# COVERING SPACES OF 3-ORBIFOLDS

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CONJECTURE :

ANY CLOSED ORIENTABLE HYPERBOLIC 3-ORBIFOLD WITH NON-EMPTY SINGULAR LOCUS HAS LARGE FUNDAMENTAL GROUP.

# LARGE GROUPS

A GROUP IS LARGE IF SOME FINITE INDEX SUBGROUP ADMITS A SURJECTIVE HOMOMORPHISM ONTO A NON-ABELIAN FREE GROUP.

LARGENESS $\Rightarrow$

1. $vb_1 = \infty$ WHERE

$$vb_1(G) = \sup\{ b_1(G_i) : G_i \underset{F.I}{\leq} G \}$$

2. SUPER-EXPONENTIAL SUBGROUP GROWTH

3. LINEAR GROWTH OF $p$-HOMOLOGY

# BASIC PRINCIPLE

EASIER $\longleftrightarrow$ HARDER

3-MANIFOLDS WITH NON-EMPTY BOUNDARY

CLOSED 3-MANIFOLDS

$\uparrow$

3-ORBIFOLDS WITH NON-EMPTY SINGULAR LOCUS

**THEOREM**: [COOPER-LONG-REID]

LET $M$ BE A COMPACT ORIENTABLE IRREDUCIBLE 3-MANIFOLD WITH NON-EMPTY BOUNDARY. THEN EITHER $M = B^3$, $S^1 \times D^2$, $T^2 \times I$ OR $K^2 \tilde{\times} I$, OR $\pi_1 M$ IS LARGE.

# THE EVIDENCE, I

THEOREM 1 [L]: LET $M$ BE A COMPACT ORIENTABLE 3-MANIFOLD, AND LET $K$ BE A KNOT IN $M$ SUCH THAT $M-K$ HAS A FINITE-VOLUME HYPERBOLIC STRUCTURE. FOR ANY $n \in \mathbb{N}$, LET $M(K,n)$ BE THE ORBIFOLD OBTAINED BY PLACING A SINGULARITY ALONG $K$ OF ORDER $n$. THEN $\pi_1 M(K,n)$ IS LARGE FOR INFINITELY MANY $n$.

# THE EVIDENCE

## II: SUBGROUP GROWTH

FOR A FINITELY GENERATED GROUP $G$, LET $s_n(G)$ BE THE NUMBER OF SUBGROUPS WITH INDEX $\leq n$, 'SUBGROUP GROWTH FUNCTION'.

TWO FUNCTIONS $f, g : \mathbb{N} \to \mathbb{N}$ HAVE THE SAME GROWTH TYPE IF

$\exists\, K > 1$ S.T $f(n)^{K^{-1}} \leq g(n) \leq f(n)^K$

$G$ LARGE $\Rightarrow$ $s_n(G)$ HAS GROWTH TYPE

$$2^{n \log n}$$

LET $O$ BE A COMPACT ORIENTABLE 3-ORBIFOLD WITH NON-EMPTY SINGULAR LOCUS AND A FINITE VOLUME HYPERBOLIC STRUCTURE.

<u>THEOREM 2</u>: [L] $\pi_1 O$ HAS AT LEAST EXPONENTIAL SUBGROUP GROWTH.

IN FACT, THE SUBNORMAL SUBGROUP GROWTH HAS EXPONENTIAL GROWTH TYPE.

ALTERNATIVE FORMULATION:

ANY LATTICE IN $PSL(2, \mathbb{C})$ WITH TORSION HAS AT LEAST EXPONENTIAL SUBGROUP GROWTH.

CONTRAST THIS WITH WHAT IS KNOWN FOR HYPERBOLIC 3-MANIFOLDS:

<u>THEOREM</u> [LUBOTZKY]

LET $M$ BE A CLOSED HYPERBOLIC 3-MANIFOLD. THEN $\forall \varepsilon > 0, \exists k > 0$ S.T

$$s_n(\pi, M) \geq 2^{k(\log n)^{3-\varepsilon}}$$

# THE EVIDENCE

## III: LINEAR GROWTH OF $p$-HOMOLOGY

LET $p$ BE A PRIME.

LET $\mathbb{F}_p$ BE THE FIELD OF ORDER $p$.

FOR A GROUP $G$,

LET $d_p(G) = \dim H_1(G; \mathbb{F}_p)$

WE SAY THAT A COLLECTION $\{G_i\}$ OF FINITE INDEX SUBGROUPS HAS LINEAR GROWTH OF $p$-HOMOLOGY IF

$$\inf_i \frac{d_p(G_i)}{[G:G_i]} > 0$$

VERY STRONG CONDITION.

$G$ LARGE $\Rightarrow$ $G$ HAS A NESTED COLLECTION OF FINITE INDEX SUBGROUPS WITH LINEAR GROWTH OF $p$-HOMOLOGY.

<u>THEOREM</u> 3: [L] $\pi_1 O$ HAS A NESTED COLLECTION OF FINITE INDEX SUBGROUPS WITH LINEAR GROWTH OF $p$-HOMOLOGY, FOR SOME PRIME $p$.

LET $\{O_i\}$ BE THE CORRESPONDING COVERING SPACES. (WE MAY IN FACT TAKE THESE TO BE MANIFOLDS.)

THE HEEGAARD GRADIENT OF $\{O_i \to O\}$ IS $\inf_i \dfrac{\chi_-^h(O_i)}{\text{degree}(O_i \to O)}$, WHERE

$$\chi_-^h(O_i) = \min\{-\chi(F) : F \text{ IS A HEEGAARD SURFACE FOR } O_i\}$$

LINEAR GROWTH OF $p$-HOMOLOGY

$\Rightarrow$ POSITIVE HEEGAARD GRADIENT

<u>THEOREM</u> : [L] LET $M$ BE A CLOSED ORIENTABLE IRREDUCIBLE 3-MANIFOLD, AND LET $\{M_i \to M\}$ BE A NESTED SEQUENCE OF FINITE-SHEETED REGULAR COVERING SPACES OF $M$. SUPPOSE THAT

i) THE HEEGAARD GRADIENT OF $\{M_i \to M\}$ IS POSITIVE, AND

ii) $\pi_1 M$ DOES NOT HAVE PROPERTY $(\tau)$ W.R.T $\{\pi_1 M_i\}$.

THEN $M_i$ IS HAKEN $\forall i \gg 0$.

# FOCUS ON THMS 2 & 3

NOTE : THM 3 $\underset{\text{ALMOST}}{\Longrightarrow}$ THM 2.

LET $G = \pi_1 O$

SUPPOSE $\{G_i\}$ HAS LINEAR GROWTH OF $p$-HOMOLOGY. THEN :

$$\begin{aligned} & N^{\circ} \text{ SUBGPS OF } G_i \text{ WITH INDEX } p \\ \geq\ & \left| H_1(G_i ; \mathbb{F}_p) - 1 \right| \\ \sim\ & p^{d_p(G_i)} \\ \geq\ & p^{[G:G_i]} \times \text{const} \end{aligned}$$

LET $O$ = CLOSED ORIENTABLE HYPERBOLIC 3-ORBIFOLD WITH NON-EMPTY SING LOCUS.

<u>KEY LEMMA</u> : IF THE SINGULAR LOCUS IS A LINK $L$ AND A PRIME $p$ DIVIDES THE ORDER OF EVERY COMPONENT OF $L$, THEN $d_p(\pi_1 O) \geq |L|$.

<u>PROOF</u> : $O - \mathring{N}(L)$ IS A COMPACT ORIENTABLE 3-MANIFOLD $M$.

$\Rightarrow d_p(M) \geq \frac{1}{2} d_p(\partial M) = |L|$.

$\pi_1 O$ IS OBTAINED FROM $\pi_1 M$ BY KILLING $p^{TH}$ POWERS OF WORDS. THIS LEAVES $d_p$ UNCHANGED. $\square$

# OUTLINE OF PROOF OF THM 2

1. PASS TO A COVERING SPACE $O'$ WHERE THE SINGULAR LOCUS IS A NON-EMPTY ~~LINK~~, AND WHERE EACH COMPONENT HAS ORDER A PRIME $p$.
   (USE SELBERG'S THM).

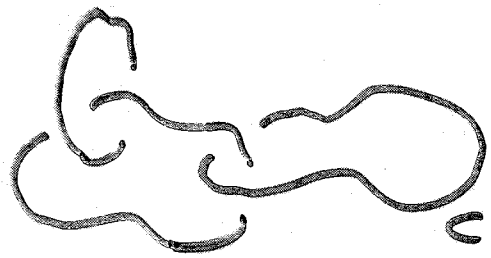
2. PASS TO A COVERING SPACE $O''$ WITH THESE PROPERTIES, AND WHERE $d_p(\pi_1 O'') \geq 11$.
   (SLIGHT GENERALISATION OF A RESULT OF LUBOTZKY)

3. FIND COVERING SPACES $O_i \to O''$ WHERE

$$\begin{array}{l} |\text{sing}(O_i)| \geq \text{degree}(O_i \to O'') \\ \quad\rotatebox{90}{$\geq$} \\ d_p(\pi_1 O_i) \end{array}$$

THIS $\Rightarrow$ LINEAR GROWTH OF $p$-HOMOLOGY.

$O''$




SET $\Gamma = \pi_1 O'' / \langle\langle \pi_1 \partial N(C) \rangle\rangle$

FINITE INDEX SUBGPS OF $\Gamma$

$\rightsquigarrow$ ——— . - - - . OF $\pi_1 O''$ CONTAINING $\langle\langle \pi_1 \partial N(C) \rangle\rangle$

$\rightsquigarrow$ FINITE COVERS OF $O''$ S.T INVERSE IMAGE OF $C$ IS A COLLECTION OF COPIES OF $C$.

<u>THM</u>: [GOLOD - SHAFAREVICH] IF $\Gamma : \langle X | R \rangle$

AND $\dfrac{d_p(\Gamma)^2}{4} > d_p(\Gamma) - |X| + |R|$

THEN $\Gamma$ HAS A NESTED SEQUENCE OF FINITE INDEX SUBGROUPS.

# MANIFOLDS COMMENSURABLE WITH ORBIFOLDS

**THM** : [L] LET $M$ BE A 3-MANIFOLD THAT IS COMMENSURABLE WITH A COMPACT ORIENTABLE FINITE-VOLUME HYPERBOLIC 3-ORBIFOLD WITH NON-EMPTY SINGULAR LOCUS. THEN:

i) $\pi_1 M$ HAS EXPONENTIAL SUBNORMAL SUBGROUP GROWTH

ii) $M$ HAS A NESTED SEQUENCE OF COVERS WITH POSITIVE HEEGAARD GRADIENT.

**EXAMPLES** :

1. MANIFOLDS WITH HEEGAARD GENUS 2
2. ARITHMETIC HYPERBOLIC 3-MANIFOLDS [REID]

# POSSIBLE APPROACHES TO THE CONJECTURE

1. IT FOLLOWS FROM A GROUP-THEORETIC CONJECTURE OF LUBOTZKY & ZELMANOV RELATING TO PROPERTY ($\tau$).

2. THERE IS A LINK WITH THE THEORY OF ERROR-CORRECTING CODES.