Minimal Hyperbolic 3-Manifolds and Character Varieties

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- 3-mflds: compact, connected, orientable, irred.

Definition

- Say V^3 *dominates* W^3, written

$$V \geq W$$

if there exists a non-zero degree map

$$f : (V, \partial V) \to (W, \partial W)$$

- Say V *strictly dominates* W if $V \geq W$ but $V \not\cong W$.

Examples

(1) $\partial V \cong S^1 \times S^1 \Rightarrow V \geq S^1 \times D^2$

(2) V, W closed and $\pi_1(W)$ finite implies $V \geq W$.

Thus "$\geq$" not a partial order (eg. $S^3 \geq P^3 \geq S^3$).

If $f : V \geq W$, Gromov's inequality

$$\|V\| \geq d(f)\|W\|$$

implies that it is a partial order when restricted to

$$\{V : V \text{ has at least one hyperbolic piece }\}$$

Definition

$$V \text{ is } \textit{minimal} \text{ if } V \gneq W \Rightarrow \begin{cases} |\pi_1(W)| < \infty & \text{if } \partial V = \emptyset \\ W \cong S^1 \times D^2 & \text{if } \partial V \cong S^1 \times S^1 \end{cases}$$

Before we discuss examples, recall that

- The $PSL_2(\mathbb{C})$ *representation variety* of a finitely generated group Γ is the set

  $$R_{PSL_2}(\Gamma) = Hom(\Gamma, PSL_2(\mathbb{C}))$$

  endowed with the compact-open topology. It is a complex, affine algebraic set in a natural way.

- $PSL_2(\mathbb{C})$ acts on $R_{PSL_2}(\Gamma)$ by conjugation.

- The $PSL_2(\mathbb{C})$ *character variety* of Γ is the set

  $$X_{PSL_2}(\Gamma) = R_{PSL_2}(\Gamma)//PSL_2(\mathbb{C})$$

  In other words, $\mathbb{C}[X_{PSL_2}(\Gamma)] = \mathbb{C}[R_{PSL_2}(\Gamma)]^{PSL_2(\mathbb{C})}$.

- $R_{PSL_2}(V) = R_{PSL_2}(\pi_1(V)), X_{PSL_2}(V) = X_{PSL_2}(\pi_1(V))$

- If V^3 is small, then

  $$\dim_{\mathbb{C}}(X_{PSL_2}(V)) = \begin{cases} 0 & \text{if } \partial V = \emptyset \\ 1 & \text{if } \partial V \cong S^1 \times S^1 \end{cases}$$

Examples

1) $\underline{\partial V \cong S^1 \times S^1}$

(a) punctured torus bundles whose monodromies are not proper powers.

(b) hyperbolic twist knot exteriors M

<u>Idea</u> There is only one non-trivial curve in $X_{PSL_2}(M)$ (Burde) and M covers only itself.

(c) the exterior M of the $(-2, 3, n)$ pretzel knot is minimal iff $n \not\equiv 0 \pmod 3$.

<u>Idea</u> There is only one non-trivial curve in $X_{PSL_2}(M)$ if $n \not\equiv 0 \pmod 3$ (Mattman). When $n \equiv 0 \pmod 3$ there are two, the extra one coming from a strict domination $M \geq T_{2,3}$. Also, M covers only itself.

(d) the exterior of the $\frac{p}{q}$ rational knot is not minimal in general (Ohtsuki-Riley, Sakuma). If $M_{\frac{p}{q}} \geq W$, then $W \cong M_{\frac{p'}{q'}}$ for some p' which divides p. Can use $PSL_2(\mathbb{C})$ character variety methods to show that if $M_{\frac{p}{q}} \geq M_{\frac{p'}{q'}}$, then either $M_{\frac{p}{q}} = M_{\frac{p'}{q'}}$ or $p > p'$. In particular

$$p \text{ prime } \Rightarrow M_{\frac{p}{q}} \text{ minimal.}$$

2) $\underline{\partial V = \emptyset}$ (Reid-Wang)

(a) $V = M \cup M'$ where $M = M'$ trefoil exteriors glued together so that $\mu = \phi'$ and $\phi = \mu'$.

Idea All representations to $PSL_2(\mathbb{C})$ have finite cyclic image. Thus V is minimal in $\mathcal{M}$ where

$\mathcal{M} = \{V : V$ Haken or admits a geometric structure$\}$

(b) $V = M(\frac{1}{2})$ where M is the exterior of the figure 8 knot.

Idea All irreducible representations to $PSL_2(\mathbb{C})$ are faithful and there are exactly two conjugacy classes of discrete representations. Further, V covers only itself. Thus V minimal in $\mathcal{M}$.

Theorem 1 *Only finitely many surgeries on a hyperbolic twist knot (eg. the figure 8 knot) can yield a manifold which is not minimal in the class $\mathcal{M}$.*

Theorem 2 *Suppose that M is a small hyperbolic 3-manifold with torus boundary such that*

- *M does not strictly dominate any hyperbolic 3-manifold.*
- *there is a slope α_0 on ∂M such that $M(\alpha_0)$ does not dominate any hyperbolic 3-manifold.*

Suppose as well that for any essential surface S_0 in M, either

(a) $|\partial S_0| \leq 2$, *or*

(b) *for each norm curve $X_0 \subset X_{PSL_2}(M)$, there is a character $\chi_\rho \in X_0$ which restricts to an irreducible character on $\pi_1(S_0)$.*

Then there are only finitely many slopes α on ∂M such that the manifold $M(\alpha)$ can strictly dominate a closed hyperbolic 3-manifold.

Examples

1) hyperbolic twist knot exteriors.

2) 2-bridge knot exteriors $M_{\frac{p}{q}}$ with p prime.

3) $(-2, 3, n)$ pretzel knot exterior.

4) many pctrd torus bundles (eg. RSS examples).

Remark First two conditions are necessary.

Idea of proof

For $n > 0$, consider non-zero degree maps

$$f_n : M(\alpha_n) \to V_n, V_n \text{ closed hyperbolic}$$

where $\alpha_n \neq \alpha_m$ for $n \neq m$ and f_n not a htpy equiv

- define $\rho_n \in R_{PSL_2}(M)$ by

$$\begin{tikzcd} \pi_1(M(\alpha_n)) \arrow[rr, "(f_n)_\#"] & & \pi_1(V_n) \arrow[rr, "\psi_n", "\cong"'] & & \Gamma_n \arrow[d] \\ \pi_1(M) \arrow[u] \arrow[rrrr, "\rho_n"] & & & & PSL_2(\mathbb{C}) \end{tikzcd}$$

- hypotheses $\Rightarrow$ $\ker(\rho_n|\pi_1(\partial M)) = \langle \alpha_n \rangle$

- set $\chi_n = \chi_{\rho_n} \in X_{PSL_2}(M)$

- characters distinct and $\dim X_{PSL_2}(M) = 1$ so WLOG

$$\chi_n \in X_0 \subset X_{PSL_2}(M)$$

where X_0 is an affine curve.

Two cases to consider:

(1) $\{\chi_n\}$ subconverges to some $\chi_{\rho_0} \in X_0$

(2) $\{\chi_n\}$ subconverges to an ideal point of X_0

Case 1. $\{\chi_n\}$ subconverges to some $\chi_{\rho_0} \in X_0$

- WLOG $\lim \rho_n = \rho_0$
- set $\Gamma_n := \rho_n(\pi_1(M))$ $(n \geq 0)$ and note Γ_n torsion-free, cocpct, non-elem, Kleinian for $n \geq 1$
- Jørgensen-Marden $\Rightarrow$

(a) Γ_0 torsion-free, non-elem, Kleinian

and after passing to a subsequence we may assume that

(b) $\{\Gamma_n\}$ converges geometrically to torsion-free, non-elem, Kleinian Γ containing Γ_0.
(c) $\lim_n V_n = V := \mathbb{H}^3/\Gamma$ (Gromov-Hausdorff)
(d) $\exists$ homoms $\theta_n : \Gamma \to \Gamma_n$ such that $\rho_n = \theta_n \circ \rho_0$

Note
(c) $\Rightarrow vol(V) = \lim vol(V_n) \leq \lim vol(M(\alpha_n)) = vol(M)$
(d) $\Rightarrow \rho_0|\pi_1(\partial M)$ injective $\Rightarrow V$ has cusps and V_n obtained by Dehn filling V for $n \gg 0$

Can construct map $g : (M, \partial M) \to (V, \partial V)$ realizing ρ_0 and show has non-zero degree. Then WLOG, g a homeomorphism and can show that for infinitely many n, f_n is induced (up to htpy) by g. For such n, $f_n : M(\alpha_n) \xrightarrow{\simeq} V_n$, which contradicts our hypotheses.

Case 2. $\{\chi_n\}$ sbconvrges to an ideal point x_0 of X_0

Let $S_0 \subset M$ be a conn. essential surface associated to x_0.

- Culler-Shalen theory $\Rightarrow \lim_n \chi_n|\pi_1(S_0) = \chi_\rho \in X_{PSL_2}(S_0)$ where ρ is reducible.

- WLOG $\lim_n \rho_n|\pi_1(S_0) = \rho$

- $\rho_n(\pi_1(S_0))$ Kleinian $\Rightarrow$ $\rho_n(\pi_1(S_0))$ elementary
  $\Rightarrow \rho_n(\pi_1(S_0))$ cyclic, loxodromic

Subcase Condition (a) holds (so $|\partial S_0| \leq 2$)

Show χ_n cannot be bent along S_0 ($n \gg 0$) so that the restriction of ρ_n to some complementary component of S_0 has cyclic image. Then the condition $|\partial S_0| \leq 2$ implies that ρ_n has cyclic image, which is false.

Subcase Condition (b) holds

Since $\chi_n|\pi_1(S_0)$ reducible for large n, can show that $\chi|\pi_1(S_0)$ reducible for all $\chi \in X_0$. Can also show that X_0 is a norm curve. This contradicts condition (b).

Theorem 3 *Suppose that M is a small hyperbolic 3-manifold with torus boundary such that*

- *M does not strictly dominate any Seifert manifold besides $S^1 \times D^2$.*
- *there is a slope α_0 on ∂M such that $\pi_1(M(\alpha_0))$ does not admit a surjective homomorphism onto a Fuchsian triangle group.*
- *the image of $\pi_1(\partial M)$ under any surjective homomorphism of $\pi_1(M)$ onto a Fuchsian triangle group is infinite.*

and that either

- (a) *$H_1(M) \cong \mathbb{Z}$ and Δ_M is not divisible by a cyclotomic polynomial, or*
- (b) *$H_1(M) \cong \mathbb{Z}$ and $X_{PSL_2}(M(\lambda))$ is finite, or*
- (c) *each non-trivial curve in $X_{PSL_2}(M)$ is a norm curve*

Suppose as well that for any essential surface S_0 in M and any norm curve $X_0 \subset X_{PSL_2}(M)$, there is a character $\chi_\rho \in X_0$ which restricts to a strictly irreducible character on $\pi_1(S_0)$. Then there are only finitely many slopes α on ∂M such that the manifold $M(\alpha)$ can strictly dominate an $\widetilde{SL_2}$ manifold.

Example A hyperbolic twist knot exterior M satisfies all the hypotheses. Moreover,

- if $M(\alpha)$ dominates a Haken manifold (eg. any $\mathbb{E}^3, \mathbb{H}^2 \times \mathbb{E}^1$, or Sol manifold) or a reducible manifold (eg. an $S^2 \times \mathbb{E}^1$ manifold), then α a bound slope. This happens for only finitely many slopes.
- a simple homological argument shows that if some filling of M dominates a non-Haken Nil manifold, then there is a surjective homomorphism $\pi_1(M) \to \Delta(2,3,6)$. But this is readily shown to be impossible. Thus Theorem 1 holds.

Addendum *If we remove the condition on representations to Fuchsian triangle groups, we can still conclude that there are proper subgroups $L_1, L_2, \ldots, L_k \subset H_1(\partial M)$, none of which contain α_0, such that for each slope α in $H_1(\partial M) \setminus (L_1 \cup L_2 \cup \cdots \cup L_k)$, $M(\alpha)$ does not strictly dominate an $\widetilde{SL_2}$ manifold.*

Examples

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2) $(-2, 3, n)$ pretzel knot exterior, $n \not\equiv 0 \pmod 3$.
3) many pctrd torus bundles (eg. RSS examples).