Roots of 3-mfds, ...

$\sigma = \{T_1, T_2, \dots\}$: a set of moves on M^3.

Df: M_0^3 is a <u>root</u> of M^3 if:

(1): $M \overset{T_i}{\rightsquigarrow} M_0$

(2): M_0 admits no T_i

Q1: Does M_0 exist?

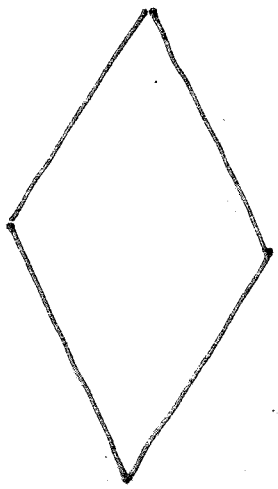
Q2: Is M_0 unique?

Theorems 1 – 10 (for diff. σ)
$\forall M$ the root exists and is unique

$\diamond$

<u>Existance</u>: Yes, if T_i simplify manifolds.
(w.r.t. a complexity function)

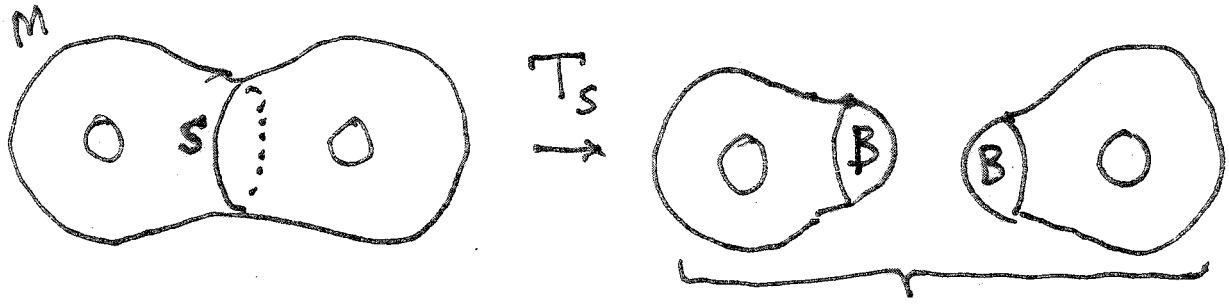
Uniqueness:
diamond trick:

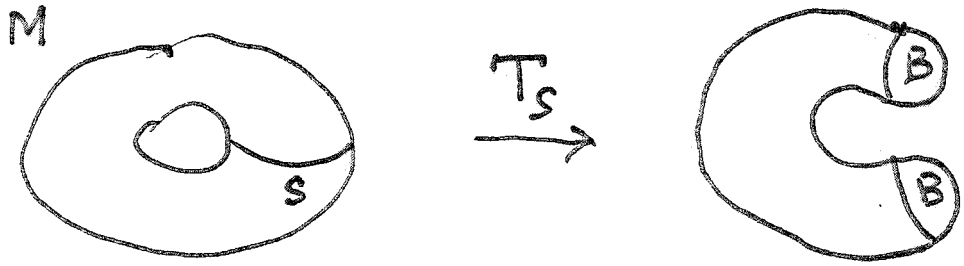


Example.

$\sigma = \{T_s\}$: spherical compressions

Given $S \subset M$, we cut M along S and add two balls.





Theorem 1. For $\sigma = \{T_s\}$ the root exists and is unique.

Theorem (Kneser - Milnor)

(1) $\forall M = \#_i M_i \# n(S^2 \tilde{\times} S^1)$

(M_i are irreducible)

(2) M_i and n are determined by M

(1) belongs to Kneser:

Any sequence of T-moves stops after $\leq N(M)$ steps.

$\Rightarrow \exists\, C_{\text{kneser}}(M)$

(Kneser complexity of M).

(2) belongs to Milnor (?)

Relation to roots:

$M = \# M_i \# (n)(S^2 \times S^1) \Longleftrightarrow$

$\bigcup M_i$ is a root of M

Proof of (2): the root is unique:

Suppose not. Let M be a minimal manifold having two roots (minimal with respect to the Kneser complexity)

CASE $A \cap B = \emptyset$

$$
\begin{tikzcd}
 & M \arrow[dl, "A"'] \arrow[dr, "B"] & \\
T_A(M) \arrow[dr, "B"] & & T_B(M) \arrow[dl, "A"'] \\
 & T_B T_A(M) = T_A T_B(M) \arrow[d, dashed] &
\end{tikzcd}
$$

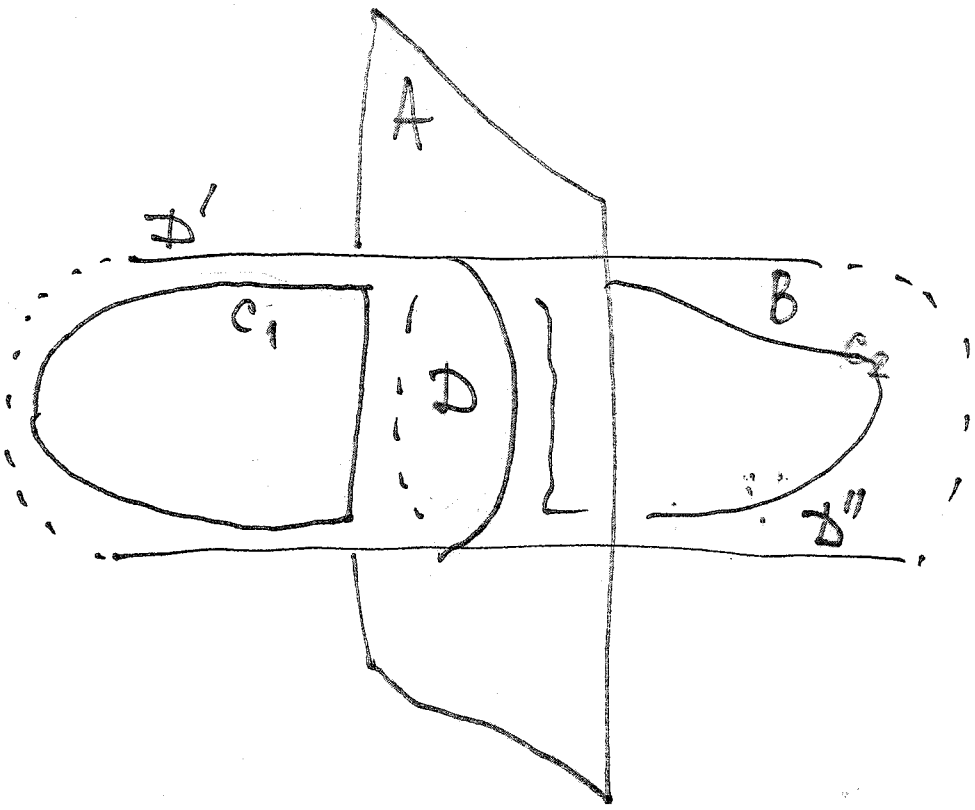
OK

OK

SAME ROOT!

Remark: $\hat{A}$ may be trivial

case $A \cap B \neq \emptyset$



$\#(c_i \cap B) = \emptyset$,

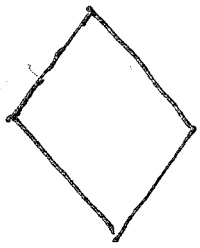
$\#(c_i \cap A) < \#(B \cap A)$

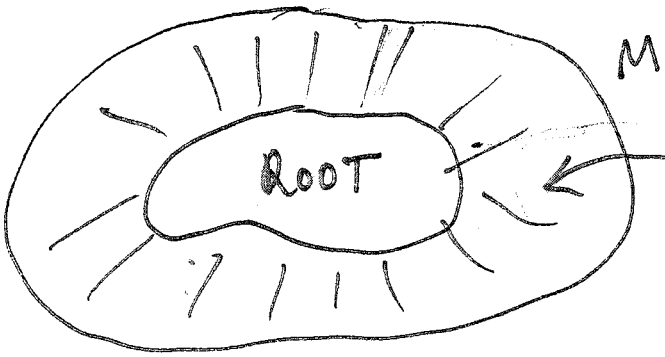
D is an innermost disc in A bounded by $A \cap B$

c_i is nontrivial

$\sigma = \{T_D\}$: cutting along $\mathcal{D}^2$.

Theorem 2: The root $\exists!$

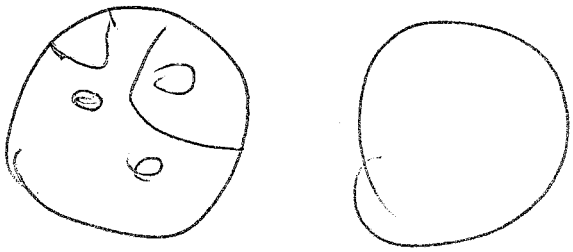
Same proof: 



characteristic Compression body for M

F. Bonahon, Cobordism of automorphisms of surfaces, Ann. Éc. Norm. sup. 83 (1983), 237-270.

$\sigma = \{T_D, T_S\}$: Theorem 3: $\exists!$



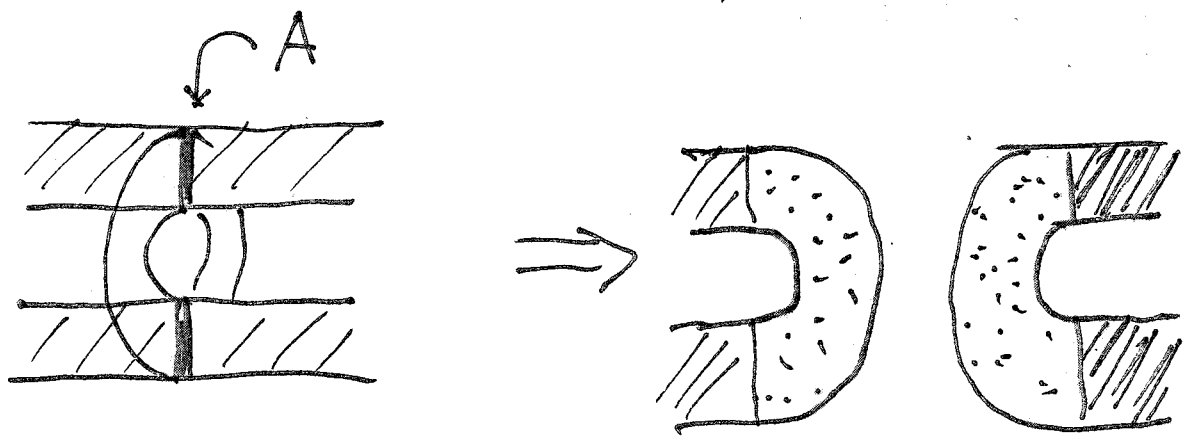
Roots of cobordisms:

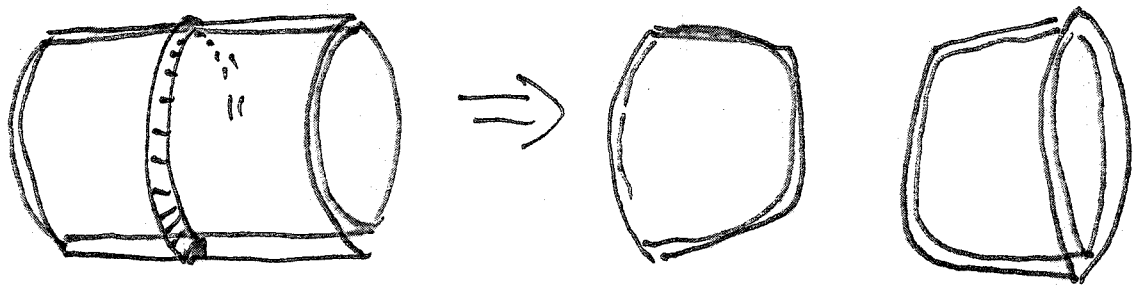
$\sigma = \{ T_S, T_D, T_A \}$

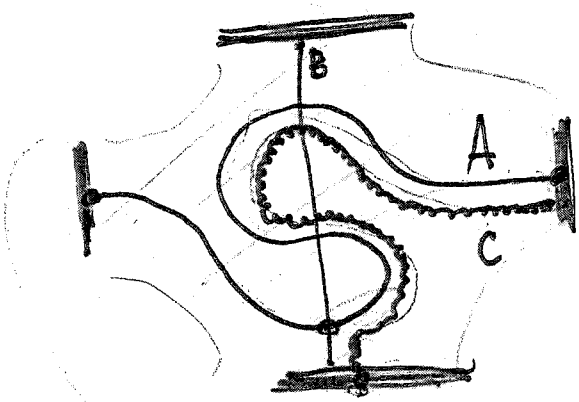
$A \subset (M, \partial_- M, \partial_+ M)$ incompessible annulus; $\partial_- A \subset \partial_- M$.

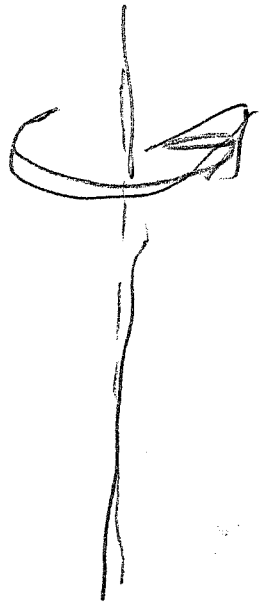
$\partial_+ A \subset \partial_+ M$

Cut M along A and attach two plates:





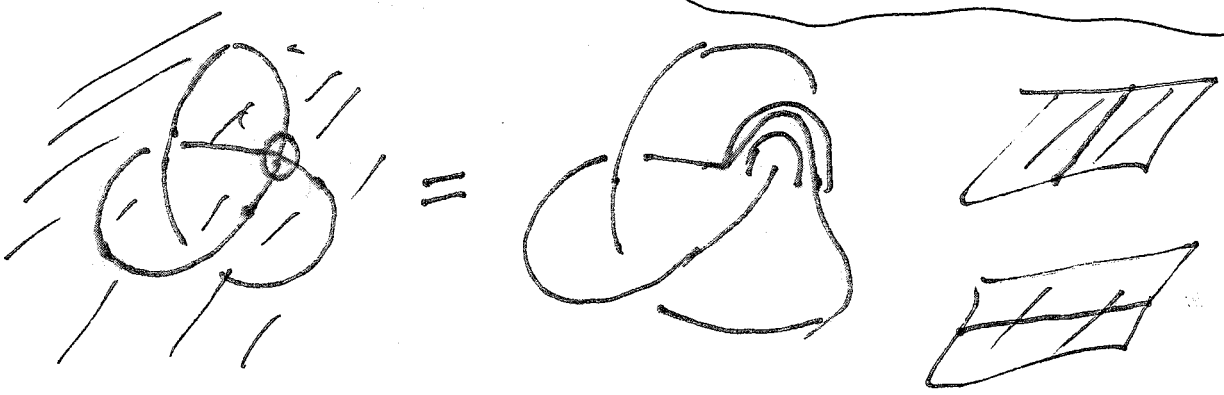




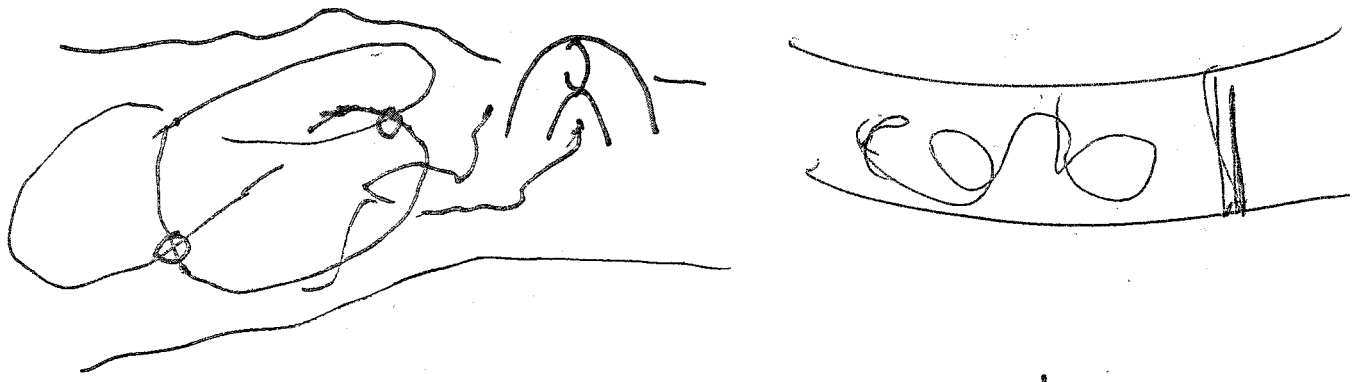
Theorem 4. $\exists!$

Cobordisms $\rightsquigarrow$ virtual knots

virtual knot = knot in $F \times I$



up to destabilization:



Theorem 5. Any virtual knot has a unique minimal genus representative

G. Kuperberg, Algebraic & Geometric Topology, 3 (2003), 587–591

"What is a virtual link"?

cobordisms $\rightsquigarrow$ manifolds

$\sigma = \{T_S, T_D, T_A\}$

A is incompressible and

either boundary incompressible !!!

* or $\partial_+ A, \partial_- A$ are in different components of ∂M.

**

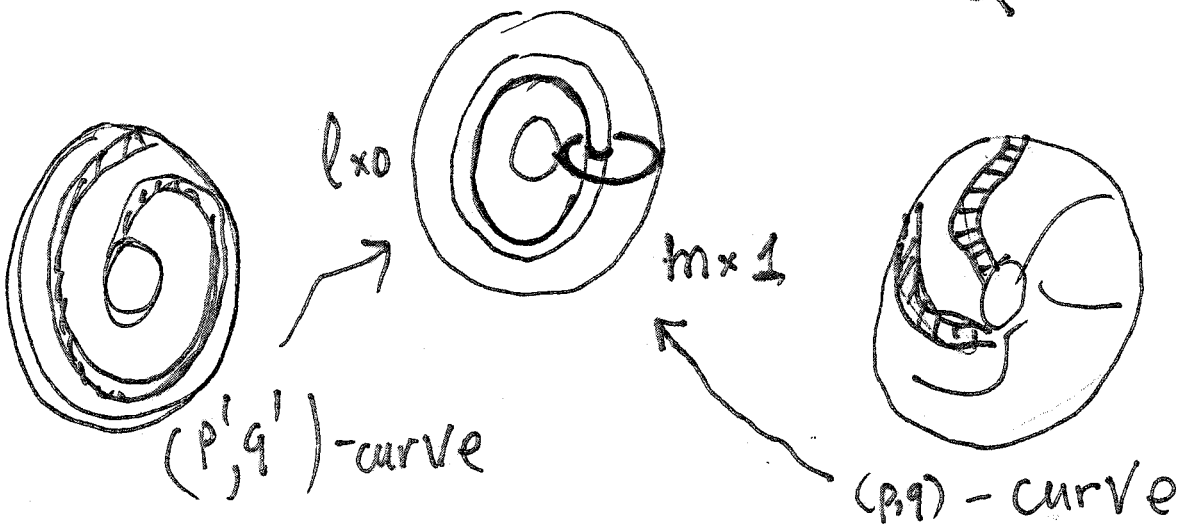
Theorem 6. For σ^{**} the root $\exists$!

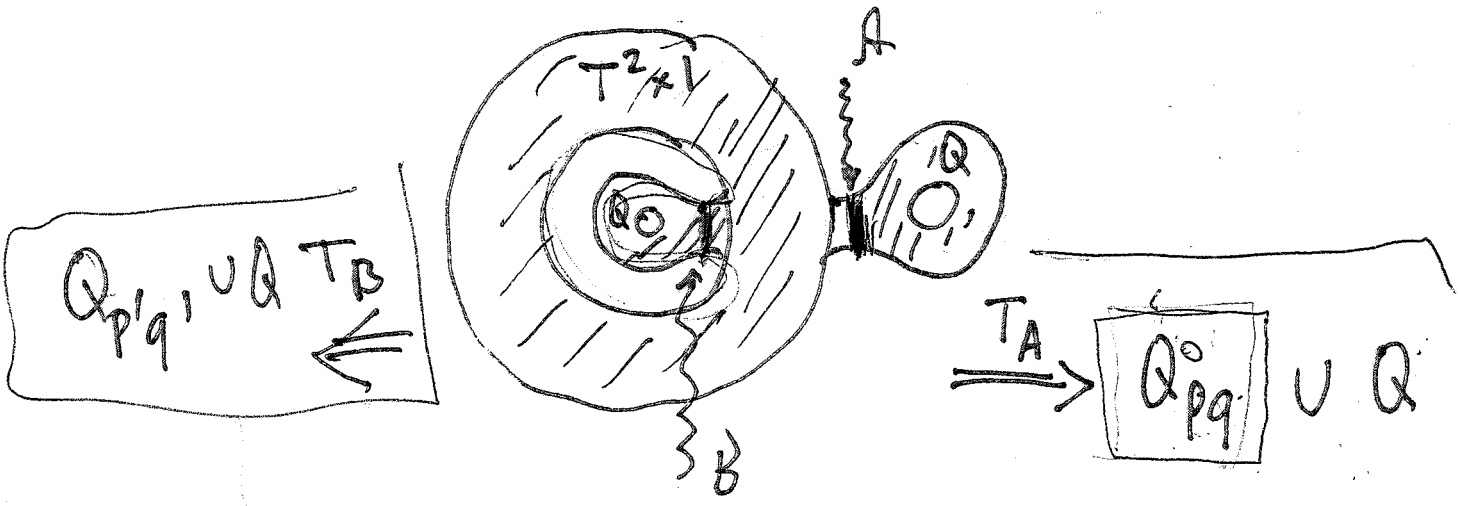
~~Theorem 6 For σ^{*} the root $\exists$!~~

False !

Example.

$$Q \xrightarrow{(p',q')} T^2 \times I \xleftarrow{(p,q)} Q$$

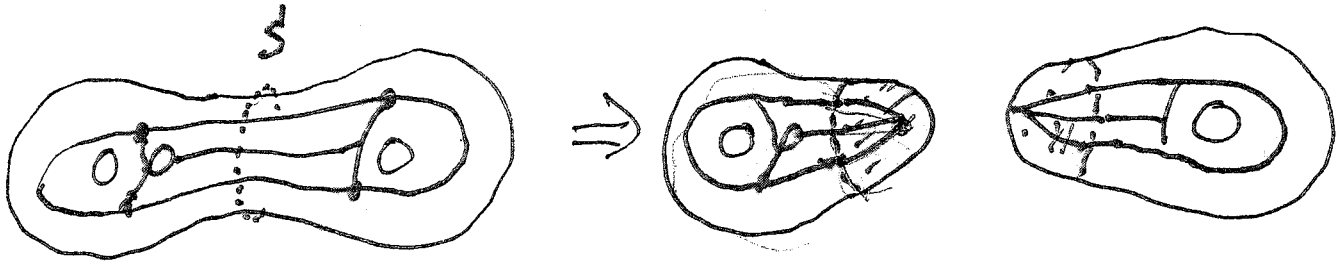




Knotted graph =

(M, G), where G is a graph in M. Compact!

$\sigma = \{T_s\}$

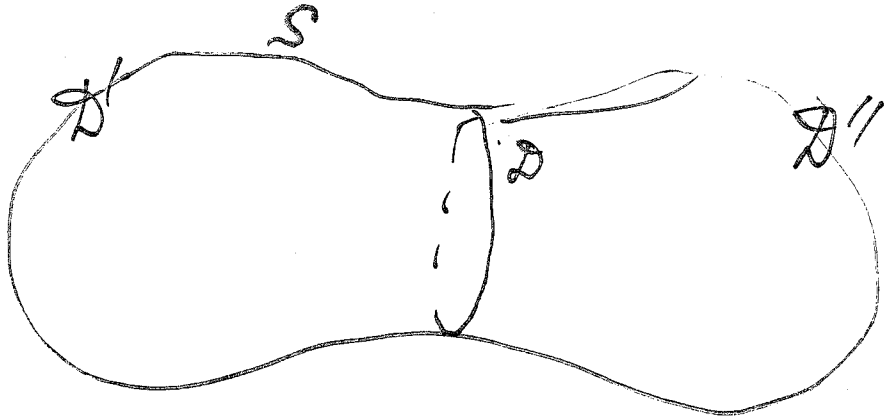


Cut along S and take cones over

$(S_\pm, S_\pm \cap G)$.

Restrictions:

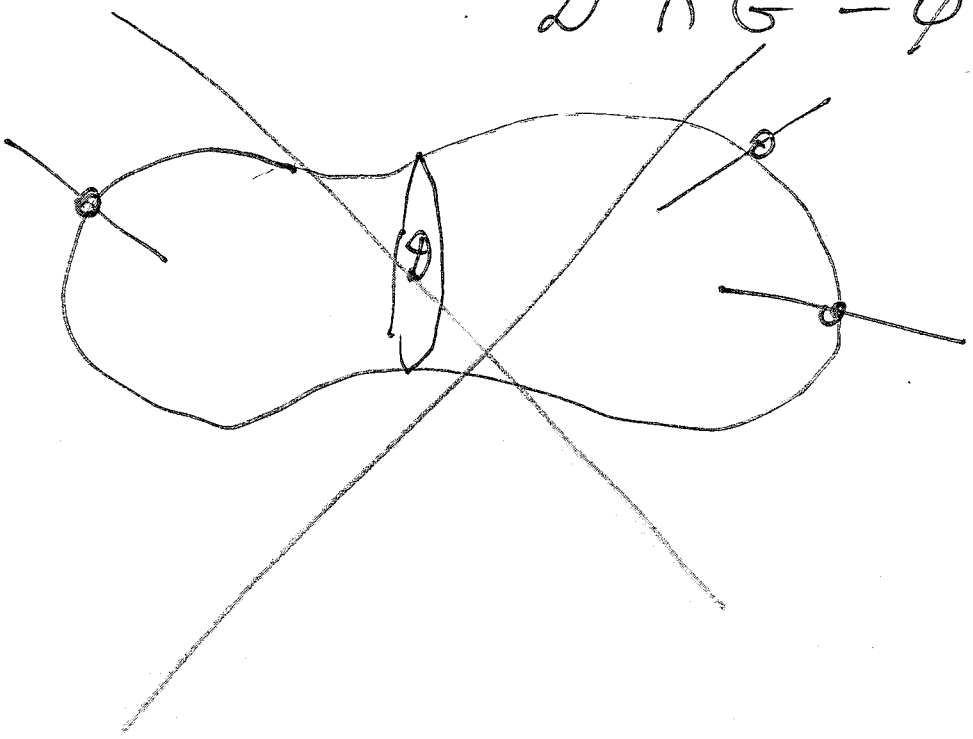
1. S should be incompressible



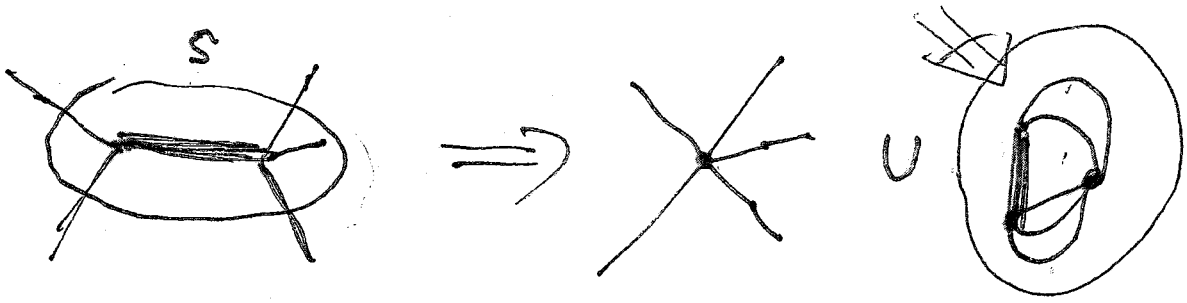
$D \cap S = \partial D$

$D \cap G = \emptyset \Rightarrow D' \cap G = \emptyset$ or

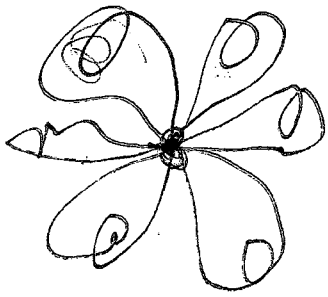
$D'' \cap G = \emptyset$

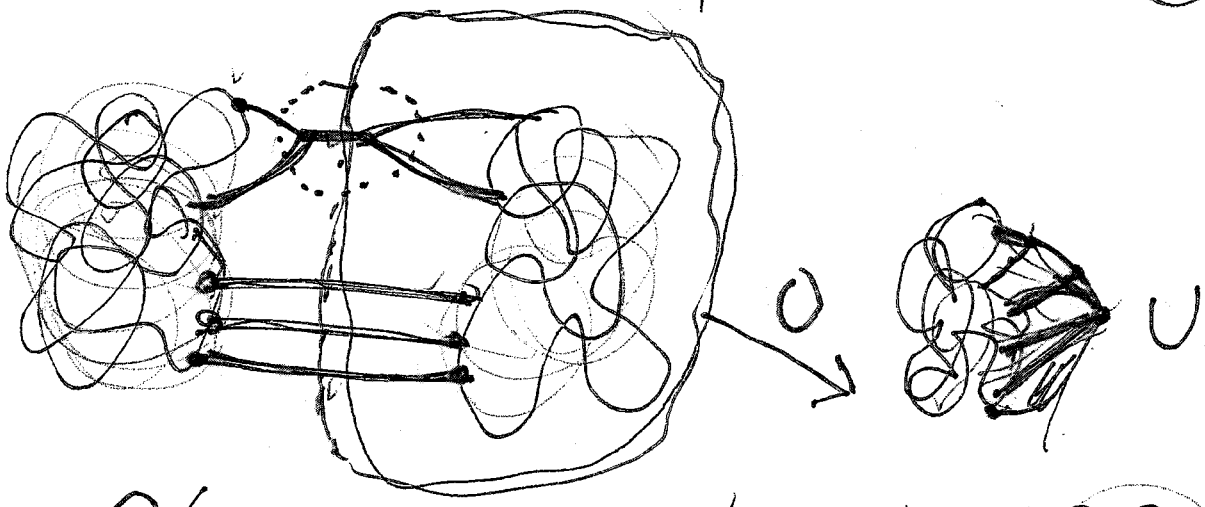


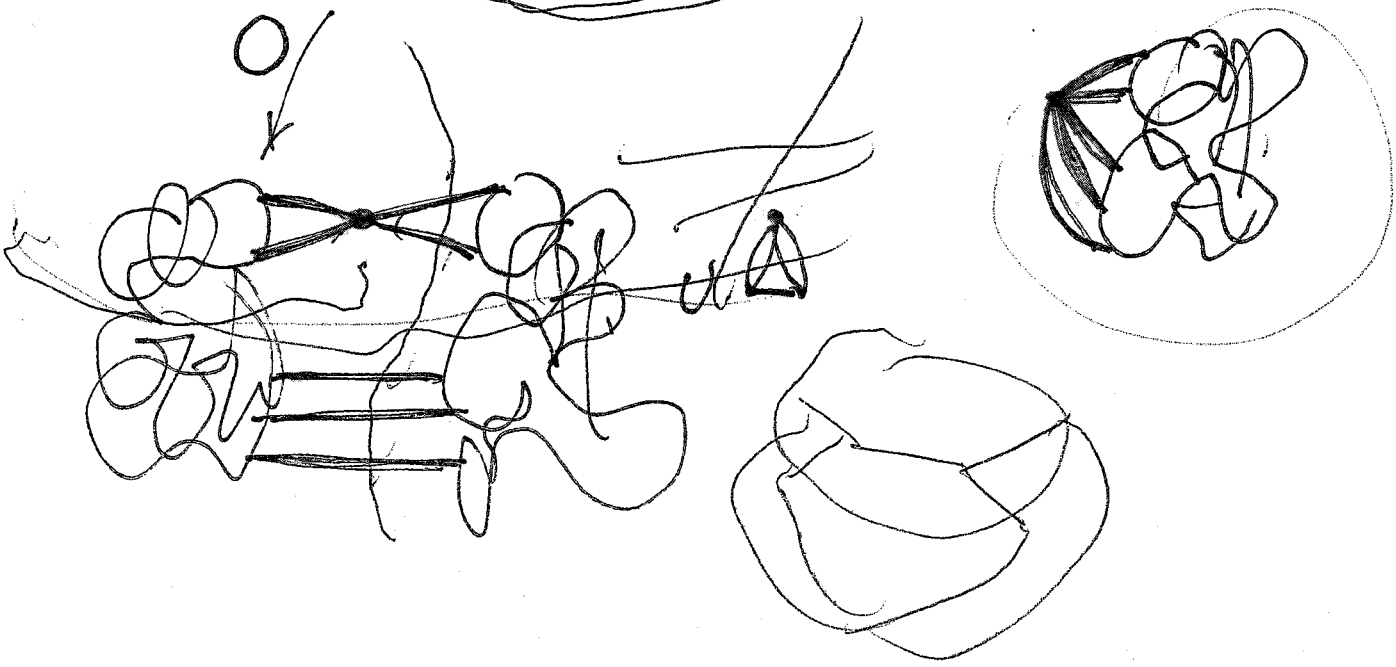
2. $\#(S \cap G)$ should be bounded



3. $\#(S \cap G) \leq \cancel{3}\ 4$







Accept: S is incompressible

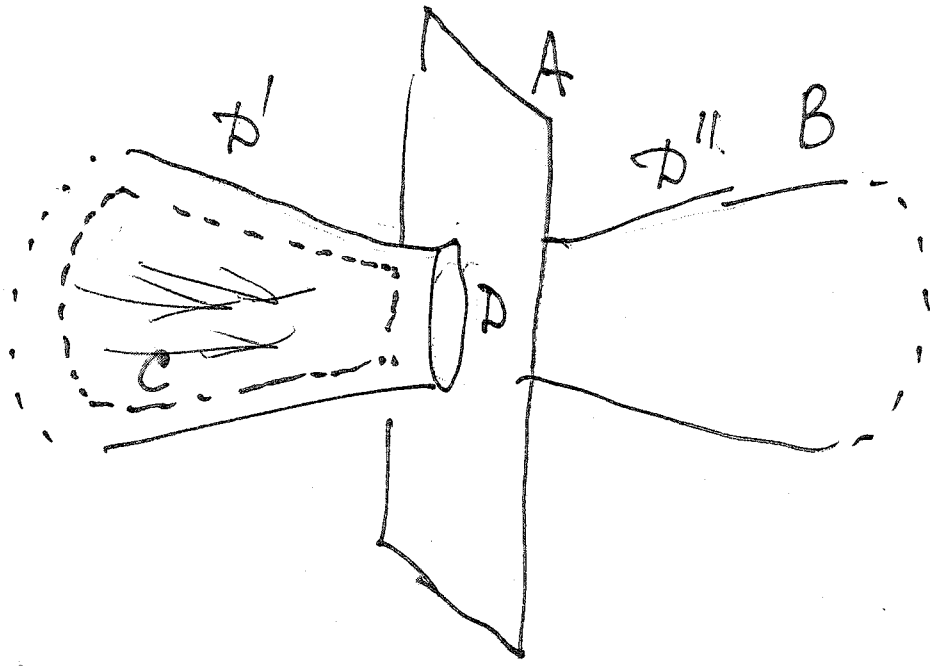
$$\#(S \cap G) \leq 3$$

Theorem 7. $\exists!$

$\exists$: Kneser works:

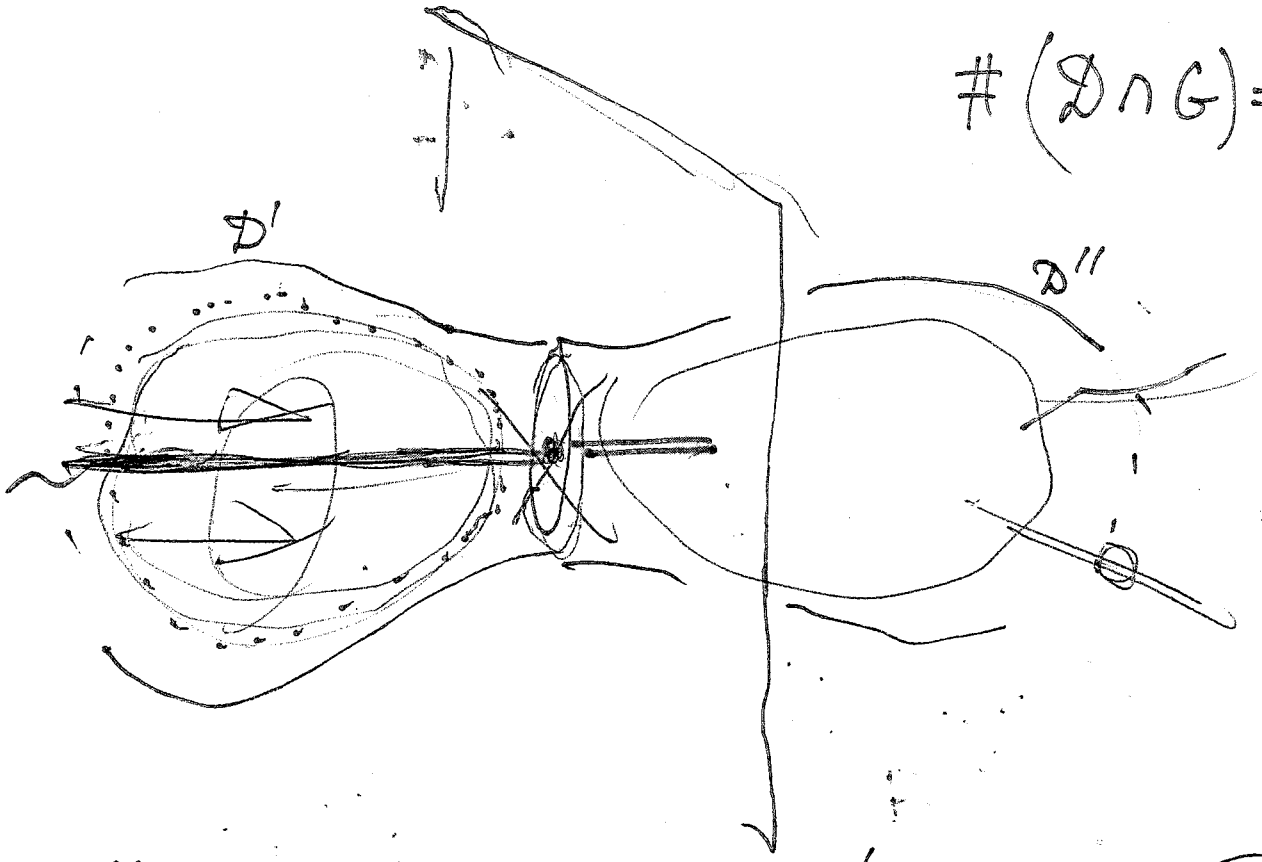
$!$ $\diamond$ works

If $D \cap G = \emptyset$

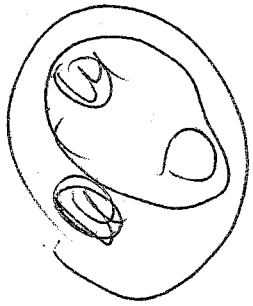


$$\#(A \cap C) < \#(A \cap B)$$

$\#(\mathcal{D} \cap G) = *$



May assume: $\mathcal{D}' \cap G \neq \emptyset$
$\mathcal{D}'' \cap G \neq \emptyset$

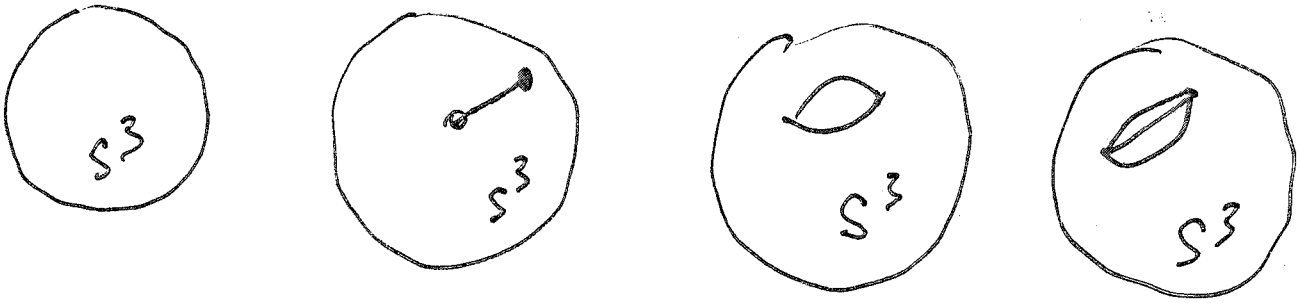


since $3 = 1 + 2$, then
$2 = 1 + 1$

$\#(\mathcal{D}' \cap G) = 1$ or $\#(\mathcal{D}'' \cap G) = 1$.

There always exists $\mathcal{D}$ with $\mathcal{D} \cap G = *$.

The root is unique up to addition / removing trivial pairs



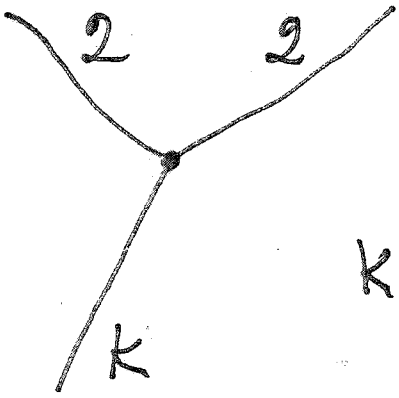
Partial cases:

Theorem 8: $\exists!$ if G is a link

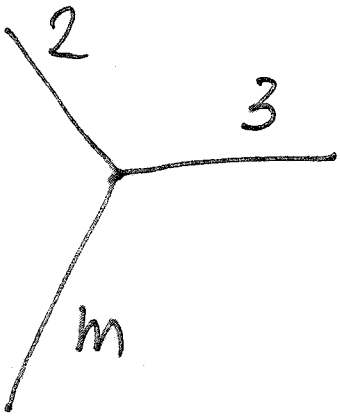
Theorem 9: $\exists!$ if G is a knot.

Orbifolds

(M^3, G, coloring by natural numbers)

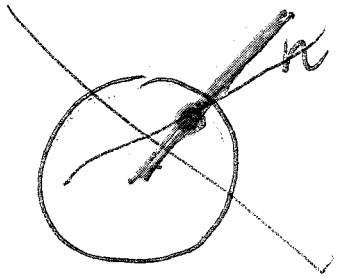


$k > 2$

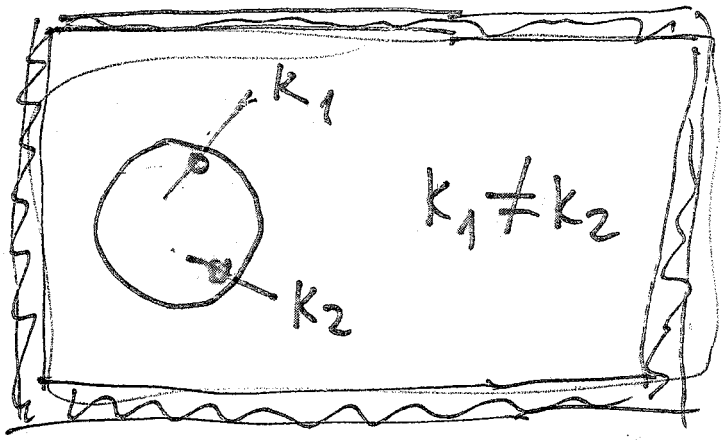


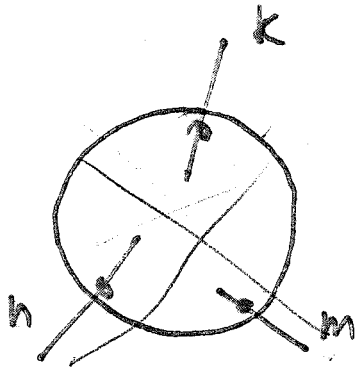
$m = 3, 4, 5$

$\sigma = \{T_s\}$ except



$S \cap G = *$





$(k, m, n) \neq (2, 2, p)$ or

$(2, 3, q)$ with $q = 3, 4, 5$

Th 10 The root $\exists$!