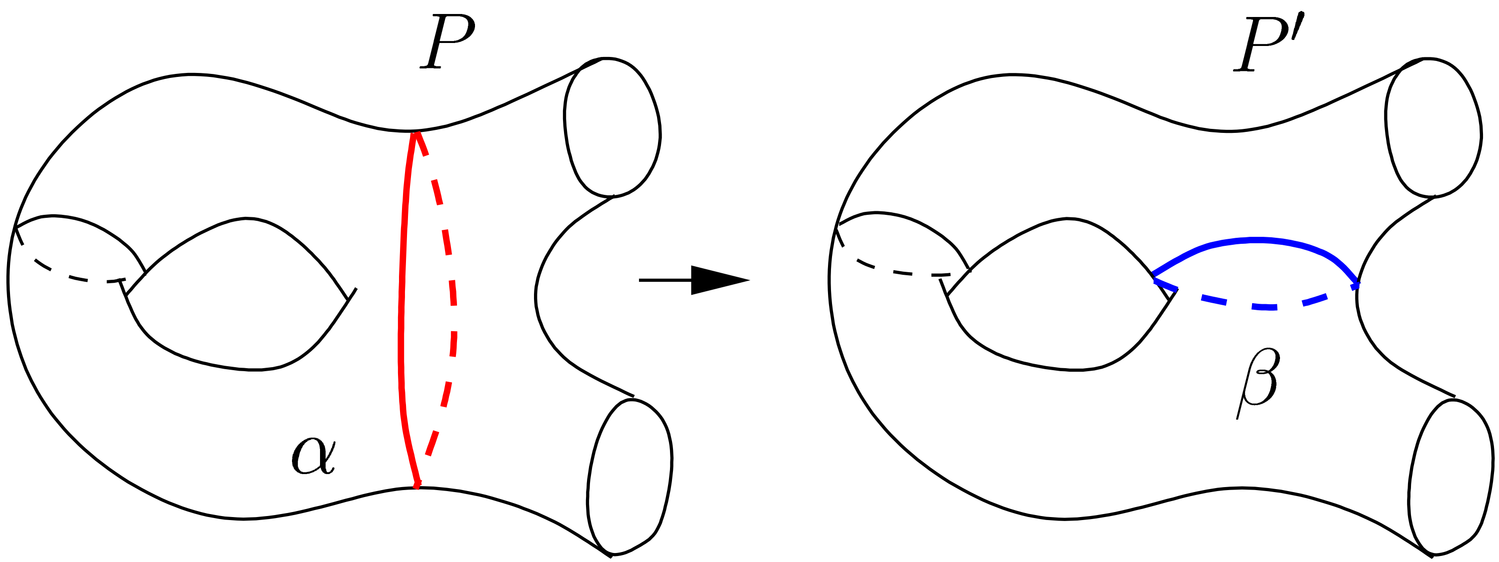
Theorem. (B-'01) *There is a K depending only on S so that*

$$\mathrm{vol}(M_\psi) \in [K^{-1}, K] ||\psi||_P.$$

Here, the quantity $||\psi||_P$ measures the *translation distance* of ψ on the *pants graph*.

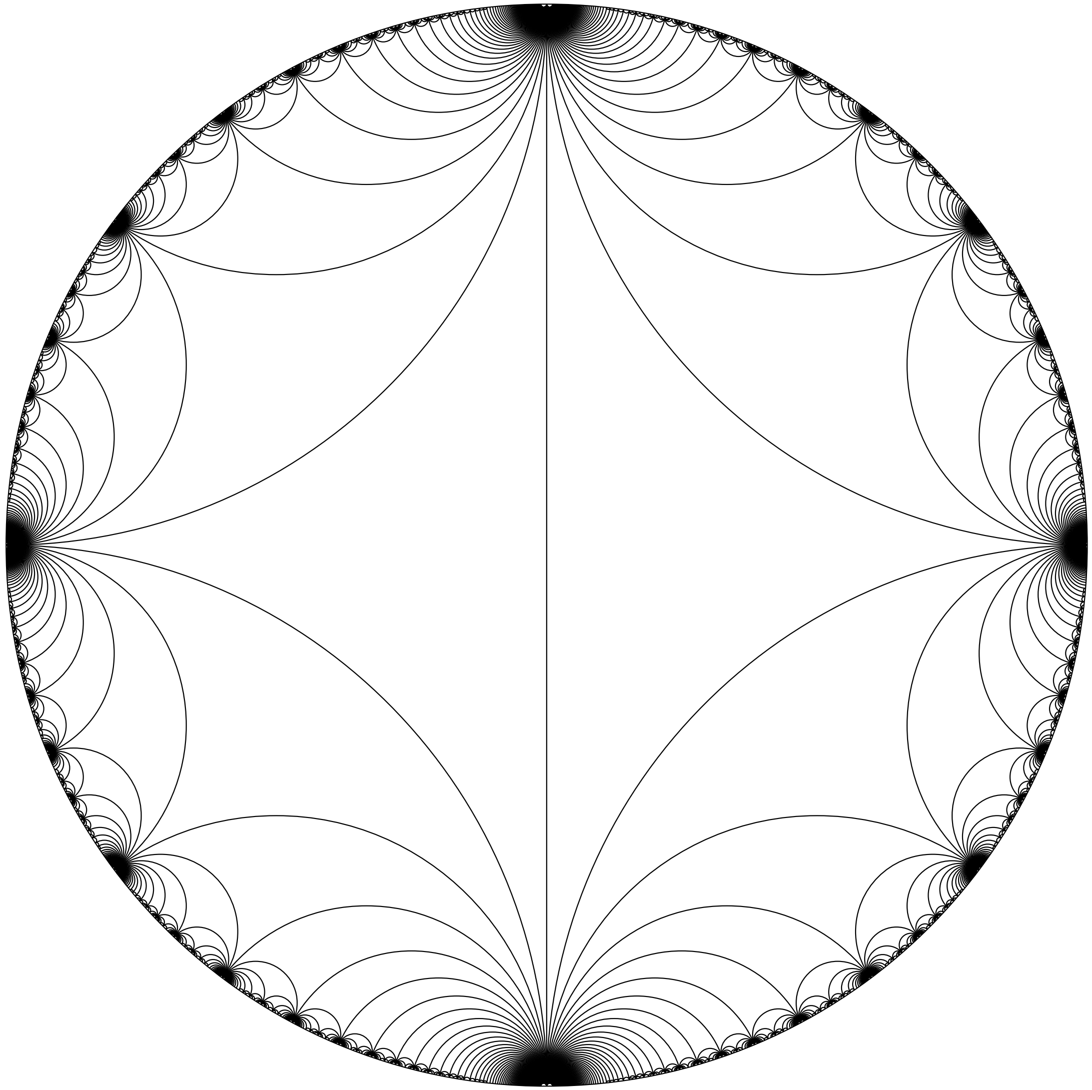
The Pants Graph

The pants graph $P(S)$ has a vertex for each pants decomposition of S and edges joining vertices whose pants decompositions differ by an elementary move:



a type-2 move

The Pants Graph



Translation Length

Giving each edge length 1, we obtain a metric space and a notion of *pants distance*

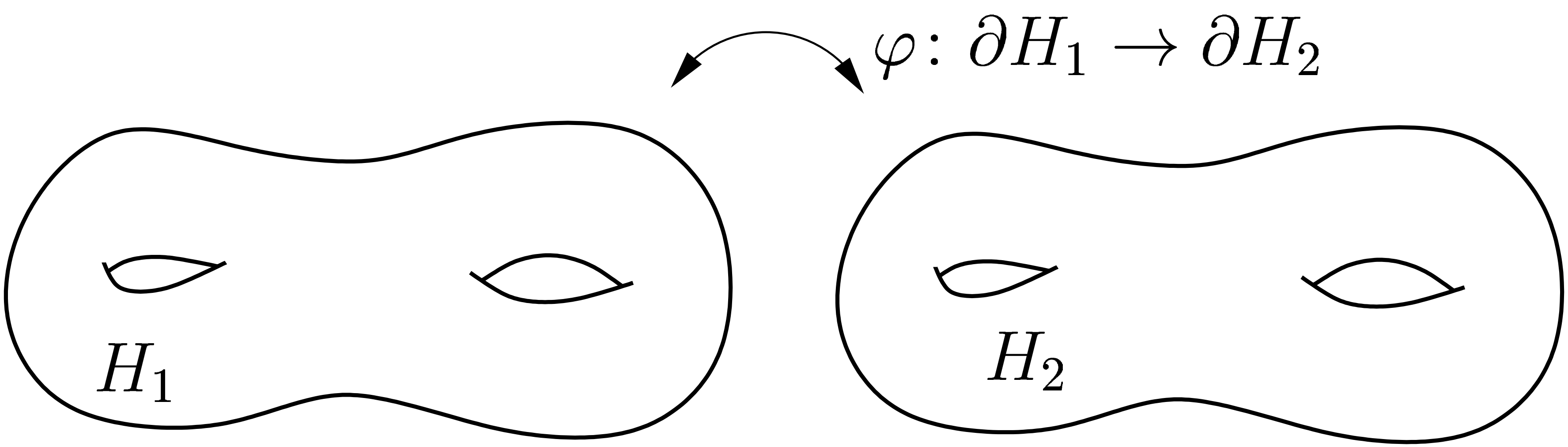
$$d_P \colon P^0(S) \times P^0(S) \to \mathbb{Z}$$

between pants decompositions of S.

- The mapping class group acts by isometries of $P(S)$.
- The translation length $\|\psi\|_P$ of ψ is defined to be the min of $d(P, \psi(P))$ over all P.

HEEGAARD SPLITTINGS

Heegaard splittings represent another instance where sufaces give a combinatorial description of 3-manifolds.



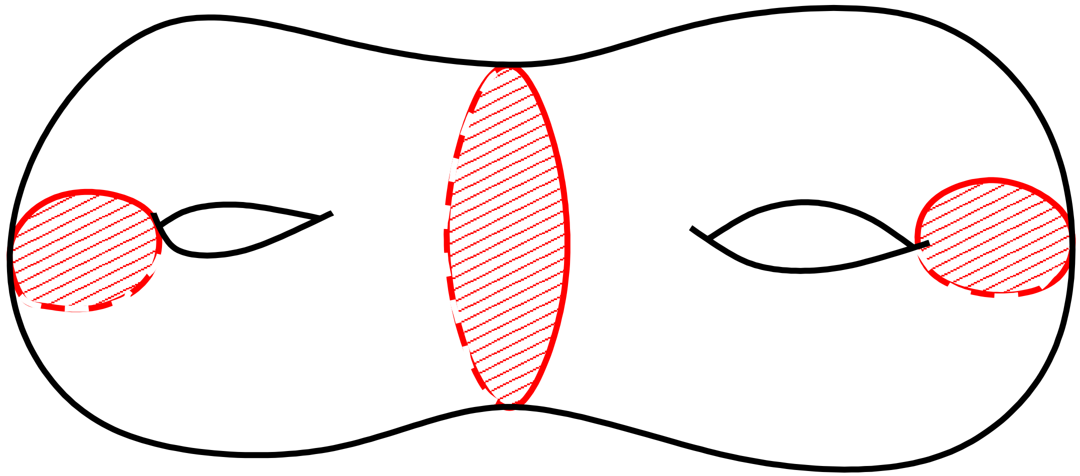
Can we estimate the volume of a hyperbolic Heegaard splitting via its combinatorics?

MERIDIANS

One question:

How do pants decompositions arise?

A *meridian* on H is an essential simple closed curve on ∂H that bounds a disk in H.



There are many meridional pants decompositions of ∂H.

Handlebody Sets

Consider the set $\mathcal{M}(H) \subset P(\partial H)$ consisting of pants decompositions consisting of meridians.

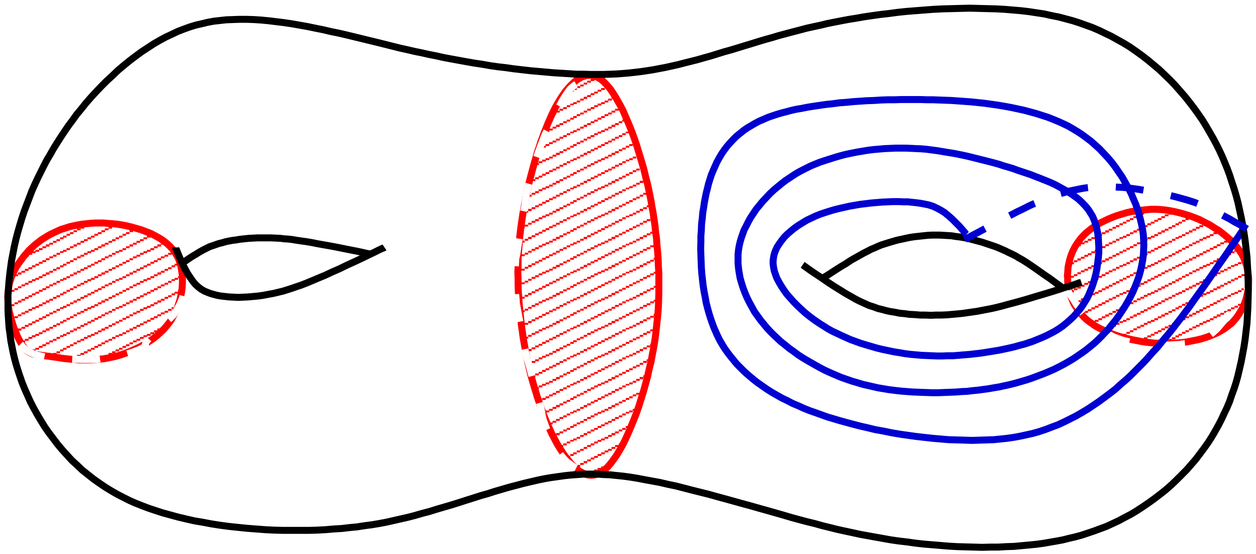
- Consider a hyperbolic Heegaard splitting $M = H_1 \cup H_2$
- One is tempted to guess that

$$d_P(\mathcal{M}(H_1), \mathcal{M}(H_2))$$

may bear some relation to volume...

NOT QUITE...

Dehn surgery about the core of a handle can increase distance to $\mathcal{M}(H)$ arbitrarily, but volume remains bounded.

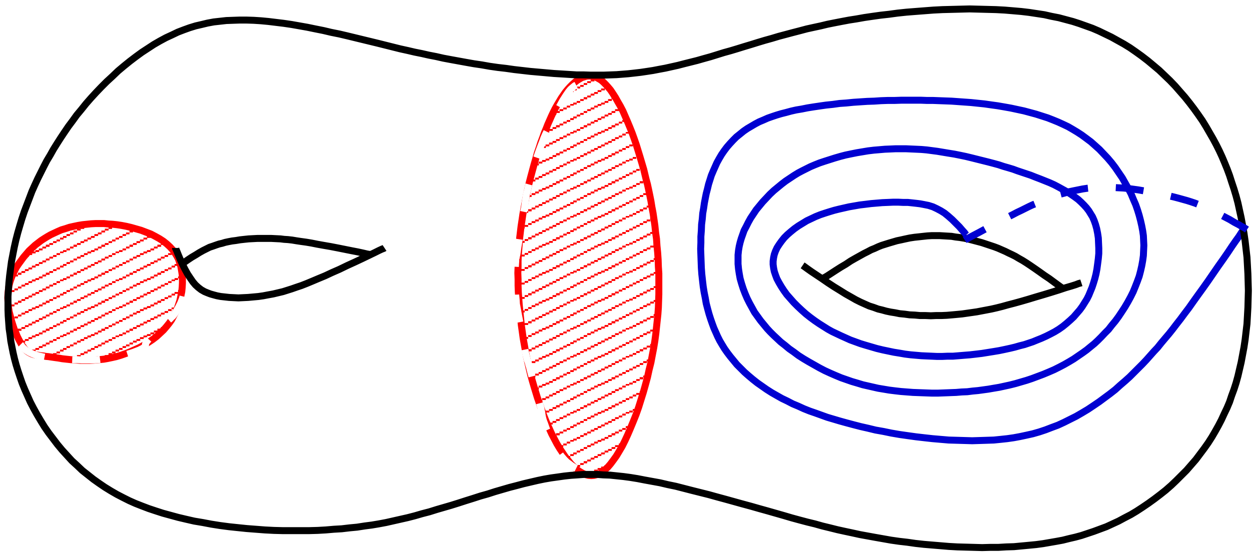


(Thanks B. Martelli...)

Handlebody Sets II

As such, we modify our handlebody set:

Definition. *Let $\mathcal{P}(H)$ denote the set of pants decompositions of ∂H so that each $P \in \mathcal{P}(H)$ contains a set of meridians whose meridian disks decompose $\text{int}(H)$ into solid tori.*



HEEGAARD VOLUMES

Then we have the following:

Theorem. (B-Souto) *Let* $M = H_1 \cup H_2$ *be a strongly irreducible Heegaard splitting of* $M = \mathbb{H}^3/\Gamma$. *Then there is a* K *depending only on* $\partial H_i = S$ *so that*

$$\mathrm{vol}(M) \in [K^{-1}, K] d_P(\mathcal{P}(H_1), \mathcal{P}(H_2)).$$

The Heegaard splitting $M = H_1 \cup H_2$ is *strongly irreducible* if every meridian of H_1 intersects every meridian of H_2 at least *twice* on S.

The Argument

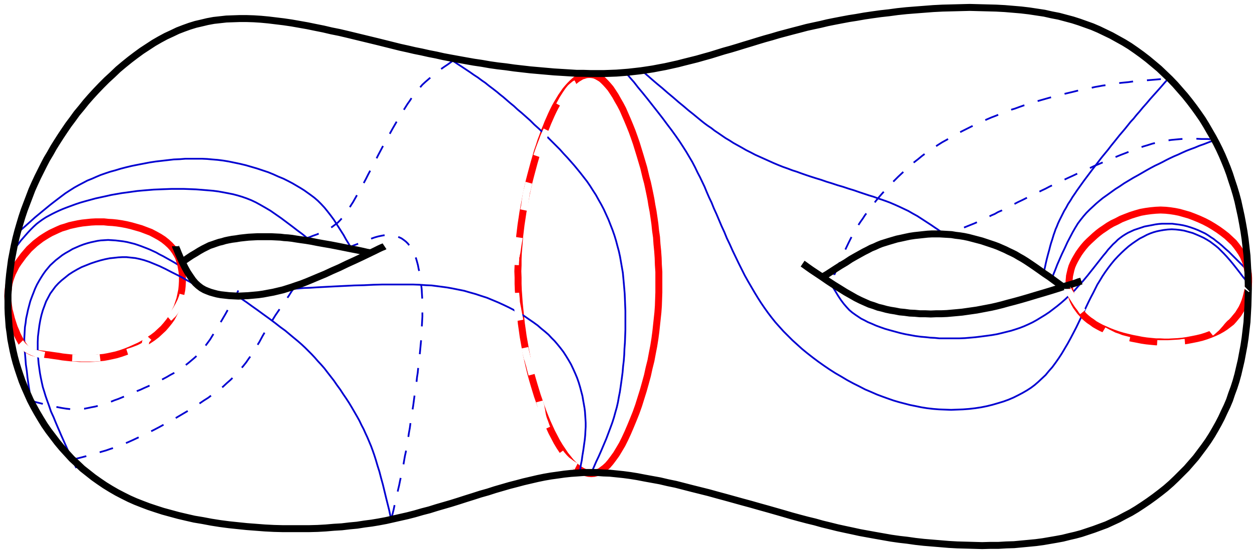
- The condition of strong irreducibility allows for a cleaner statement.
- In the *weakly reducible* case, there is a decomposition into strongly irreducible 'pieces' each of which admits a similar estimate.
- Here are some elements of the proof.

THE UPPER BOUND

- To bound of volume from above in terms of $\delta(H_1, H_2) = d_P(\mathcal{P}(H_1), \mathcal{P}(H_2))$ is a fairly standard argument using *ideal triangulations*.
- An ideal triangulation adapted to the pants decompositions in a shortest path in $P(S)$ joining $\mathcal{P}(H_1)$ to $\mathcal{P}(H_2)$ has a bounded number of tetrahedra for each elementary move.

IDEAL TRIANGULATIONS

The triangulation interpolates between ideal surface triangulations.



Each ideal hyperbolic tetrahedron has volume at most that of the regular ideal tetrahedron, giving the upper bound.

The Lower Bound

- The lower bound is somewhat more delicate.
- Strong irreducibility provides a minimal Heegaard surface X (Pitts-Rubinstein).
- Then $M \setminus X$ is a union of two hyperbolic handlebodies with boundary a suface of negative curvature.

THE HANDLEBODY CASE

This essentially reduces the theorem to estimating the volume of the convex core of a hyperbolic handlebody $N = \mathbb{H}^3/\Gamma \cong \mathrm{int}(H)$.

Theorem. (Bers) *There is an L depending only on ∂H so that $\partial CC(N)$ admits a pants decomposition of length less than L.*

If $\partial CC(N) = Y$, then we let P_Y be this bounded length pants decomposition.

Then we have...

Handlebody Volume

Theorem. (B-Souto) *The convex core volume of N satisfies*

$$\text{vol}(CC(N)) \in [K^{-1}, K] d_P(P_Y, \mathcal{P}(H)).$$

Similar statements exist for compression bodies.

Bounded Geometry Case

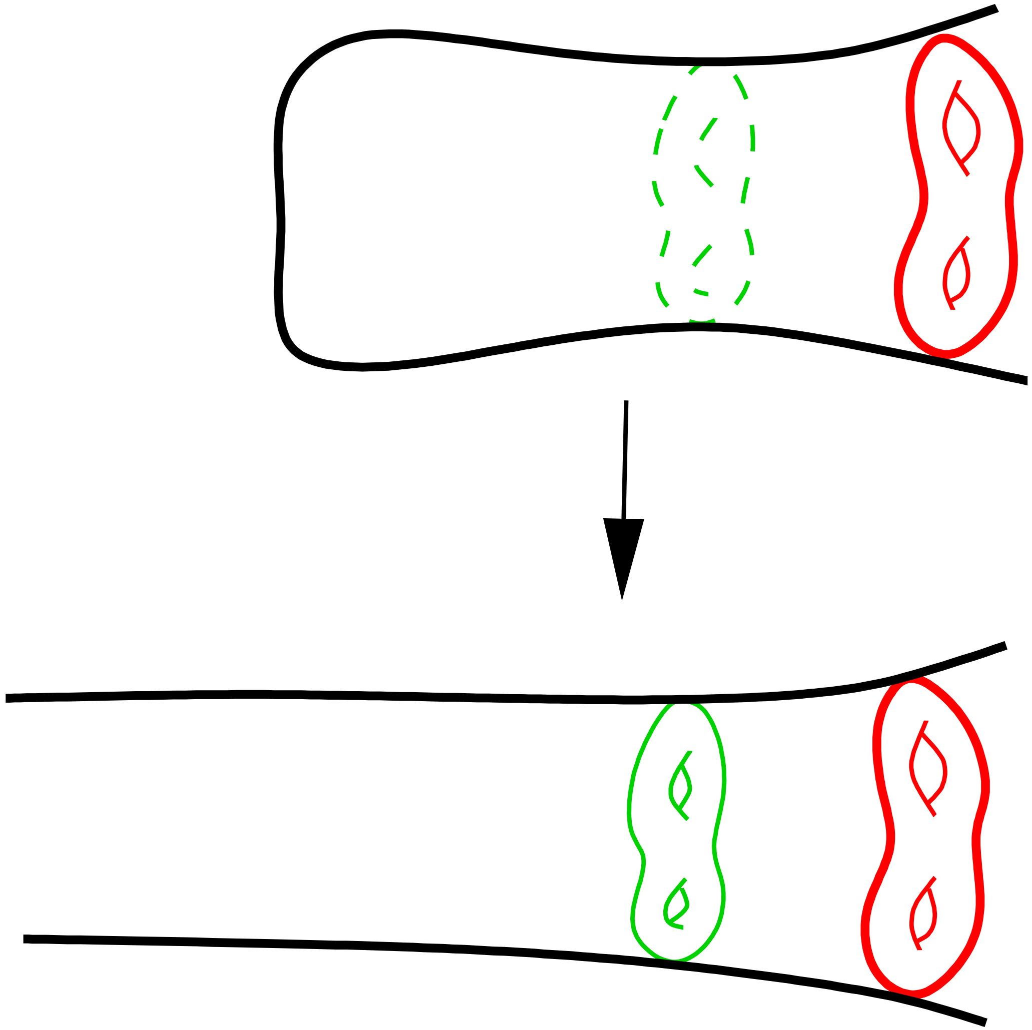
When inj$(N) > \epsilon_0$ we have the following:

Claim. *There exist $D, L > 0$ so that if the shortest meridian on Y has length at least L there is an embedded uniformly Lipschitz hyperbolic surface Y' in $CC(N)$ so that*

- Y and Y' cut off a product with volume at least 1, and
- We have $d_P(P_Y, P_{Y'}) \leq D$.
- (A geometric limit argument...)

Geometric Limits

The worse things are the better they are...



Bounds from Below

- Proceeding inductively, we bound the pants distance by the volume.
- When N has short geodesics, we apply a theorem of Souto guaranteeing that short geodesics are unknotted and unlinked (they can all be put on disjoint *level surfaces*)
- Drilling (effectively) reduces to the bounded geometry case.

Other Questions

- Can one predict short curves from the combinatorics of a Heegaard splitting?
- Is there a natural model manifold arising from these combinatorics? (cf. B-Canary-Minsky)
- Can combinatorics yeild hyperbolic structure? (cf. Namazi)

Cone-Manifolds

Recent collaborations in cone-manifolds: