

Dehn surgery along links in 3-manifolds

N compact orientable

$\partial N = T_1 \cup \dots \cup T_k$ disjoint union of tori

A Dehn filling of N is the following operation:

take H_1, \dots, H_k distinct solid tori

$\varphi_i: \partial H_i \xrightarrow{\sim} T_i$ homeomorphisms $\forall i$

and glue along the φ_i 's to get a closed

$M = N \cup H_1 \cup \dots \cup H_k / \sim$ with $x \sim \varphi_i(x)$

A Dehn surgery on a link $L \subset M$
(contained in a closed M)

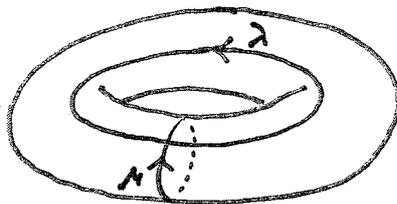
is a Dehn filling of $N := M \setminus N(L)$

Classical result:

Teo: Every closed M is obtained via Dehn surgery
along some link $L \subseteq S^3$

How to encode Dehn fillings:

take on each $H_i (T_i, \mathbb{Z})$ an arbitrary pair of generators (μ_i, λ_i)



Every oriented simple closed curve in T_i

gives some $p_i \mu_i + q_i \lambda_i \in H_1$ with $(p_i, q_i) = 1$

$$\{\text{simple closed curves}\} \mapsto \left\{ \frac{q_i}{p_i} \right\} = \mathbb{Q} \cup \{\infty\}$$

The meridian of H_i is glued via φ_i to some curve $\frac{q_i}{p_i}$

Fact:

The Dehn filling depends only on the numbers $(\frac{q_1}{p_1}, \dots, \frac{q_k}{p_k})$

Therefore the Dehn fillings space consists of all rational points in the k -torus $(S^1)^k$

We give this space the induced topology.

(not depend of choices of generators μ_i, λ_i)

Question: Which closed M we can obtain from a fixed N ?

... And how many times we get it?

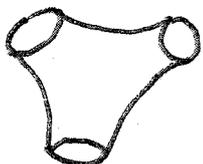
examples:

1. $N =$ solid torus:

we get all lens spaces, all but $S^2 \times S^1$

infinitely many times, because $L(p, q)$

$$\parallel \begin{matrix} L(p, q) \\ L(p, q+np) \end{matrix} \forall n$$

2. $N =$  $\times S^1$:

we get many Seifert manifold

$$(S^2, (p_1, q_1), (p_2, q_2), (p_3, q_3))$$

$\parallel \parallel$

$$(S^2, (p_1, q_1), (p_2, q_2 + np_2), (p_3, q_3 - np_3)) \forall n$$

so ∞ many times if $p_i \neq 0 \forall i$

3. $N =$ hyperbolic with 1 cusp:

we get each M finitely many times.

Pf:

take $q_i/p_i \in S^1$ all distinct

Thurston:

$N(q_i/p_i)$ is hyperbolic $\forall i \gg 0$

and $\text{Vol}(N(q_i/p_i)) \rightarrow \text{Vol}(N)$
(on a subsequence)

Try to extend somehow this argument for any N :

Def: N is geometrizable if it is irreducible and decomposed by its JSS into geometric pieces.

Def: the volume of N is the sum of the volumes of these pieces, where:

- if ~~it's~~ it's Sol or Seifert with $\chi \geq 0$, set $\text{Vol} = 0$
- if it is Seifert with $\chi < 0$, set $\text{Vol} = -\chi$
- if hyperbolic, take usual volume

Def: the Euler number of N is:

- if N is Seifert with $\chi > 0$, set $e(N) = |\pi_1(N)|$ if finite and " = 0 if infinite
- if N is Seifert with $\chi \leq 0$, set $e(N) = |\chi|$
- if N is hyperbolic, set $e(N) = 0$
- if N is Sol torus-fibered with monodromy ψ , set
$$e(N) = \min \{ \Delta(m, \psi(m)) + \Delta(l, \psi(l)) \}$$
 among pairs (m, l) with $\Delta(m, l) = 1$
- otherwise, N has a nontrivial JSS, and $e(N)$ measures how complicatedly the blocks are glued together

Thm: $\mathcal{M} = \{ \text{geometrizable 3-manifolds} \}$

1. $\text{Vol}(\mathcal{M})$ is well-ordered
2. the set $\{ e(M) \mid \text{Vol}(M) = v_0 \}$ does not accumulate $\forall v_0$
3. there are finitely many $M \in \mathcal{M}$ with fixed $(\text{Vol}(M), e(M))$

Theorem: N irreducible, bounded by k tori;

$\{s_i\}_{i \in \mathbb{N}}$ essential sequence of rational points in the Dehn filling space $\cong (S^1)^k$;

suppose $s_i \rightarrow \lambda = (\lambda_1, \dots, \lambda_k)$ (not nec. rational) such that:

1. no λ_j is a slope bounding a disc in N
2. no $\lambda_j, \lambda_{j'}$ (distinct) bound an annulus in N

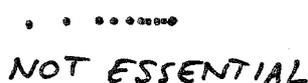
then:

on a subsequence, every $N(s_i)$ is geometrizable and

either $\text{Vol}(N(s_i)) \nearrow \text{Vol}(N)$ or $e(N(s_i)) \nearrow \infty$

Remarks:

1. Essential means $s_i = (s_{i1}, \dots, s_{ik})$ with $s_{ij} \neq \lambda_j \forall i$
i.e. for $k=2$:



2. Since $(S^1)^k$ is compact, every set of infinite points contains a subsequence

3. slope := simple closed curve

4. both hypothesis are needed:

take $N = \text{solid torus}$ $N\left(\frac{q+np}{p}\right) = L(p, q+np) \cong L(p, q)$
 $\searrow \infty$ meridian

and $N = \text{Seifert with two boundary components}$

$N\left(\frac{q_1+np_1}{p_1}, \frac{q_2-np_2}{p_2}\right)$ is the same mfd $\forall n$

pf (sketch):

1. if N is solid torus, I-bundle, Seifert: check
example: if N Seifert with base orbifold having $\chi < 0$

$$N(s^i) = N\left(\frac{q_1^i}{p_1^i}, \dots, \frac{q_k^i}{p_k^i}\right) \quad \frac{q_j^i}{p_j^i} \rightarrow \lambda_j$$

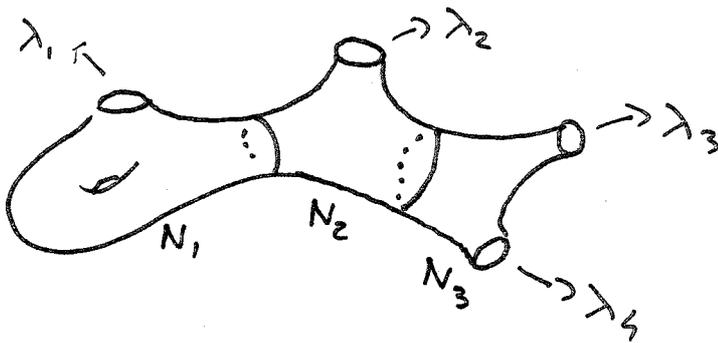
by assumption: at most one λ_j equals ∞

if none: $|p_j^i| \rightarrow \infty \quad \forall j \Rightarrow \chi_i \rightarrow \chi \Rightarrow \text{Vol}_i \nearrow \text{Vol}$

if one: $e = \left| e(N) + \sum_j \frac{q_j^i}{p_j^i} \right| \nearrow \infty$

2. induction on number of blocks in the JSJ:

$$N = N_1 \cup \dots \cup N_\epsilon \text{ blocks}$$



on a subsequence, can suppose geometries persist:

$N_j(i) := N_j$ filled by adjacent fillings.

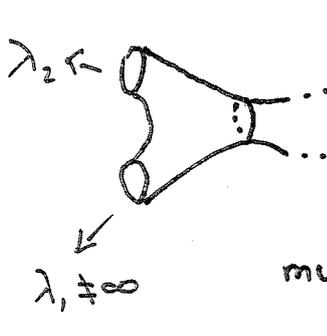
$$N(s^i) = \cup N_j(i)$$

- N_j Seifert $\Rightarrow \begin{cases} N_j(i) \text{ solid torus, or} \\ \text{" " I-bundle, or} \\ \text{with } \chi < 0 \end{cases} \quad \forall i$
- N_j Hyperbolic $\Rightarrow N_j(i)$ hyp. $\forall i$

5. only true geometric blocks:

- if some λ_j is some fiber of some Seifert block then $e_i \nearrow \infty$
- if not then $\text{Vol}_i \nearrow \text{Vol}(N)$

3. solid tori:

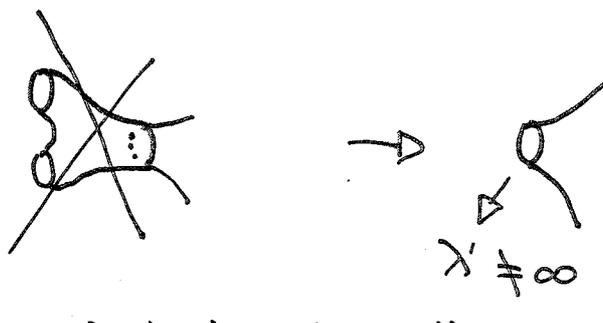


$\frac{q_1^i}{p_1^i} \rightarrow \infty \Rightarrow |p_1^i| \rightarrow \infty$
 must be: $\frac{q_2^i}{1} \rightarrow \infty$
 $\lambda_1 \neq \infty$

meridian of solid torus is

$$\frac{q_1^i}{p_1^i} + q_2^i \rightarrow \infty$$

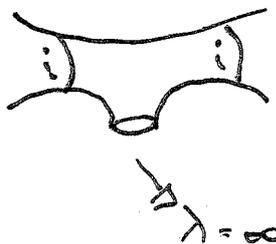
so can use induction:



$\lambda' \neq \infty$
 because fibers of adjacent blocks do not match

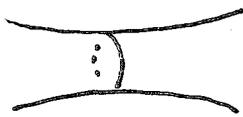
induction hypothesis on discs and annuli is preserved.

4. I-bundles



must be $q^i \rightarrow \lambda = \infty$
 $\lambda = \infty$

homeomorphic to

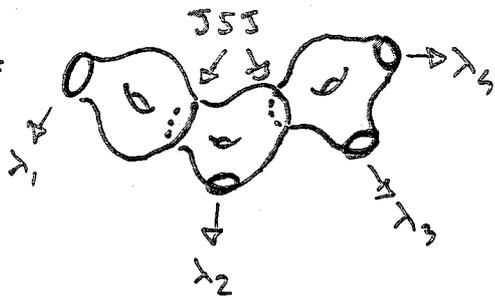


with map twisted q^i times

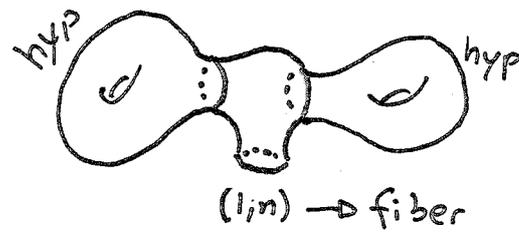
$$\Rightarrow e_i \rightarrow \infty$$

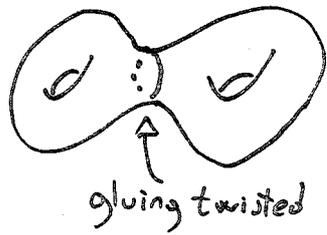
Possible phenomena:

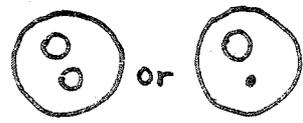
1. $N = \text{solid torus}$, $\lambda \neq \text{meridian}$ $e = |\pi_1| \nearrow \infty$

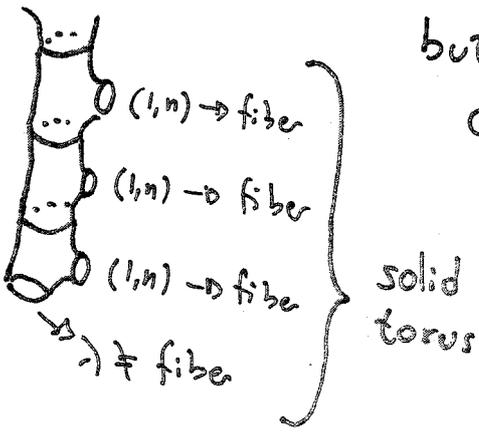
2. $N =$  λ_i is never the fiber of a Seifert block

then $N(s_i) =$  and $\text{Vol}(N(s_i)) \nearrow \text{Vol}(N)$

3.  $(1,n) \rightarrow \text{fiber}$

then $N(s_i) =$  $e \nearrow \infty$

4. N can have many adjacent hyperbolic small Seifert blocks, i.e.  or  degenerating ~~altogether~~ to a solid torus $\forall i$ but ~~without~~ "localized" and in a controlled way (not too much nested)



Applications:

Moves preserving the filled manifold:

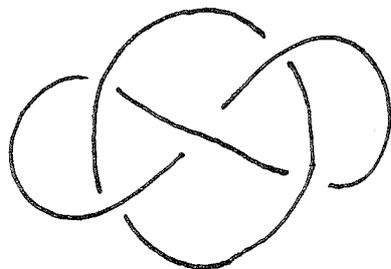
$s = (s_1, \dots, s_k) \in (S^1)^k$ (rational) point on D.f.s. of N

D: if $N(s_1, \dots, s_{k-1}, \cdot)$ contains a disc then
can modify s_k to some s'_k by twisting along it
 $s \xrightarrow{D} s' = (s_1, \dots, s_{k-1}, s'_k)$

A: if $N(s_1, \dots, s_{k-2}, s_{k-1}^\circ, \cdot)$ contains annulus then
 $s \xrightarrow{A} s' = (s_1, \dots, s_{k-2}, s'_{k-1}, s'_k)$ by twisting along it

Thm: finitely many points $s \in (S^1)^k$ give the
same fixed $M = N(s)$, up to moves D and A

Example:



N

$$N(s_1, s_2) = S^3 \Leftrightarrow$$

$$(s_1, s_2) = (\infty, 1/n) \text{ or } (1/n, \infty)$$

they are all related by D-moves

P-f: suppose $\{s_i\}_{i \in \mathbb{N}}$ pairwise not related

$s_i \rightarrow \lambda$, can suppose is essential (or we permanently fill one boundary torus)

(Thm) $\Rightarrow \lambda$ bounds disc or annulus

disc case $\Rightarrow N$ solid torus \Rightarrow easy

annulus case \Rightarrow twist along it and create a new
sequence with different limit $\lambda' \Rightarrow$ by iterating find
so many annuli that it must be $T \times [0, 1] \Rightarrow$ easy

□ 2

As usual, statements on Dehn fillings are translated into statements on link complements thanks to

$$\left\{ \text{links in } M \text{ sharing same complement } N \right\} / \text{Aut}(M)$$

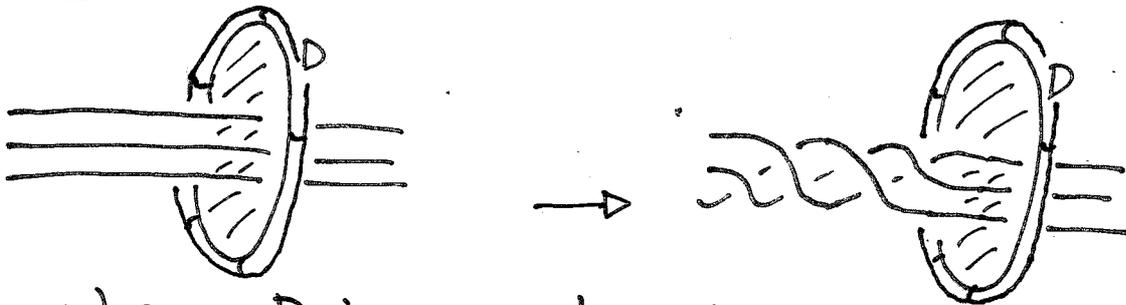
$$\updownarrow$$

$$\left\{ s \mid N(s) = M \right\} / \text{Aut}(N)$$

which translates the previous result into:

Cor: M closed 3-mfd (not nec. irreducible)
 There are finitely many links sharing the same fixed complement, up to self-homeomorphisms of M and twists along discs and annuli

Remark: we mean



where D is properly embedded in $M \setminus N(K)$ for some component K of L .

Cor: There are finitely many links in S^3 sharing the same complement up to twisting along discs and annuli spanning unknotted and coaxial components

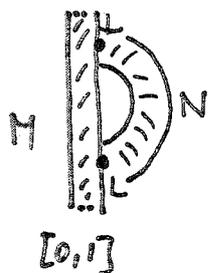
extends result of [Gordon '01]

K unknotted: bounds a disc

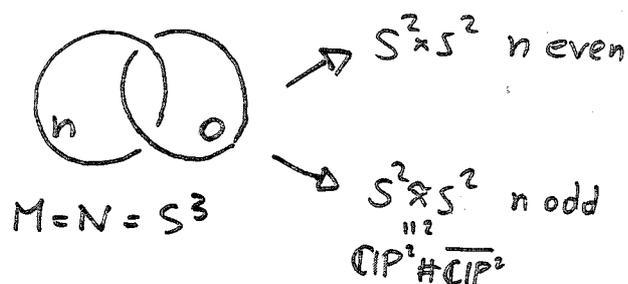
K_1, K_2 coaxial: ~~annuli~~ $K_1 \subseteq \partial N(K_2)$

2-Handler and smooth 4-manifolds:

Thm: M, N closed 3-manifolds. $L \subseteq M$ link.
 \exists finitely many smooth cobordisms between M and N obtained by adding 2-handlers to M along L .



Example:



Cor: Finitely many smooth 4-manifolds have Kirby diagram with at most K crossings, $\forall K$

Pf: links underlying

\exists finitely many such diagrams. For each link,

$\dim H_1(N) \leq \dim (H_1(S^3, L))$ therefore

$N \not\cong \#_h S^2 \times S^1$ for big h .

For each remaining h we have finitely many possibilities.

[Laudenbach-Poenaru] = \emptyset 3-, 4-handlers are attached "uniquely" \square

pf of Thm:

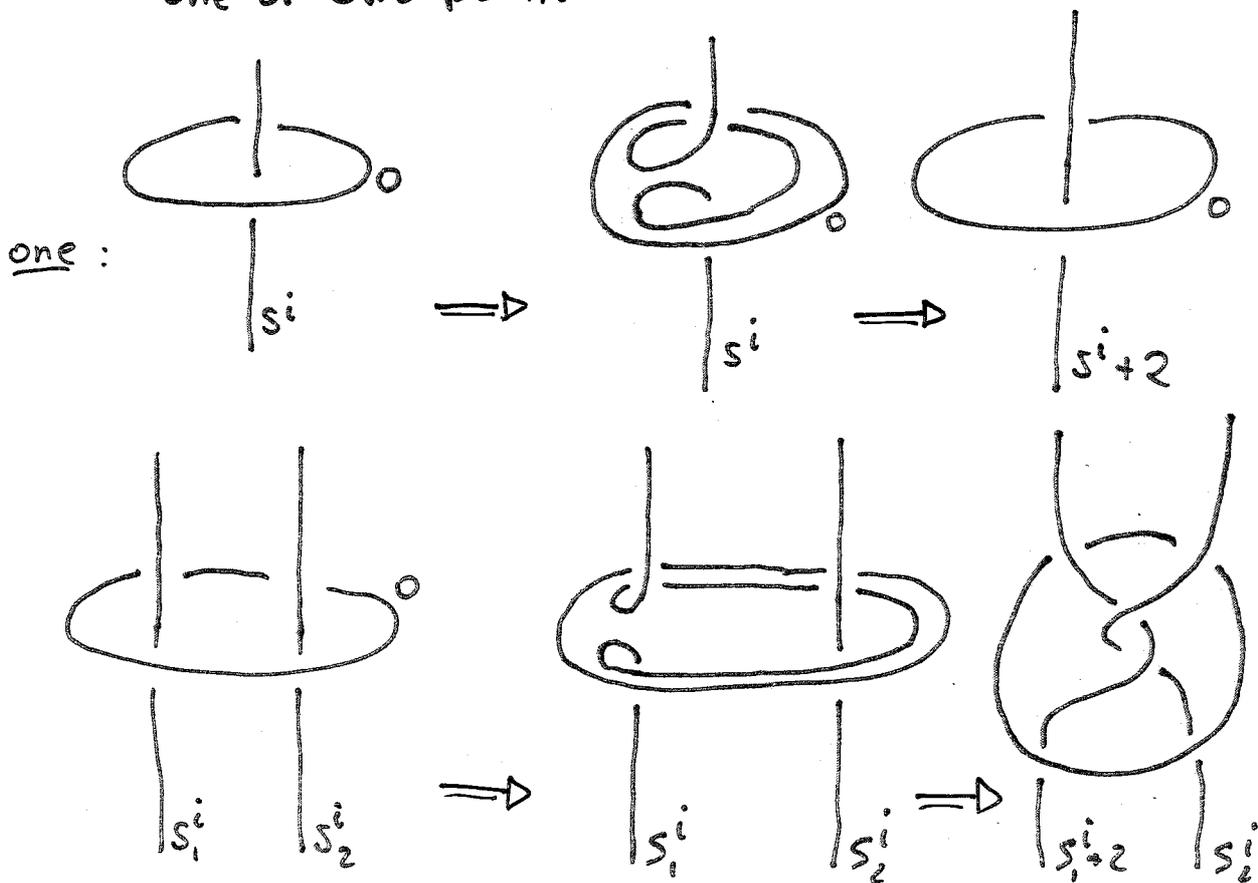
we prove that among all cobordisms, only finitely many are not related by handle slides.

By contradiction: (fix a frame)

$$L \subset M, M_L(s_i) = N \quad \forall i \quad s_i \in \mathbb{Z}$$

can restrict to essential sequence $\{s_i\}_{i \in \mathbb{N}} \rightarrow \infty$

Thm \Rightarrow L intersects 2-sphere Σ in one or two points:



with these moves I can force all numbers on one component to be zero or one, and I can proceed by induction with one component less.

