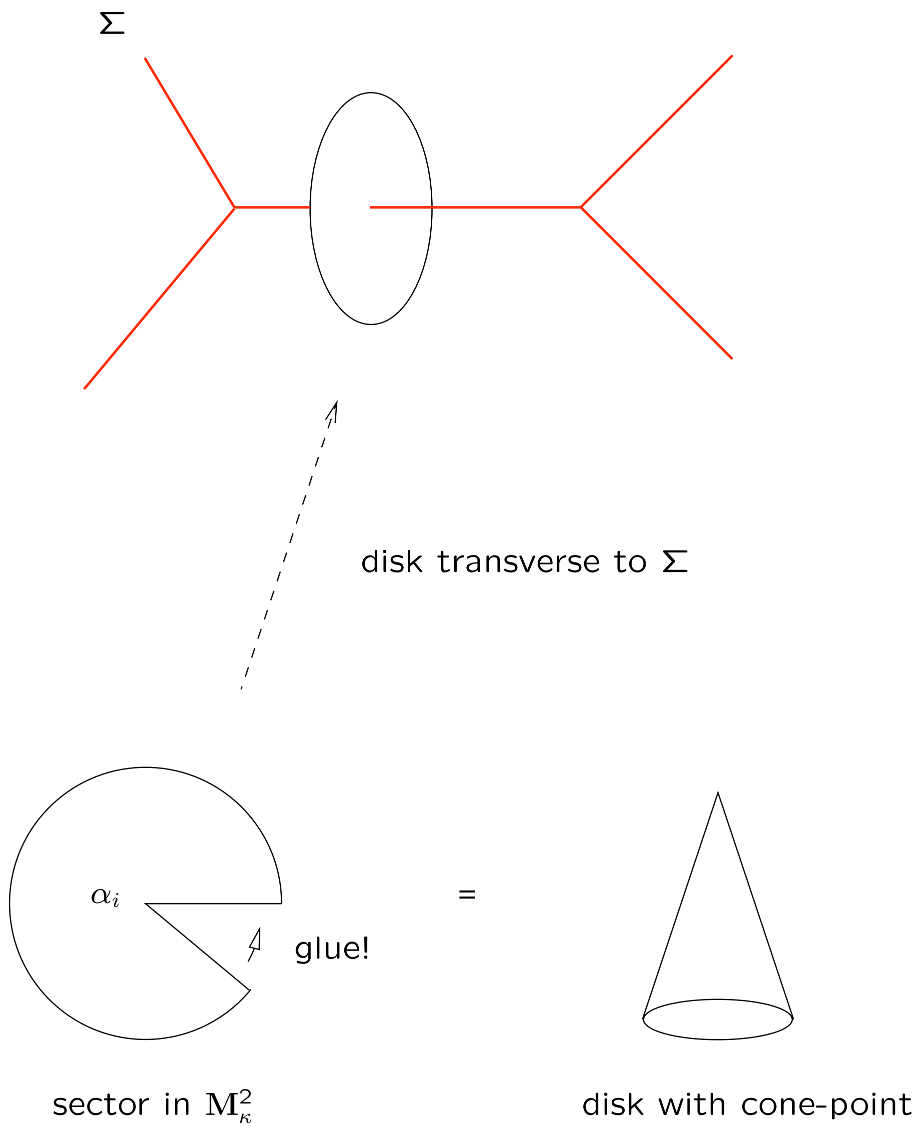
Rigidity of cone-3-manifolds

ICTP Trieste

June 2005

Hartmut Weiß (LMU München)

cone-3-manifold of curvature κ: metric space locally modelled on the κ-cone over a spherical cone-surface $\cong S^2$



generalizes notion of geometric orbifold, where cone-angles are of the form $2\pi/n$, $n \geq 2$

cone-angles $\leq 2\pi \Rightarrow$ curvature bounded below by κ in the triangle comparison sense

cone-angles $\leq \pi \Rightarrow$ singular locus Σ a trivalent graph

topological type of $C :=$ homeomorphism type of the pair (C, Σ)

local rigidity holds $:\Leftrightarrow$ the deformation space of hyperbolic (spherical) cone-manifold structures is locally parametrized by the vector of cone-angles

global rigidity holds $:\Leftrightarrow$ the isometry type of C is determined by the topological type of C and the vector of cone-angles

1. The hyperbolic case

Theorem [Kojima]: Global rigidity holds for hyperbolic cone-3-manifolds with cone-angles $\leq \pi$ and singular locus a link.

Proof:

Decrease cone-angles to 0 using

Theorem [Hodgson-Kerckhoff]: Local rigidity holds for closed hyperbolic cone-3-manifolds with cone-angles $\leq 2\pi$ and singular locus a link.

and the techniques used in the proof of the cyclic case of the Orbifold Theorem.

Then use Mostow rigidity for complete hyperbolic 3-manifolds of finite volume to deduce global rigidity.

Theorem [W.]: Global rigidity holds for closed hyperbolic cone-3-manifolds with cone-angles $\leq \pi$ in the general case.

Proof:

Follow the same strategy as Kojima: decrease cone-angles to 0, then use Mostow rigidity.

cone-angles 0 $\Leftrightarrow$ complete hyperbolic 3-manifold of finite volume, possibly with totally geodesic boundary consisting of thrice punctured spheres

Geometry of links of singular vertices changes:

$\alpha + \beta + \gamma > 2\pi \Leftrightarrow$ spherical $\mathbf{S}^2(\alpha, \beta, \gamma)$

$\alpha + \beta + \gamma = 2\pi \Leftrightarrow$ horospherical $\mathbf{E}^2(\alpha, \beta, \gamma)$

$\alpha + \beta + \gamma < 2\pi \Leftrightarrow$ hyperspherical $\mathbf{H}^2(\alpha, \beta, \gamma)$

Local deformation theory:

Theorem [W.]: Local rigidity holds for hyperbolic cone-3-manifolds of finite volume with cone-angles $\leq \pi$, at most finitely many ends which are (smooth or singular) cusps with compact cross-sections $\neq \mathbf{E}^2(\pi,\pi,\pi,\pi)$, and possibly with totally geodesic hyperbolic turnover boundary.

Proof:

Let $M = C \setminus \Sigma$ be the smooth part and

$$\begin{aligned}\mathcal{E} &= \mathfrak{so}(TM) \oplus TM \\ &= \tilde{M} \times_{Ad\circ hol} \mathfrak{sl}_2(\mathbb{C})\end{aligned}$$

the flat bundle of infinitesimal isometries.

Step 1: Prove $H^1_{L^2}(M, \mathcal{E}) = 0$ using analysis on manifolds with conical singularities (Cheeger, Brüning-Seeley).

Step 2: Analyze the variety of representations of $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$ near the holonomy of a hyperbolic cone-manifold structure.

Study of degenerations:

Geometry of hyperbolic cone-manifolds with $diam(C) \geq D > 0$ and cone-angles $\leq \alpha < \pi$ according to Boileau, Leeb and Porti:

thin parts: $\exists$ a short list of local models for the thin part of C (smooth Margulis tubes, tubes around closed singular geodesics, umbilic tubes with turnover cross-sections)

thickness: $\exists\, r = r(D, \alpha) > 0$ such that C contains an embedded smooth standard ball of radius r.

thickness $\Rightarrow$ no collapse

finiteness: $vol(C) < \infty \Rightarrow C$ has at most finitely many ends, all of which are (smooth or singular) cusps with compact cross-sections, i.e. T^2 or $\mathbf{E}^2(\alpha, \beta, \gamma)$.

Finishing the proof (the essential step):

Given a family of hyperbolic cone-3-manifolds $(C_t)_{t\in(t_\infty,1]}$ with cone-angles $(t\alpha_1,\ldots,t\alpha_N)$ and $C_1 = C$, show that this family extends to the closed interval $[t_\infty, 1]$!

Schläfli's formula: $vol(C_t) \nearrow$ as $t \searrow t_\infty$

$\Rightarrow diam(C_t) \geq D$

Kojima's straightening argument: $vol(C_t) \leq V$

Boileau, Leeb and Porti: The only possible degenerations are tubes around closed (smooth or singular) geodesics opening into rank-2 cusps.

These cusps can be closed via hyperbolic Dehn surgery (in the setting of hyperbolic cone-3-manifolds).

2. The finite-volume case

The same proof yields the following result in the finite-volume case:

Theorem [W.]: Global rigidity holds for hyperbolic cone-3-manifolds of finite volume with cone-angles $\leq \pi$, at most finitely many ends which are (smooth or singular) cusps with compact cross-sections $\neq \mathbf{E}^2(\pi, \pi, \pi, \pi)$, and possibly with totally geodesic hyperbolic turnover boundary.

Remark: If cone-angles are $< \pi$, by the finiteness result of Boileau, Leeb and Porti, this is is the general finite-volume case.

3. The spherical case

Theorem [W.]: Global rigidity holds for closed spherical cone-3-manifolds with cone-angles $\leq \pi$ which are not Seifert fibered.

Proof:

Use the spherical version of local rigidity, i.e.

Theorem [W.]: Local rigidity holds for closed spherical cone-3-manifolds with cone-angles $\leq \pi$ which are not Seifert fibered.

and the fact that spherical cone-3-manifolds don't collapse according to Boileau, Leeb and Porti to deform cone-angles to π.

Global rigidity follows from

Theorem [de Rham]: A spherical structure on a closed 3-orbifold is unique.