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A Crash Course on Kleinian Groups

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Introduction

A Kleinian group is a discrete group of isometries of hyperbolic 3-space $\mathbb{H}^3$. Any hyperbolic 3-manifold is the quotient of $\mathbb{H}^3$ by a Kleinian group. In the 1960s, the school of Ahlfors and Bers studied Kleinian groups mainly analytically, in terms of their action on the Riemann sphere. Thurston revolutionised the subject in the 1970s by taking a more topological viewpoint and showing that in a certain sense 'many' 3-manifolds, perhaps one could say 'most', are hyperbolic. He also introduced many wonderful new concepts, some of which we shall meet here.

In the last five years, our understanding of Kleinian groups has advanced by leaps and bounds with the proofs of three great conjectures: the Density Conjecture, the Ending Lamination Conjecture and the Tameness Conjecture. Combined, they give a remarkably complete picture of Kleinian groups.

The aim of these lectures is to give a very rapid introduction to this vast subject. Our goal is that by the end of the lectures we should be able to appreciate the statements and significance of these wonderful conjectures. The first two lectures contain general background on the algebra, geometry and topology of Kleinian groups. Lecture 3 presents the 'classical' and well understood picture of 'geometrically finite' groups. In the last lecture we describe spaces of groups and the Thurston-Bonahon picture of 'geometrically infinite' ends.

There are only a few books which systematically take the modern viewpoint, notably [MT, Kap]. I have been fortunate to have a sneak preview of the forthcoming book by Marden [Mar]. Since I found it so useful, I have given numerous references. The book should appear around the end of 2005 and I am sure will become an essential handbook for anyone wanting to work on this subject seriously.

The first lecture contains rather a lot of material. I shall spend very little time on the model of hyperbolic space and properties of Möbius maps, which I imagine most people will have seen before. It would be helpful if possible to look at this before the lecture.

Lecture 1: Kleinian group basics

A Kleinian group is a discrete group of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. There are many reasons to study such groups. One important motivation is that they arise as the *holonomy representation* of a hyperbolic structure on a 3 manifold M. More precisely:

Theorem 1.1 *Let M be a complete hyperbolic manifold. Then $M = \mathbb{H}^3/G$ where G is a Kleinian group.*

Here *complete* means that every geodesic can be extended indefinitely.

Since M looks locally like hyperbolic space, G contains no elements of finite order. So we will often find it convenient to simplify by assuming that G is *torsion free*, in other words that there is no $g \in G$ with $g^k = \mathrm{id}$.

In this lecture we look at some basic properties of Kleinian groups.

Modelling hyperbolic space We shall work with the upper half space and Poincaré ball models of hyperbolic 3-space $\mathbb{H}^3$. This allows us to identify the orientation preserving isometry group $Isom^+\mathbb{H}^3$ with the group $PSL(2,\mathbb{C})$.

The upper half space $\mathcal{H}$ consists of points $\{(z,t) : z \in \mathbb{C}, t > 0\}$. The metric is $ds^2 = \frac{|dz|^2 + dt^2}{t^2}$. In the unit ball $B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : r^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$ the metric is $ds^2 = \frac{4|dr^2|}{(1-r^2)^2}$. Using stereographic projection (or rather its inverse) we can easily map $\mathcal{H}$ to B^3 so usually we consider these two models to be equivalent.

The model is conformal, that is, angles are correctly seen as Euclidean angles. In $\mathcal{H}$, hyperbolic planes are either vertical Euclidean planes of hemispheres centred on $\mathbb{C}$. Geodesics are vertical Euclidean lines or arcs of great circles on these hemispheres.

The hyperbolic distance $d_{\mathbb{H}^3}(P, Q)$ between $P = (z_1, t_1)$ and $Q = (z_2, t_2)$ in $\mathcal{H}$ is given by
$$\cosh d_{\mathbb{H}^3}(P, Q) = 1 + \frac{|z_1 - z_2|^2 + |t_1 - t_2|^2}{2t_1 t_2}.$$
In particular, we find
$$d_{\mathbb{H}^3}((0, t_1), (0, t_2)) = |\log \frac{t_1}{t_2}|.$$

If $P \in \mathcal{H}$ is fixed, then $d_{\mathbb{H}^3}(P, Q) \to \infty$ as $t_2 \to 0$. This justifies calling the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ the *boundary at infinity*. When the specific model is not important, we write this boundary as $\partial\mathbb{H}^3$.

Representation of isometries using $SL(2,\mathbb{C})$. As in Euclidean geometry, isometries of $\mathbb{H}^3$ are generated by reflections in (hyperbolic) planes. Orientation preserving isometries correspond to even numbers of relections. Hyperbolic planes meet $\hat{\mathbb{C}}$ in circles and hyperbolic reflection in a plane extends to *inversion* in a circle. (Inversion in the circle $|z - c| = r$ is given by the formula $z \mapsto c + \frac{r^2}{\overline{z-c}}$.) Thus $Isom^+\mathbb{H}^3$ is generated by products of even numbers of inversions. These are just the *Möbius maps* $z \mapsto \frac{az+b}{cz+d}$. We briefly recall some basic facts on Möbius maps. You can find details in many places, for example [MSW].

- Möbius maps compose by matrix multiplication. Multiplying all coefficients in $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by a non-zero $\lambda \in \mathbb{C}$ does not affect the action. Hence dividing all coefficients by $\pm\sqrt{\det A}$ we may assume A is *normalised* so that $\det A = 1$. The ambiguity ± 1 means that the group of Möbius maps $\text{Mob}\,\hat{\mathbb{C}}$ can be identified with $PSL(2,\mathbb{C})$.

- $\text{Mob}\,\hat{\mathbb{C}}$ acts transitively on triples of distinct points in $\hat{\mathbb{C}}$.

- Transforming $\hat{\mathbb{C}}$ by $S \in \text{Mob}\,\hat{\mathbb{C}}$ induces the action of conjugation on $A \in \hat{\mathbb{C}}$. For example, if A has a fixed point z_0 then SAS^{-1} has a fixed point $S(z_0)$.

- $\text{Tr}\,A = a + d$ is invariant under conjugation in $SL(2,\mathbb{C})$. Because of the ambiguity of sign, strictly speaking only $\text{Tr}^2 A$ is defined on $PSL(2,\mathbb{C})$.

- A Möbius map has one or two fixed points. If there are two fixed points, one is attracting and one is repelling. The classification up to conjugation is given below.

Types of isometries

One fixed point; Parabolic: $\text{Tr}\,A = \pm 2$. Canonical form: $z \mapsto z + c, c \in \mathbb{C}$.

Two fixed points $\text{Tr}\,A \neq \pm 2$. Canonical form: $z \mapsto \kappa z, \kappa \in \mathbb{C}$. From the canonical form, $\text{Tr}\,A = \sqrt{\kappa} + 1/\sqrt{\kappa}$. This subdivides into:

(Purely) hyperbolic: $\kappa \in \mathbb{R}$. $\text{Tr}\,A \in \mathbb{R}, \text{Tr}^2 A > 4$.

Elliptic: $|\kappa| = 1$. $\text{Tr}\,A \in \mathbb{R}, \text{Tr}^2 A < 4$.

Loxodromic: $\text{Tr}\,A \notin \mathbb{R}$.

In $\mathbb{H}^3$, a parabolic $z \mapsto z+c$ extends to the Euclidean translation $(z,t) \mapsto (z+c,t)$. All the other types $z \mapsto \kappa z$ extend to a homothety $(z,t) \mapsto (\kappa z, \kappa t)$.

If $A \in \mathrm{Mob}\,\hat{\mathbb{C}}$ has two fixed points, then the line in $\mathbb{H}^3$ joining them is called its *axis*. By moving to the standard position with fixed points at $0, \infty$, it is easy to check that A moves points along the axis by a distance l_A and rotates around the axis by an angle θ_A given by

$$\mathrm{Tr}\,A = 2\cosh((l_A + i\theta_A)/2).$$

(The ambiguity in sign caused by the fact that θ_A and $\theta_A + 2\pi i$ represent the same rotation corresponds to the ambiguity in sign of $\mathrm{Tr}\,A$.) l_A is called the *translation length* of A and $\lambda_A := l_A + i\theta_A$ is called the *complex translation length*.

A is purely hyperbolic iff $\theta_A = 0$ and elliptic iff $l_A = 0$. Note that an elliptic is just rotation about the axis, so has finite order iff $\theta_A \in 2\pi\mathbb{Q}$. If A is parabolic then by convention we set $l_A = \theta_A = 0$.

Elementary groups   The above discussion gives our first examples of Kleinian groups:

1. $\langle g^n, n \in \mathbb{Z} \rangle$, g parabolic.

2. $\langle g^n h^m, n, m \in \mathbb{Z} \rangle$, g, h parabolic with same fixed point but different translation directions.

3. $\langle g^n, n \in \mathbb{Z} \rangle$, g loxodromic.

4. $\langle g^n, 0 \leq n < k \rangle$, g elliptic, $g^k = \mathrm{id}$.

Actually these examples are not so special as they seem.

Lemma 1.2 *Let G be Kleinian and say $g, h \in G$. Then g, h have either both fixed points in common or neither.*

PROOF. Normalise so that one common fixed point is at ∞ and then study $g^{-n}hg^n$. For example, if $h(z) = z + 1$ and $g(z) = \kappa z$ with $\kappa > 1$ then $g^n h g^n(z) = z + \kappa^{-n}$. It follows that G is not discrete. $\square$

From this we deduce that if g, h are loxodromics with a common fixed point, then they have the same axis.

Lemma 1.3 *Let G be Kleinian and let $g, h \in G$ be purely hyperbolic with the same axis. Then G is cyclic (see example (3) above).*

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PROOF. Normalise such that $g(z) = \kappa z, h(z) = \mu z$. The set of powers $\kappa^n \mu^m$ is discrete in $\mathbb{R}$ if and only if there exists $a > 0$ such that $\kappa = a^r$ and $\mu = a^s$; moreover using the Euclidean algorithm we can assume that $a = \kappa^p \mu^q$ for suitable $p, q \in \mathbb{Z}$. $\square$

These and other similar results lead one to make the following

Definition 1.4 *A Kleinian group is* elementary *if it is virtually abelian; that is, it has an abelian subgroup of finite index.*

Proposition 1.5 *A Kleinian group is elementary iff there is a finite G-orbit for its action on $\hat{\mathbb{C}}$.*

Theorem 1.6 *The list (1)-(3) above contains all the torsion free elementary Kleinian groups.*

If a Kleinian group is not elementary, then it is quite complicated:

Proposition 1.7 *If a Kleinian group is not elementary then it contains infinitely many loxodromic elements with pairwise distinct axes.*

There are very few explicit tests to check if a group is discrete. Essentially the best we have is:

Theorem 1.8 *(Jørgensen's inequality) Let $A, B \in SL(2, \mathbb{C})$. If $\langle A, B \rangle$ is Kleinian and non-elementary, then*

$$|\operatorname{Tr}^2 A - 4| + |\operatorname{Tr} ABA^{-1}B^{-1} - 2| \geq 1.$$

As we shall see, this has many important implications.

Details of the above results can be found in many texts on Kleinian groups, for example [Bea, Mas, Mar].

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More on discreteness   Recall our definition: A subgroup $G \subset Isom^+\mathbb{H}^3$ is Kleinian if it is discrete. With our identification of $Isom^+\mathbb{H}^3$ with $SL(2, \mathbb{C})$, this is equivalent to: *G is Kleinian iff there exists $\epsilon > 0$ such that $||g \pm \mathrm{I}|| > \epsilon$, where $||.||$ is any suitable norm on $SL(2, \mathbb{C})$.* This condition is enough to ensure that G has no accumulation points, since $g_n \to g \in G$ is equivalent to $g^{-1}g_n \to I$.

You may be worried about the $PSL(2, \mathbb{C})$ versus $SL(2, \mathbb{C})$. Here is a rather deep theorem:

Theorem 1.9 ([Cul]) *Let Γ be an abstract group. Suppose given a representation $\rho : \Gamma \to PSL(2, \mathbb{C})$ such that $\rho(\Gamma)$ is Kleinian and non-elementary. Then ρ lifts to $\hat{\rho} : \Gamma \to SL(2, \mathbb{C})$ iff Γ contains no elements of order 2.*

If G acts on a topological space X, then in order to have a decent quotient space X/G we need to know that the G acts in a 'reasonable' way, such that, for example, the orbits form a Hausdorff space. This leads to

Definition 1.10 *Suppose that G acts by homeomorphisms on a topological space X. Then H acts* properly discontinuously *if for all compact subsets $K \subset X$, $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.*

Lemma 1.11 *If $G \subset PSL(2, \mathbb{C})$ acts properly discontinuously on a topological space X, then G is discrete.*

PROOF. If the result is false, then there exist $g_n \in G$, $g_n \to I$. Take K to be a ball with compact closure. Then $g_n K \cap K \neq \emptyset$ for infinitely many g_n. $\square$

In general, the converse is false. For example, let $G = SL(2, \mathbb{Z}[i])$, the subgroup of $SL(2, \mathbb{C})$ whose entries are *Gaussian intergers* $m + in, m, n \in \mathbb{Z}$. This is clearly discrete. However the G-orbits on $\hat{\mathbb{C}}$ 'pile up', as one can see by looking at the orbit of 0 under the subgroup $SL(2, \mathbb{Z})$. It is easy to see this consists of the extended rational numbers $\mathbb{Q} \cup \infty$.[1] However we do have the following crucial theorem:

Theorem 1.12 *$G \subset PSL(2, \mathbb{C})$ is Kleinian iff it acts properly discontinuously on $\mathbb{H}^3$.*

PROOF. The main idea is that for any $r > 0$, and $P = (0, 1) \in \mathbb{H}^3$, $\{\gamma \in SL(2, \mathbb{C}) : d_{\mathbb{H}^3}(P, \gamma P) < r\}$ is compact in $SL(2, \mathbb{C})$. For details see for example [Bea, Thu]. $\square$

Corollary 1.13 *If $G \subset PSL(2, \mathbb{C})$ is Kleinian and torsion free then $\mathbb{H}^3/G$ is a hyperbolic 3-manifold.*

There are lots of possible variations of the definition of proper discontinuity. You can find an excellent discussion in [Thu] §3.5.

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[1]Groups with entries in number fields form a large and important class of Kleinian groups which we shall not have time to touch on in these lectures.

Action on $\hat{\mathbb{C}}$ What about the action of a Kleinian group on $\hat{\mathbb{C}}$? The above example of $SL(2, \mathbb{Z}[i])$ shows it may not be properly discontinuous.

Definition 1.14 *The* ordinary set *$\Omega = \Omega(G) \subset \hat{\mathbb{C}}$ of a Kleinian group G is the maximal open subset of $\hat{\mathbb{C}}$ on which G acts properly discontinuously. The* limit set *$\Lambda = \Lambda(G)$ is $\hat{\mathbb{C}} \setminus \Omega$.*

Ω is also called the *regular set* and the *domain of discontinuity*. It is the maximal set on which the elements of G form a normal family in the sense of complex analysis. McMullen calls Λ *'the chaotic set'*. It is the analogue of the Julia set for a rational map. If G consists entirely of elliptic elements (and hence is finite), Λ is empty. Otherwise, Λ is non-empty, because $\hat{\mathbb{C}}$ is compact and the (infinite) G-orbits have to accumulate somewhere. Ω may or may not be empty. In some rather old fashioned terminology, G is called 'of the first kind' if $\Omega = \emptyset$ and 'of the second kind' otherwise. $SL(2, \mathbb{Z}[i])$ is a good example of a group of the first kind. The elementary groups are all of the second kind; in fact:

Lemma 1.15 Λ *is finite iff G is elementary. In this case, Λ contains either $0, 1$ or 2 points.*

Suppose G is non-elementary. Then:

1. Λ is uncountably infinite, closed and minimal, that is, the G-orbit of any point in Λ is dense in Λ.

2. Λ is the closure of the loxodromic fixed points (and equally of the parabolic fixed points, if G contains any parabolic).

3. Λ is the set of accumulation points of the G-orbit of any point in $\hat{\mathbb{C}} \cup \mathbb{H}^3$.

4. Suppose that H is a subgroup of G. If either H is of finite index, or H is normal in G, then $\Lambda(H) = \Lambda(G)$.

5. If $\Lambda(G) \neq \hat{\mathbb{C}}$, then its interior is empty.

The last item suggest Ahlfors' conjecture:

Conjecture 1.16 *If $\Lambda(G) \neq \hat{\mathbb{C}}$, then it has Lebsgue measure 0.*

A consequence of the three great new theorems mentioned in the introduction, is that Ahlfors conjecture is now proved [BBES]. More precisely, Ahlfors proved it for geometrically finite groups (see Lecture 3). Following on work of Thurston and Bonahon about geometrically infinite groups (see Lecture 4), Canary proved it for tame groups [Can]. Thus the Ahlfors conjecture now follows from the tameness conjecture (see Lecture 3).

Here are some properties of Ω. Assume that G is finitely generated, non-elementary and that $\Omega \neq \emptyset$. Then:

1. Ω has either $1, 2$ or infinitely many connected components.

2. Each connected component of Ω is either simply connected or infinitely connected.

3. If there are two G-invariant components Ω_1 and Ω_2, then each is simply connected and $\Omega = \Omega_1 \cup \Omega_2$.

4. If there is one G-invariant component Ω_0, then all other components are simply connected.

Properties (1)-(3) are elementary but (3,4) are much deeper and are most easily proved using 3-dimensional topology, see [Mar74, Mar]. The groups in (3) are exactly the *quasifuchsian groups* we shall meet in the next lecture. In the older literature, groups with an invariant component are called *function groups*. Groups with a simply connected invariant component are called *B-groups*.

The Ahlfors finiteness theorem. We end with two deep theorems. Recall that a Riemann surface is a surface with local charts to $\mathbb{C}$ such that the overlap maps are complex analytic. A puncture is a neighbourhood of a boundary component for which we have a chart to a punctured disk. If a surface carries a hyperbolic structure, it automatically carries a conformal structure by 'forgetting' the metric and remembering only the angles. It is a consequence of the famous uniformisation theorem that:

Theorem 1.17 (Uniformisation theorem) *Suppose that S is a Riemann surface, possibly with punctures, which has negative Euler characteristic. Then there is a unique hyperbolic structure on S inducing the given conformal structure.*

Theorem 1.18 (Ahlfors finiteness theorem) *Suppose that the Kleinian group G is non-elementary and finitely generated. Then Ω/G is a finite union of Riemann surfaces of finite type. Moreover each of these surfaces has negative Euler charateristic and so is hyperbolisable.*

There are (at least) two modern proofs of the Ahlfors' finiteness theorem which are much easier than Ahlfors' original version, see [Kap] and [Mar05].

Lecture 2: Geometry of hyperbolic 3-manifolds

In this lecture we look at some basic features of a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ with the action of G on $\hat{\mathbb{C}}$. As we shall see, what we learnt in the last lecture gets us quite a long way.

First, consider loops in $M = \mathbb{H}^3/G$. We can identify G with $\pi_1(M)$. Take any homotopically non-trivial loop $\gamma \subset M$. Let $L([\gamma]) = inf\{l(\gamma') : \gamma' \in [\gamma]\}$, where $[\gamma]$ is the free homotopy class of γ. Then either

- $L([\gamma]) > 0$, in which case the infimum is attained by a unique closed geodesic, in the homotopy class and $L([\gamma])$ is the translation length of the associated isometry $g \in G$, or

- $L([\gamma]) = 0$, in which case γ is represented by a parabolic element in G.

From Theorem 1.6 and Proposition 1.7 in the last lecture we obtain:

Corollary 2.19 *Suppose that G is a non-elementary Kleinian group. Then:*

1. *$\mathbb{H}^3/G$ contains infinitely many distinct closed geodesics.*

2. *The only rank 2 abelian subgroups correspond to a pair of parabolics with a common fixed point.*

As we shall see, (2) implies that M is *atoroidal*.

Incompressible and compressible boundary If G is a Kleinian group, it is often convenient to add on the surfaces at infinity to obtain the *Kleinian manifold* $M_G = (\mathbb{H}^3 \cup \Omega)/G$. In the last lecture we saw that each connected component Ω_i of Ω is either simply connected or multiply connected. This has a nice interpretation in terms of the topology of M_G.

Let Ω_i be a connected component of Ω. There is a natural inclusion map from the surface Ω_i/G into M_G which induces a map $\iota : \pi_1(\Omega_i/G) \to \pi_1(M) = G$.

Lemma 2.20 *ι is injective iff Ω_i is simply connected.*

PROOF.$\Rightarrow$ Suppose Ω_i is not simply connected. Then it contains a closed loop γ which is not trivial in Ω_i and hence not trivial in $\pi_1(\Omega_i/G)$. However the 'roof' sitting over γ in $\mathbb{H}^3$ provides a null homotopy of γ in $\mathbb{H}^3 \cup \Omega$ and hence in M_G. So ι is not injective.

$\Leftarrow$ Suppose ι is not injective. This means there is a loop on Ω_i/G which is non-trivial in $\pi_1(\Omega_i/G)$ but trivial in G. By the loop theorem, there is a loop on Ω_i/G which bounds a disk in M_G. This disk lifts to a disk in $\mathbb{H}^3$ whose boundary meets Ω_i in a closed non-trivial loop in Ω_i. $\square$

If ι is injective, the corresponding boundary component is called *incompressible*, otherwise it is *compressible*. Usually the incompressible case is easier to handle.

More on Cusps   Suppose that G contains a parabolic element. We want to see what M looks like in a neighbourhood of its fixed point, which necessarily lies on the boundary $\partial \mathbb{H}^3$. Since the parabolic is represented by arbitrarily short loops in M, we know that this neighbourhood is in the thin part of the manifold and hence, by the Margulis lemma, is either a cusp cylinder or a cusp torus.

We can get a more explicit picture of cusp neighbourhoods as follows. In the upper half space model, a ball tangent to $\partial \mathbb{H}^3 = \hat{\mathbb{C}}$ at $\xi \in \hat{\mathbb{C}}$ is called a *horoball* based at ξ. In particular, if $\xi = \infty$, a horoball based at ∞ is a set $H_s = \{(z, t) : t \geq s\}$ for some $s > 0$. We sometimes call s the *height* of the horoball. We easily see by conjugating that a parabolic with fixed point at ξ maps horoballs based at ξ to themselves.

Let P be a Kleinian group all of whose elements are parabolic with a common fixed point. As we saw in Lecture 1, P is isomorphic either to $\mathbb{Z}$ (rank 1) or $\mathbb{Z}^2$ (rank 2). By conjugation, we can put the fixed point at ∞ and scale so that P has generators $A(z) = z + 1$, and additionally, in the rank 2 case, $B(z) = z + b$ where $\operatorname{Im} b \neq 0$. With this normalisation it is not hard to see that, in the rank 1 case, H_s/P is a solid infinite cylinder with missing core (a *cusp cylinder*) and, in the rank 2 case, a solid torus with missing core (a *cusp torus*). The hyperbolic distance from ∂H_s to the core of the cusp cylinder or cusp torus is infinite. The cusp cylinder is also infinite in the direction parallel to the core curve. Travelling along ∂H_s in either direction parallel to the core curve we limit on the boundary at infinity $\hat{\mathbb{C}}$.

Now suppose that P is a subgroup in a larger Kleinian group G. We want to see that this is also the correct picture in $\mathbb{H}^3/G$. For this, we need to show that for some choice of s, $H_s/P = H_s/G$. This will be the case if we can find s such that H_s is *precisely invariant* under (P, G). This means:

- $g(H_s) = H_s$ for all $g \in P$ and
- $g(H_s) \cap H_s = \emptyset$ for all $g \in G \setminus P$.

That we can do this is the content of the Margulis lemma. Jørgensen's inequality 1.8 allows us to quantify s explicity.

Theorem 2.21 *Suppose that the Kleinian group G contains a parabolic $A(z) = z+1$. Then the interior of H_1 is precisely invariant under the parablic subgroup P which fixes ∞.*

The proof uses *isometric circles*. Suppose that $g \in SL(2, \mathbb{C})$ does not fix ∞. The *isometric circle* I_g of g is the circle in $\hat{\mathbb{C}}$ on which $|g'(z)| = 1$. If $g = (az + b)/(cz + d)$

we compute that $g'(z) = (cz + d)^{-2}$ and hence that I_g is the circle centre $-d/c$ and radius $1/|c|$.

Lemma 2.22 *(Shimizu's lemma) Suppose that the non-elementary group G contains the parabolic $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have $|c| \geq 1$.*

PROOF. Jørgensen's inequality says that

$$|\mathrm{Tr}^2 A - 4| + |\mathrm{Tr}\, ABA^{-1}B^{-1} - 2| \geq 1.$$

We compute $\mathrm{Tr}\, ABA^{-1}B^{-1} = 2 + c^2$ so $|c| \geq 1$. $\square$

PROOF OF THEOREM 2.21. Shimizu's lemma implies that if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$ then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ not fixing ∞, the isometric circle I_g has radius at most 1. Let $\hat{I}_g$ be the hemisphere in $\mathcal{H}$ sitting above I_g. It is easy to check that g maps the region in $\mathcal{H}$ outside $\hat{I}_g$ to the region in $\mathcal{H}$ inside $\hat{I}_{g^{-1}}$. This implies that the interior of the closed horoball H_1 is precisely invariant under (P, G). $\square$

Cusps on Ω/G   How does a parabolic in $\mathbb{H}^3/G$ relate to a puncture on Ω/G? Recall from the uniformisation theorem 1.17 that each surface in Ω/G carries a hyperbolic metric inducing its Riemann surface structure. A more general question is, how does the hyperbolic metric on Ω/G relate to the hyperbolic metric on $\mathbb{H}^3/G$?

Theorem 2.23 *(Ahlfors lemma) Suppose that Ω_0 is a simply connected component of Ω. Let γ be a geodesic on Ω_0/G, represented by an element $g \in G$. Then*

$$l_{\mathbb{H}^3/G}(g) \leq 2l_{\Omega_0/G}(\gamma).$$

In particular, if γ is parabolic on Ω_0/G (so that $l_{\mathbb{H}^3/G}(g) = 0$), then g is parabolic in G.

By the two dimensional analogue of the discussion above, the neighbourhood of a puncture on a hyperbolic surface looks like the quotient of a horoball neighbourhood of ∞ by a translation; in other words, a cusp annulus. (Notice that on a hyperbolic surface, there are no rank 2 parabolic subgroups, because no such a subgroup has an invariant disk in $\hat{\mathbb{C}}$.) Such a neighbourhood projects to a topological disk in Ω_0 tangent to the parabolic fixed point. The hemisphere above this disk is also precisely

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invariant under P. This means that on the end of the cusp cylinder we can glue in a punctured annulus, a neighbourhood of the puncture on Ω_0/G. If this can be done on both ends of the cylinder, we call the quotient a *pairing tube*. This happens, for example, if G is Fuchsian or quasifuchsian.

Notice that the inequality in Ahlfors' lemma only goes one way. This means that we could have a loop in Ω_0/G which was not parabolic (so not a loop round a puncture) which was nevertheless repesented by a parabolic element in G. Such curves are called *accidental parabolics* on Ω_0/G. Typically, a curve which is accidentally parabolic on one component of Ω/G will represent a cusp on another.

The Kleinian manifold   Recall we defined the Kleinian manifold associated to G as $M_G = (\mathbb{H}^3 \cup \Omega)/G$. If G has cusps, M_G is not compact. It is usually easier to deal with a compact manifold, so it is often convenient to remove solid cusp annuli and tori round the cusps to create a compact manifold M_G^0. Corollary 2.19 shows that M_G^0 is always *atoroidal*, that is, all the $\mathbb{Z}^2$ subgroups come from boundary tori. Because of the hyperbolic structure it is also *irreducible*, that is, every 2 sphere bounds a ball. Compare a version of Thurston's hyperbolisation theorem:

Theorem 2.24 *(Thurston, see [Kap]) Suppose that a compact 3-manifold V is irreducible and atoroidal and that $\partial V \neq \emptyset$. Then V has a hyperbolic structure, that is, V is homeomorphic to M_G^0 for some Kleinian group G.*

One can also specify the *parabolic locus*, which we can think of as the toral boundary components and in addition a collection of annuli in the non-toral boundary components which are to become parabolic. The only condition is that the boundary of any essential annulus in V should not contained in the parabolic locus. Such a specification is called a *pared manifold*, see [Kap] for a precise statement.

Fundamental domains   We have seen in Zimmerman's lectures how one can form hyperbolic manifolds by gluing the faces of a polyhdron (or polygon in the surface case) using hyperbolic isometries. Poincaré's theorem tells us that the resulting manifold will be $\mathbb{H}^3/G$ for some Kleinian group G iff
- The sum of angles around each edge is 2π and
- The resulting manifold with a suitable induced metric is complete.

The construction in $\mathbb{H}^2$ is simpler: we glue the sides of a polygon ensuring that the angles sum around each vertex is 2π. In this case, if the polygon is finite sided

there is a simple condition for completeness: Any cycle corresponding to an ideal vertex of the polygon (that is, a vertex where two sides meet on $\partial \mathbb{H}^2$) is *parabolic*.

Suppose that conversely we are given a hyperbolic 2- or 3-dimensional manifold. Can we find a corresponding polyhedron? The answer is provided by the *Dirichlet domain*. The following nice description comes from [Mar]. Pick a point $a \in M$ and start blowing up a balloon centred at a. Eventually one side of the balloon will touch another. If you keep blowing, the parts of the ballon near these touching points will flatten out forming the beginnings of a planar face. As you keep blowing, other points will touch starting further faces. Eventually different faces will meet at vertices. Keep going until you have filled up all of M, lift to $\mathbb{H}^3$, and you have the Dirichlet domain $D_a(G)$ centred on a. More formally:

$$D_a(G) = \{q \in \mathbb{H}^3 : d(q, a) < d(q, g(a)) \text{ for all} g \in G \setminus I\}.$$

One can show that $D = D_a(G)$ is a convex *fundamental polyhedron* for the action of G on $\mathbb{H}^3$. This means that the images $g(D)$, $g \in G$ tesselate $\mathbb{H}^3$, in other words, $g(D) \cap h(D) = \emptyset$ unless $g = h$, and the images of $\overline{D}$ cover $\mathbb{H}^3$.

D is locally finite, more precisely only finitely many faces meet any compact set in $\mathbb{H}^3$. The faces of D are pieces of hyperplanes which bisect the distance between a and any of its neighbouring orbit points. They come naturally in pairs: if D meets $g(D)$ along a common face F, then $g^{-1}(D)$ meets $D = g^{-1}(g(D))$ along the common face $g^{-1}(F)$. We say that g^{-1} 'pairs' the face F to the face $g^{-1}(F)$, and g^{-1} is called a *side pairing* of D. It is easy to see that if g^{-1} is a side pairing, then so is g.

Proposition 2.25 *For any $a \in \mathbb{H}^3$, a Kleinian group G is generated by the side pairings of $D_a(G)$.*

PROOF. Take $h \in G$ and join a to $h(a)$ by a path α which avoids all vertices of $D = D_a(G)$ and its images under G. Notice that if $g(D)$ meets $g'(D)$ in a common face, then $g'^{-1}g(D)$ meets D in a common face so that $g'^{-1}g$ is a side pairing. Suppose that the polyhedra crossed by α in order along its path are $D, g_1(D), \ldots, g_{k-1}(D), h(D)$. Then $h = (I^{-1}g_1)(g_1^{-1}g_2)\ldots(g_{k-1}^{-1}h)$, expresses h as a product of side pairings. $\square$

Corollary 2.26 *If for some $a \in \mathbb{H}^3$ $D_a(G)$ has finitely many faces, then G is finitely generated.*

The converse is true in dimension 2 but not in dimension 3. Groups for which $D_a(G)$ has finitely many faces are called *geometrically finite* and are the subject of the next lecture.

It is high time to have some examples.

Fuchsian groups   By definition, a *Fuchsian group* is a discrete group of isometries of $\mathbb{H}^2$. Embedding $\mathbb{H}^2$ as a plane in $\mathbb{H}^3$, we see that any isometry of the plane extends to an isometry of $\mathbb{H}^3$. So a Fuchsian group G can be considered as a special case of Kleinian group. The quotient $\mathbb{H}^2/G$ is a hyperbolic surface Σ.

Think of $\mathbb{H}^2$ as the equatorial plane in the ball B^3. Let G be a Fuchsian group and take a fundamental polygon D_G for G acting in $\mathbb{H}^2$. Now extend each of its sides into a plane in $\mathbb{H}^3$. This extends D_G to a 'chimney' which forms a fundamental polyhedron for G acting in $\mathbb{H}^3$. From this picture, we see that $\mathbb{H}^3/G$ is homeomorphic to $S \times (0, 1)$, where S is a *topological* surface homeomorphic to the *hyperbolic* surface Σ.

Assuming that Σ is closed or possibly has finitely many punctures (so that G is of the first kind) then $\Lambda(G)$ is the equatorial circle in which $\mathbb{H}^2$ meets $\hat{\mathbb{C}}$. Ω has two connected components $\Omega^{\pm}$ each of which is simply connected and G-invariant. The quotients $\Omega^{\pm}/G$ each have conformal structures whose corresponding hyperbolic structures are identical with that of Σ.

Isometries of $\mathbb{H}^2$ are exactly the Möbius map which map $\mathbb{H}^2$ to itself. An element in $SL(2, \mathbb{C})$ maps $\mathbb{H}^2$ to itself iff all coefficients are real. Thus $Isom^{+}\mathbb{H}^2$ is naturally identified with $SL(2, \mathbb{R}) \subset SL(2, \mathbb{C})$. Conjugating by a Möbius map sends $\mathbb{H}^2$ to another disk in $\hat{\mathbb{C}}$. Even though the matrix entries in the conjugated group are no longer real, all the traces remain real. This gives another characterization of a Fuchsian group:

Definition 2.27 *A Fuchsian group is a Kleinian group which leaves invariant a disk in $\hat{\mathbb{C}}$. A Kleinian group is Fuchsian iff the traces of all its elements are real.*

We can construct lots of examples of Fuchsian groups by starting with a finite sided hyperbolic polygon, pairing sides in some specified way, and checking the conditions of the Poincaré theorem. For example, identifying opposite sides of a regular $4g$-gon with interior angle $\pi/2g$, we obtain a hyperbolic structure on a closed surface of genus g whose associated Fuchsian group is generated by the isometries which pair the opposite sides.

Quasifuchsian groups   By definition, a quasifuchsian group is a quasi-isometric or quasiconformal deformation of a Fuchsian group. We shall see in more detail what this means in the next lecture. For the moment, let's just assume we have a Fuchsian group Γ, a Kleinian group G, a group isomorphism $\chi : \Gamma \to G$ and a homeomorphism $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\psi(gz) = \chi(g)\psi(z)$ for all $z \in \hat{\mathbb{C}}$ and $g \in \Gamma$. It is a deep result of Marden [Mar74] that ψ can always be taken to be *quasiconformal*.

From the construction, the limit set of a quasifuchsian group is always a topological circle and the regular set has two simply connected G-invariant components $\Omega^\pm$. The 3-manifold $\mathbb{H}^3/G$, being homeomorphic to $\mathbb{H}^3/\Gamma$, is homeomorphic to $S \times (0,1)$. However the hyperbolic or conformal structures on the two components $\Omega^\pm/G$ are now different. The following is a famous result of Bers [Ber]:

Theorem 2.28 (Simultaneous uniformisation) *Given any two conformal structures $\omega^\pm$ on Σ, there exists a unique quasifuchsian group for which $\Omega^\pm/G$ have the structures $\omega^\pm$. (Here 'unique' means 'up to conjugation in $SL(2,\mathbb{C})$'.)*

The limit set of a quasifuchsian group is a fractal:

Theorem 2.29 *(Bowen) Suppose that G is quasifuchsian but not Fuchsian. Then the Hausdorff dimension of $\Lambda(G)$ is strictly greater than 1.*

Schottky groups Take $2k$ pairwise disjoint round disks $E_i, E_i', i = 1, \ldots, k$ in $\hat{\mathbb{C}}$ and choose $e_i \in SL(2,\mathbb{C})$ such that e_i maps the inside of E_i to the outside of E_i'. The group G generated by $e_1, \ldots, e_k$ is called a (classical) Schottky group. If we place a hemisphere over each disk, then we can view the region D in $\mathbb{H}^3$ outside all the hemispheres as a polyhedron in $\mathbb{H}^3$. The e_i match the sides of D in pairs and D is a fundamental domain for G acting in $\mathbb{H}^3$. It follows that from Poincaré's theorem that G is free and discrete. Klein gave a nice direct proof of this fact, often called the ping-pong theorem. It is good exercise to convince oneself that the 3-manifold obtained by gluing the faces of D using the side pairings is a genus k handlebody.

As explained in great detail in [MSW], each 'infinite reduced word' in the generators in G corresponds to a nested sequence of images of the disks E_i, E_i'. Each limit point is the infinite intersection of such a nested sequence. It follows that the limit set $\Lambda(G)$ is a Cantor set and the ordinary set Ω has a single G-invariant component which is not simply connected. It is another good exercise to find which curves in Ω correspond to compressing disks in $\mathbb{H}^3/G$.

Any Kleinian group has lots of Schottky subgroups. This is based on the fact that sufficiently high powers of loxodromics have disjoint isometric circles which serve as the disks from which to construct the group. Based on this we find:

Proposition 2.30 *(see [MT] Theorem 2.9)) G is non-elementary iff it contains a free subgroup on 2 generators.*

Just like Fuchsian groups, Schottky groups can be deformed using homeomorphisms of $\hat{\mathbb{C}}$. This does not change the topology of the three manifold. Such groups are also known as (non-classical) Schottky groups. In fact we have:

Theorem 2.31 *(Maskit, see [MT] Theorem 4.23) The following are equivalent:*

1. *M_G is a handlebody*

2. *G is Schottky*

3. *G is free and purely loxodromic*

Combination theorems   These are important theorems which enable one to build up complicated groups from simple ones by gluing along surfaces in the boundary. The fundamental principal is the same as that in the ping-pong theorem. We don't have time to go into this here; good accounts can be found in [MT, Kap] and for all the details see [Mas].

Lecture 3: Geometrically finite groups

The Mostow rigidity theorem says that a closed hyperbolic 3-manifold is rigid, in fact is determined up to isometry by the isomorphism class of $\pi_1(M)$. In this lecture we study a very important class of hyperbolic manifolds with boundary. These are the geometrically finite manifolds. As we shall see, their deformations are well understood.

Geometrically finite groups

Definition 3.32 *A Kleinian group is* geometrically finite *if it has a finite sided fundamental domain.*

As we saw in Corollary 2.26, the side pairings generate the group, so a geometrically finite group is always finitely generated.

Theorem 3.33 *A Fuchsian group is finitely generated iff it is geometrically finite.*

For the proof, see [Bea]. As was first pointed out by Greenberg, Theorem 3.33 fails in dimension 3. We shall see why in Lecture 4. Marden systematically investigated geometrically finite groups in [Mar74]. The first point is to understand the cusps.

Proposition 3.34 *([Mar74, Mar]) If G is geometrically finite then there are only finitely many cusps. The rank 1 cusps come in pairs; each pair corresponds to a pairing tube which matches two punctures on Ω/G.*

This leads easily to

Theorem 3.35 *(Marden) G is geometrically finite iff M_G^0 is compact.*

Here M_G^0 is the Kleinian manifold $M_G = (\mathbb{H}^3 \cup \Omega)/G$ with cusp neighbourhoods removed, as explained in Lecture 2. Proofs can also be found in [MT].

Definition 3.36 *G is (topologically) tame iff $\mathbb{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold.*

Theorem 3.37 *(Marden) A geometrically finite group G is tame.*

Marden conjectured that if G is finitely generated, then G is tame, see Lecture 4.

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Ahlfors-Bers deformation theory It turns out that the deformations of geometrically finite groups are completely described in terms of the deformations of Ω/G.

Definition 3.38 *Let (X_1, d_1) and (X_2, d_2) be metric spaces. A homeomorphism $f : X_1 \to X_2$ is a* quasi-isometry *if there exists $k > 0$ such that*

$$d_1(p, q)/k \leq d_2(f(p), f(q)) \leq k d_1(p, q)$$

for all $p, q \in X_1$.

Definition 3.39 *Let $U \subset \hat{\mathbb{C}}$ be an open set. A homeomorphism $f : U \to f(U) \subset \hat{\mathbb{C}}$ is* quasiconformal *if*

$$K(z) = \lim_{r \to 0} \frac{\max_\theta |f(z + re^{i\theta}) - f(z)|}{\min_\theta |f(z + re^{i\theta}) - f(z)|}$$

is bounded on U. It is called K-quasiconformal if $K(z) \leq K$ for almost all $z \in U$. The dilatation K_f *of f is the infimum of K for which f is K-quasiconformal.*

Proposition 3.40 *Every quasi-isometry $\mathbb{H}^3 \to \mathbb{H}^3$ extends to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$.*

Suppose that G_1, G_2 are Kleinian groups and that $\chi : G_1 \to G_2$ is an isomorphism. A quasiconformal map $\psi : \Omega(G_1) \to \Omega(G_2)$ is said to induce χ if $\psi(gz) = \chi(g)\psi(z)$ for all $g \in G_1$ and $z \in \hat{\mathbb{C}}$. χ is called *type preserving* if $\chi(g)$ is parabolic in G_2 iff g is parabolic in G_1. It is possible to have a quasiconformal map $\psi : \Omega(G_1) \to \Omega(G_2)$ which induces an isomorphism $\chi : G_1 \to G_2$ but for which there is no corresponding quasi-isometry $\mathbb{H}^3/G_1 \to \mathbb{H}^3/G_2$. However we have:

Theorem 3.41 (Marden's isomorphism theorem [Mar74]) *Let G_1 be a geometrically finite group with $\partial M_G^0 \neq \emptyset$. Suppose that $\chi : G_1 \to G_2$ is a type preserving isomorphism induced by an orientation preserving quasiconformal map $\psi : \Omega(G_1) \to \Omega(G_2)$. Then:*

1. *G_2 is geometrically finite.*

2. *ψ extends to a quasi-isometry $\mathbb{H}^3/G_1 \to \mathbb{H}^3/G_2$.*

3. *ψ extends to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$.*

4. *If ψ is conformal on $\Omega(G_1)$ then it is Möbius, so that $\mathbb{H}^3/G_1$ is isometric to $\mathbb{H}^3/G_2$.*

This result reduces the problem of understanding deformations of hyperbolic structures on a geometrically finite manifolds to that of understanding the structures on Ω/G. The following extension of Bers' simultaneous uniformisation theorem 2.28 describes the deformation theory of geometrically finite groups, see [MT, Kap]).

Theorem 3.42 *Let $M = \mathbb{H}^3/G$ be a geometrically finite hyperbolic 3-manifold with incompressible boundary. Then the quasi-isometric deformation space $\mathcal{QI}(M)$ of M is $\Pi_i \text{Teich}(S_i)$, where the product runs over the surfaces S_i making up Ω/G.*

If S_0 is a topological surface, the Teichmüller space $\text{Teich}(S_0)$ describes the possible conformal structures on the *marked* surface S_0. More precisely, a point in $\text{Teich}(S_0)$ is a pair (S, ϕ) such that S is Riemann surface and $\phi : S_0 \to S$ is a homeomorphism. (S, ϕ), (S', ϕ') are equivalent if there is a conformal map $f : S \to S'$ homotopic to $\phi'\phi^{-1}$. Teichmüller's theorem states that for any two points $(S_1, \phi_1), (S_2, \phi_2) \in \text{Teich}(S_0)$, there is a unique quasiconformal map which minimises the dilatation, called the extremal map. This provides a metric on $\text{Teich}(S_0)$. Teichmüller also showed that if S_0 has genus g with b punctures, then $\text{Teich}(S_0)$ is homeomorphic to $\mathbb{R}^{6g-6+2b}$. There are many books on Teichmüller theory. Brief summaries suitable to our viewpoint can be found in [O, Kap].

The *quasi-isometric deformation space* $\mathcal{QI}(M)$ of a 3-manifold M can be defined in a similar way, see [MT]. We always need the base manifold M (or base surface S_0) to keep track of the *marking* on M; otherwise, we might be seeing the same structure on a surface which differed from the original one by a diffeomorphism which was not homotopic to the identity. (This is why we restricted to incompressible boundary in the statement of Theorem 3.42.)

Triply punctured spheres Note that if S_0 is a sphere with 3 punctures (so $g = 0$ and $b = 3$) then $6g - 6 + 2b = 0$. This means that S_0 is *rigid*. In other words, it carries a unique hyperbolic structure which cannot be deformed. In fact it is not hard to prove the following lemma, see [MSW] Note 7.1:

Lemma 3.43 *Suppose that $A, B \in SL(2, \mathbb{C})$ are parabolic and that AB is also parabolic. Suppose their fixed points are all distinct. Then there exists $C \in SL(2, \mathbb{C})$ such that $CAC^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $CBC^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.*

Convince yourself that this implies that S_0 is rigid!

Convergence of Kleinian groups There are several different ways in which one might say that Kleinian groups are close. The topology implied in Theorem 3.42 is essentially that of Gromov-Hausdorff convergence. It is important to specify basepoints. Roughly, hyperbolic manifolds M, M' with basepoints $x \in M, x' \in M'$ are close if for large r, the r-balls in M, M' centred on x, x' look 'almost' the same. A more formal definition is that of *polyhedral convergence*[2]:

Definition 3.44 *Let G_n be a sequence of Kleinian groups. Then G_n converges polyhedrally to the Kleinian group H if the Dirichlet domains $D_a(G_n)$ with base point $a \in \mathbb{H}^3$ converge to the Dirichlet domain $D_a(H)$, uniformly on compact sets in $\mathbb{H}^3$, and if in addition the side pairing transformations of $D_a(G_n)$ converge to those of $D_a(H)$ in the following sense:*
    (i) Each face pairing of $D_a(H)$ is the limit of face pairings of $D_a(G_n)$ and
    (ii) The limit of any convergent subsequence of face pairings of $D_a(G_n)$ is a face pairing of $D_a(H)$.

This leads to:

Definition 3.45 *Let G_n be a sequence of Kleinian groups. Then G_n converges geometrically to the Kleinian group H if:*
    (i) For each $h \in H$, there is a sequence $g_n \in G_n$ which converges to h and
    (ii) If a subsequence $g_{n_r} \in G_{n_r}$ which converges to $\gamma \in SL(2, \mathbb{C})$, then $\gamma \in H$.

Polyhedral and geometric convergence turn out to be the same and, modulo a discussion about base points and conjugation in $SL(2, \mathbb{C})$, are the same as Gromov-Hausdorff convergence of the corresponding manifolds. It is also the same as convergence of the limit sets:

Theorem 3.46 *(Kerckhoff-Thurston, see [MT, Mar]) Suppose the groups G_n are geometrically finite and boundary incompressible, and suppose that G_n converge geometrically to H. Then $\Lambda(G_n)$ converges to $\Lambda(H)$ in the sense of Hausdorff convergence of closed subsets of $\hat{\mathbb{C}}$.*

There is also a formulation in terms of convergence of regular sets, see for example [MT, Mar].

There is another definition of convergence which is more natural if we are thinking in terms of groups of matrices:

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[2]General references for this section are [MT, Kap, Mar, Thu80].

Definition 3.47 *Let Γ be a fixed abstract group (think $\Gamma = \pi_1(M)$). Suppose we have a sequence of homomorphisms $\rho_n : \Gamma \to G_n$ where G_n is Kleinian. Suppose that for each $\gamma \in \Gamma$, the sequence $\rho_n(\gamma)$ converges to a Möbius map $\rho_\infty(\gamma)$. Then G_n converges* algebraically *to the group $G_\infty = \{\rho_\infty(\gamma) : \gamma \in \Gamma\}$. G_∞ is called the* algebraic limit *of the G_n.*

It is an important and non-obvious fact that the algebraic limit of discrete groups is discrete. More precisely:

Theorem 3.48 *(Chuckrow) Suppose that Γ is a non-elementary Kleinian group and that each ρ_n is an isomorphism to a Kleinian group G_n. Suppose that G_n converges algebraically to G_∞. Then G_∞ is Kleinian and non-elementary and ρ_∞ is an isomorphism.*

PROOF. This follows from several applications of Jørgensen's inequality Theorem 1.8. For example, if ρ_∞ is not injective, there exists $g \in \Gamma$ with $\rho_\infty(g) = I$. Pick a loxodromic $h \in \Gamma$ such that g, h have distinct fixed points. Then $\langle \rho_n(g), \rho_n(h) \rangle$ is non-elementary and $\rho_n(g) \to I$. So

$$|\mathrm{Tr}^2 \rho_n(g) - 4| + |\,\mathrm{Tr}\, \rho_n(g)\rho_n(h)\rho_n(g)^{-1}\rho_n(h)^{-1} - 2| \to 0$$

which is impossible. $\square$

The relationship between algebraic and geometric convergence is rather subtle.

Theorem 3.49 *(Jørgensen-Marden) Suppose that G_n converges to G_∞ algebraically. Then there exists a subsequence G_{n_r} which converges geometrically to a Kleinian group H. Moreover any geometric limit of any subsequence of the groups G_n contains G_∞.*

If G_∞ and H are finitely generated, then there is a sequence of surjective homomorphisms $\psi_n : H \to G_n$ such that $\lim \psi_n(h) = h$ for all $h \in H$.

The best situation is when G_n converges both algebraically and geometrically to the same limit. In this case the convergence is called *strong*. In general the algebraic and geometric limits may not agree. What happens is that 'extra parabolics' appear in the limit. Roughly, one can have a sequence of loxodromics h_n converging to a parabolic h such that certain powers $h_n^{m_n}$ also converge to a parabolic h' with the same fixed point but with a different translation direction. This mechanism is described in various places, for example [MT]. There is an explicit example, with pictures, in [MSW] P. 340 ff. Kerckhoff and Thurston produced a famous example in which the algebraic and geometric limits differ, on which much subsequent work is based.

It seems likely that the apperance of extra parabolics is the only reason the limit may not be strong:

Theorem 3.50 *Suppose that $\rho_n(\Gamma)$ converges to $G_\infty = \rho_\infty(\Gamma)$ algebraically. The convergence is strong if either:*

(i) $\rho_n(\Gamma)$ is geometrically finite for each n and there are no new parabolics (ie $g \in G_\infty$ is parabolic iff $\rho_n(g)\rho_\infty^{-1}(g)$ is parabolic for each n) or

(ii) There are no new parabolics and $\Omega(G_\infty) \neq \emptyset$.

A lot of effort has gone into proving that various properties persist in the algebraic limit. The following great result is the outcome of many years' work by Thurston, Brock, Bromberg, Canary, Evans, Ohshika, Souto and others:

Theorem 3.51 *([BS]) The algebraic limit of geometrically finite groups is tame.*

Cusp groups   Here is a much easier result which we shall need in the next lecture. It is a simple consequence of Theorem 3.41.

Theorem 3.52 *Suppose that $G_n = \rho_n(G_0)$ is a sequence in $\mathcal{QI}(M)$ for some geometrically finite manifold $M = \mathbb{H}^3/G_0$. Suppose also that the algebraic limit $\rho_\infty(G_0)$ is geometrically finite but not in $\mathcal{QI}(M)$. Then there is an element $g \in G_0$ such that $\rho_n(g)$ is loxodromic but $\rho_\infty(g)$ is parabolic.*

Such groups are called *cusp groups*, because a geodesic loop which had some definite length in $\mathbb{H}^3/G_0$ has been 'pinched' to become parabolic. These new parabolics all correspond to rank 1 cusps. The existence of cusp groups was first proved by Bers and Maskit using sequences of quasiconformal deformations. Any 'reasonable' collection of curves on the boundary of a hyperbolic 3-manifold can be pinched in this way. If there are no more loops which can be pinched the group is called *maximally parabolic*. Each component of the boundary of such a group is either a cusp torus or a triply punctured sphere. Such groups are rigid: to see this, double across the triply punctured spheres in the boundary and apply Mostow rigidity. The limit set of each triply punctured sphere group is a circle and every component of Ω is a round disk.

If G is quasifuchsian, then $\mathbb{H}^3/G$ is homeomorphic to $S \times (0,1)$ for some topological surface S. Choose a maximal set of pairwise disjoint non-homotopic curves on S (a pants decomposition). All these curves can be made parabolic by pinching the corresponding loops on Ω^-/G. Ω^+/G remains an unpinched surface homeomorphic to S. By Theorem 3.42, the deformation space of this family of groups is exactly Teich$S$. One can parameterise it as a $(3g - 3 + 2b)$ complex dimensional subspace of the space of all representations $\pi_1(S) \to SL(2, \mathbb{C})$. This is called the *Maskit embedding of Teichmüller space*. If S is a torus with one puncture, the Maskit embedding has complex dimension one. This is examined in great detail in [MSW].

Lecture 4: Geometrically infinite groups

What can one say about groups which are not geometrically finite? For many years they were a mystery, but as a result of remarkable work over the last few years, we now have an essentially complete picture of what they all are. A systematic reference for this section will be [Mar].

Recall from the last lecture that a group is called tame if $\mathbb{H}^3/G$ is homeomorphic to the interior of a compact 3-manifold. As we saw, any geometrically finite group is tame.

Conjecture 4.53 (Marden's tameness conjecture/theorem) *Every finitely generated group is tame.*

Conjecture 4.54 (Bers' density conjecture/theorem) *Every Kleinian group is an algebraic limit of geometrically finite groups.*

Conjecture 4.55 (Ending lamination conjecture/theorem) *Every tame group is determined up to isometry by its 'end invariants'.*

We shall explain the meaning of 'end invariants' later in the lecture.

These conjectures are now all essentially proved. Under the hypothesis of tameness, Bromberg introduced some beautiful ideas to prove Conjecture 4.54 in special cases. This was extended to the general incompressible boundary case in [BB]. A completely different proof follows by combining Conjectures 4.53 and 4.55.

Our starting point is a geometrically finite group G with corresponding manifold $M = \mathbb{H}^3/G$. What are all the groups which can be obtained as algebraic limits of groups in the quasi-isometric deformation space $\mathcal{QI}(M)$? We think of $\mathcal{QI}(M)$ as embedded in the larger space of representations $\pi_1(M) \to SL(2, \mathbb{C})$.[3] The space of representations, taken up to conjugacy in $SL(2, \mathbb{C})$, turns out to be a smooth complex variety $\mathcal{R}(M)$ of the 'expected' dimension, namely the sum of the dimensions of the Teichmüller spaces associated to the components of ∂M_G. It follows from the simultaneous uniformisation theorem that $\mathcal{QI}(M)$ is an open subset of $\mathcal{R}(M)$.

In the last lecture we discussed the geometrically finite groups on $\partial \mathcal{QI}(M)$. They are the cusp groups formed by pinching one or more loxodromic elements until they become parabolic. This leads to:

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[3]We have to be careful to specify whether or not the representations should be type preserving, see eg [Kap] for details.

Theorem 4.56 *(Greenberg) There exist Kleinian groups which are not geometrically finite.*

PROOF. A cusp group in which the element g is parabolic lies in the subvariety of $\mathcal{R}(M)$ on which $\mathrm{Tr}^2 g = 4$. Since there are only countably many possible loops we can pinch to make cusps, and since each subvariety has complex codimension 1, the union of these sets cannot be all of $\partial \mathcal{R}(M)$. $\square$

Here is a famous recent result, conjectured by Bers and orginally proved in a special case by McMullen.

Theorem 4.57 *([CCHS, CH]) Cusp groups are dense on $\partial \mathcal{QI}(M)$.*

$\partial \mathcal{QI}(M)$ is a fascinating object which sadly we do not have time for here. It appears to have complicated fractal structure. For pictures and an account of the boundary of the Maskit embedding of Teichmüller space for the once punctured torus, see [MSW].

The geometrically finite groups on $\partial \mathcal{QI}(M)$ are produced by pinching simple closed curves. Thurston had the wonderful idea that the remaining groups on $\partial \mathcal{QI}(M)$ are produced by 'pinching' some other objects which in a certain sense 'complete' the space of simple closed curves on ∂M_G. The objects in question are *projective measured laminations*, which Thurston used to compactify Teichmüller space. We digress to explain very briefly what these are.

Geodesic laminations   A standard reference for this section is [FLP], see also [EM, Thu80]. There is a good brief summary in the appendix of [O].

Let S be a hyperbolic surface. A geodesic (not necessarily closed) on S is *simple* if it does not intersect itself. A *geodesic lamination* on S is a closed set which is the disjoint union of simple geodesics called its *leaves*. Such a lamination forms a partial foliation of the surface. For example, it might consist of finitely many pairwise disjoint simple closed geodesics. More typically, however, a transversal to a lamination intersects the lamination in a Cantor set and the components of the complement are ideal polygons. A lamination is *arational* if all complementary components are ideal triangles or punctured bigons. A result of Birman and Series says that any lamination has Hausdorff dimension 1.

A *measured geodesic lamination* is a geodesic lamination together with a transverse invariant measure. That means, an assignment of a (finite Borel) measure to each transversal, which is invariant under the 'push forward' map along leaves. For example, a closed simple geodesic γ has an associated transverse measure δ_γ which

assigns to a transversal T the measure $i(T, \gamma)$, ie the number of times T intersects γ. More generally, I call a lamination *rational*[4] if it is a sum $\sum_i a_i \delta_{\gamma_i}$ where γ_i are pairwise disjoint closed geodesics and $a_i > 0$. Any transverse measure can be scaled by multiplying by positive scalar. This defines an equivalence relation on measured laminations. The equivalence classes are called *projective measured laminations*.

Let $\mathcal{ML}$ denote the space of measured laminations on S with the weak topology. That is, a sequence $\nu_n \in \mathcal{ML}$ converges to $\nu \in \mathcal{ML}$ iff $\nu_n(T)$ converges to $\nu(T)$ for every transversal T. $\mathcal{ML}$ turns out to be the completion of the rational laminations in this topology. Thurston showed that $\mathcal{ML}(S)$ is a ball of (real) dimension $6g - 6 + 2b$. This dimension is no coincidence! The following remarkable result of Thurston shows that Teichmüller space can be compactified by adjoining the space of projective laminations $\mathcal{PML}(S)$ [FLP, O].

Theorem 4.58 *Suppose that $\omega_n \in \text{Teich}(S)$. Then either:*
   (i) A subsequence ω_{n_r} converges to a point $\omega_\infty \in \text{Teich}(S)$ or
   (ii) There exists $c_n \to 0$ and $\xi \in \mathcal{ML}$ such that

$$c_n l_{\omega_n}(\gamma) \to i(\gamma, \xi)$$

for every simple curve γ on S.

Here $l_\omega(\gamma)$ is the hyperbolic length of γ in the hyperbolic surface whose associated conformal structure is ω, and $i(\gamma, \xi)$ denotes intersection number; it is the continuous linear extension of the geometric intersection number between geodesics to $\mathcal{ML}$. Since the second condition is unchanged if we scale everything by $a > 0$, the limit ξ really only depends on its projective class $[\xi]$. We say that ω_n converges to $[\xi]$.

As a special case, suppose that $\omega_n \to [\xi]$ in such a way that along the sequence ω_n the length $l_{\omega_n}(\gamma)$ of a curve γ stays bounded. Then, since $c_n \to 0$, we have $c_n l_{\omega_n}(\gamma) \to 0$. This means that $i(\gamma, \xi) = 0$, which means that either γ is disjoint from ξ or that ξ contains γ as a closed leaf. This is suggestive that if we degenerated S by pinching γ, the structures on S would converge to $[\delta_\gamma] \in \mathcal{PML}$. (To make a correct formal statement we also have to control the degeneration of $S \setminus \gamma$.)

Recall that a quasifuchsian group is uniquely specified by the structures $\omega^\pm$ on the two components $\Omega^\pm / G$. Denote this group $G(\omega^+, \omega^-)$.

Theorem 4.59 (Thurston's double limit theorem) *Suppose that $\rho_n : \pi_1(S) \to SL(2, \mathbb{C})$ is a sequence of respresentations to the quasifuchsian groups $G_n = G(\omega_n^+, \omega_n^-)$*

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[4]This is not universally accepted terminology. It is not the opposite of *arational*.

and suppose that $\omega_n^\pm$ converge to points $[\xi]^\pm \in \mathcal{PML}$. Suppose also that $\xi^\pm$ are arational and that $i(\xi^+, \xi^-) \neq 0$. Then ρ_n has a subsequence which converges algebraically to a geometrically infinite group. The limit group is doubly degenerate and has ending laminations ξ^+, ξ^-.

We will explain the meaning of the last statement below.

The convex core Before getting to geometrically infinite groups, it is also useful to look at the *convex core* of $\mathbb{H}^3/G$. By definition, this is the smallest closed convex subset of $\mathbb{H}^3/G$ containing all closed geodesics. Alternatively, let $\mathcal{C}(G)$ be the convex hull in $\mathbb{H}^3$ of the limit set $\Lambda(G)$, sometimes called the *Nielsen region* of G. The convex core of $\mathbb{H}^3/G$ is just $\mathcal{C}(G)/G$. The Nielsen region is $\mathbb{H}^3$ iff $\Lambda(G) = \hat{\mathbb{C}}$.

Geometrically finite groups can be neatly characterised in terms of their convex core:

Proposition 4.60 *A non-Fuchsian group Kleinian group G is geometrically finite iff its convex core has finite volume.*

The restriction to non-Fuchsian groups is because the convex core of a Fuchsian group is contained in a single hyperbolic plane, so always has zero volume even if the group is infinitely generated. A Fuchsian group is geometrically finite iff its Nielsen region has finite 2-dimensional area. The Nielsen region equals $\mathbb{H}^2$ unless the group is of the second kind, that is, $\Lambda(G)$ is a proper subset of the circle $\partial \mathbb{H}^2$. In this case, and assuming that G is finitely generated, the Nielsen region is bounded by axes of hyperbolic elements which project to closed geodesics which cut off the infinite ends or funnels on the quotient surface $\mathbb{H}^2/G$.

In 3-dimensions, the boundary of the convex core is a union of surfaces. There is a nice *retraction* from $r : \mathbb{H}^3 \cup \Omega \to \mathcal{C}$: from a point $x \in \mathbb{H}^3 \cup \Omega$ outside $\mathcal{C}$, blow out expanding balloons (spheres if $x \in \mathbb{H}^3$ and horoballs if $x \in \Omega$). Then $r(x)$ is the point at which you first hit $\mathcal{C}$. Since $\mathcal{C}$ is convex, $r(x)$ is well defined and it is pretty obviously continuous and G-invariant. In fact, it provides a homeomorphism between each component of Ω/G and a component of $\partial \mathcal{C}/G$ which it 'faces'.

Thurston showed that $\partial \mathcal{C}/G$ has much more structure. It is an example of a *pleated surface*:

Definition 4.61 *A pleated surface in a hyperbolic 3 manifold M consists of a hyperbolic surface Σ, a geodesic lamination λ on Σ, and a map $f : \Sigma \to M$ such that:*

(i) f is an isometry between the given metric on Σ and the induced metric on $f(M)$.

(ii) The restriction of f to each leaf of λ is an isometry to a geodesic in $\mathbb{H}^3$.

(iii) Each component of $\Sigma \setminus \lambda$ maps isometrically to a piece of totally geodesic plane in M.

The images of the leaves of λ are called the *bending lines* of the pleated surface. Roughly speaking, you can think of a pleated surface as a bent surface in $\mathbb{H}^3/G$ whose lift to $\mathbb{H}^3$ rolls out onto $\mathbb{H}^2$ to give the hyperbolic metric on Σ.

Theorem 4.62 *(Sullivan, see[EM]) If ∂M is incompressible, then there is a universal bound on the Teichmüller distance between the hyperbolic metrics on $\partial\mathcal{C}/G$ and on the corresponding components of Ω/G.*

Geometrically infinite ends   The formal definition of an end of an open 3-manifold M is a bit messy. It is simplified when there is a *compact core*: this is a compact submanifold M_c such that the inclusion $M_c \to M$ is a homotopy equivalence. It is a deep result of Scott, that any 3-manifold with finitely generated fundamental group has a compact core. Thist was refined by McCullough to show that if M is hyperbolic, one can choose the core to have a 'standard' shaped boundary in the neighbourhood of cusps. This is called a relative compact core, see eg [Kap] for details.

We can now define an *end* of M to be a component of $M \setminus M_c$. The ends are in bijective correspondence with the components of ∂M_c. An end is *topologically tame* if it has a neighbourhood U which is homeomorphic to $S \times [0,1)$ for some surface S; it is incompressible if S is incompressible. M is called *boundary incompressible* if every component of ∂M_c is incompressible.

Recall that M is tame if it is homeomorphic to the interior of a compact 3-manifold. We have:

Lemma 4.63 *M is tame iff each of its ends are tame.*

If M is geometrically finite without cusps, then the retraction r provides a homotopy equivalence between $\mathbb{H}^3/G$ and the compact manifold $\mathcal{C}/G$. This can easily be modified in case of cusps to a retraction to the compact manifold $(\mathcal{C} \cap M_G^0)/G$. Thus we can take $(\mathcal{C} \cap M_G^0)/G$ to be the core, so the ends correspond bijectively to components of $\partial\mathcal{C}/G$. In general, an end is called *geometrically finite* if it contains a neighbourhood which is disjoint from $\mathcal{C}/G$ and *geometrically infinite* otherwise. Thurston and Bonahon described the structure of geometrically infinite incompressible ends.

Definition 4.64 *A sequence of closed geodesics γ_n exits the end E if $\gamma_n \subset E$ and if only finitely many γ_n intersect any compact set $K \subset M$.*

Here is the key theorem:

Theorem 4.65 *(Bonahon [B]) Let M be a boundary incompressible hyperbolic 3-manifold. Let E be a geometrically infinite end corresponding to a component of ∂M_c homeomorphic to a surface S. Then there exists a sequence of closed geodesics which exit E. The γ_n can be taken to be homotopic to simple closed curves on S. Moreover the projective measured laminations $[\delta_{\gamma_n}]$ converge to a measured lamination $[\lambda] \in \mathcal{PML}(\mathcal{S})$. The underlying support of λ is arational and independent of the choice of sequence γ_n.*

The 'unmeasured' lamination λ_E whose support is the leaves of $[\lambda]$ is called *the ending lamination* of E. If ξ is any other measured lamination whose support is different from that of λ, then $i(\lambda, \xi) > 0$. We can choose the γ_n to be of bounded hyperbolic length.

Bonahon used the existence of this ending lamination to prove that such an end E is topologically tame. The idea is to construct pleated surfaces Σ_n in E for which the γ_n are contained in the bending lamination. Being hyperbolic surfaces, these pleated surfaces have bounded diameter[5] and also exit the end. They allow one to give E the required product struture $S \times [0, 1)$.

Suppose that M is a boundary incompressible hyperbolic 3-manifold. Its *end invariants* consist of:

(i) The conformal structures on the components of Ω/G (one for each geometrically finite end) and

(ii) The ending laminations of the geometrically infinite ends.

Theorem 4.66 *(Ending lamination theorem [Min03, BCM]) Suppose that $M = \mathbb{H}^3/G$ is a boundary incompressible hyperbolic 3-manifold. Then M is uniquely determined up to isometry by its end invariants.*

Marden rightly calls this a 'blockbuster theorem'. Expositions of parts can also be found in [Min01, Min02] The recent work on the tameness conjecture allows one to get rid of the assumption that M is a boundary incompressible. This we have to leave for Souto's lectures next week. Here is a closely related consequence.

Theorem 4.67 *Suppose that G_1 and G_2 are finitely generated Kleinian groups, and that there is a homeomophism $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which induces a type preserving isomorphism $\chi : G_1 \to G_2$. Suppose also that ψ is conformal on $\Omega(G_1)$. Then ψ is Möbius.*

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[5]This statement has to be suitably modified if there are cusps.

This should be compared to Sullivan's rigidity theorem (see eg [MT, Kap]) which asserts the same result but only under the much stronger assumption that ψ is quasicomformal.

Manifolds which fibre over the circle Ending laminations are very well illustrated with the example of hyperbolic structures on manifolds which fibre over the circle. Start with a hyperbolisable surface S, and a pseudo-Anosov map $\phi : S \to S$. By definition, such a map does not fix any closed curve on the surface. Instead, there is a pair of measured laminations $\lambda^\pm$ with the property that $\phi_*^{\pm n}(\gamma) \to [\lambda^\pm]$ in $\mathcal{PML}$. Thurston constructed a hyperbolic structure on the 3-manifold $M = (S \times [0, 1])/\sim$, where $\sim$ is the equivalence relation which identifies $(x, 1)$ with $(\phi(x), 0)$, see [O]. The cyclic cover $\tilde{M}$ of this manifold has fundamental group $\pi_1(S)$ and is a limit of quasifuchsian groups $G(\phi_*^n \omega_0, \phi_*^{-n} \omega_0)$ where ω_0 is any choice of initial conformal structure on S. Thus $\tilde{M} = \mathbb{H}^3/G$ for some Kleinian group G isomorphic to $\pi_1(S)$. The map ϕ induces an isometry $\tilde{M} \to \tilde{M}$. $\tilde{M}$ is homeomorphic to $S \times (-\infty, \infty)$ and so has two ends. The geodesic representatives of $\phi_*^n(\gamma)$ in $\tilde{M}$ exit one end of $\tilde{M}$ and those of $\phi_*^{-n}(\gamma)$ the other. Thus the ending laminations of $\tilde{M}$ are exactly the laminations $[\lambda^\pm]$.

G provides a famous example of a Kleinian group whose limit set is a space filling curve. Involved in the construction of G is the fact that $\Lambda(G) = \hat{\mathbb{C}}$. Let G_0 be the Fuchsian such that $\mathbb{H}^2/G_0$ has the conformal structure ω_0. The group G is isomorphic to G_0. There is a continuous map from $S^1 = \Lambda(G_0)$ to $\Lambda(G)$, providing a continuous mapping from the unit circle S^1 onto the Riemann sphere $\hat{\mathbb{C}}$.

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