

A Homotopic intersection theory on surfaces with boundary.

Applications to mapping class group and Braids

(To appear in "l'Enseignement MATH")

Let $S_{g,b}$ be a compact, connected oriented surface of genus $g \geq 0$, with $b > 0$ boundary components.

Let $S_{g,b,n} = S_{g,b} - \{P_1, \dots, P_n\}$,

P_i point in Interior of $S_{g,b}$.

Let $\Gamma = \pi_1(S_{g,b,n}; s_0)$ $s_0 \in \partial S$.

$H = H_1(S_{g,b,n}; \mathbb{Z})$

We have the well-known algebraic intersection form (anti-symmetric):

$\langle , \rangle : H \times H \longrightarrow \mathbb{Z}$

defined as follows. Let $\alpha, \beta \in H$ be represented by immersed loops (oriented). Make them transversal and set:

$$\langle \alpha, \beta \rangle = \sum_{P \in \alpha \cap \beta} \varepsilon_P \quad \text{where}$$

$$\gamma \varepsilon_p = +1 \quad \text{if } (\overrightarrow{\tau_p \alpha}, \overrightarrow{\tau_p(\beta)}) > 0 \\ = -1 \quad \text{otherwise.}$$

Let $\mathbb{Z}[\Gamma]$ be the group ring of Γ
e.g. $= \left\{ \sum_{\text{finite}} n_i g_i ; n_i \in \mathbb{Z}, g_i \in \Gamma \right\}$

with obvious addition and multiplication induced by the law on Γ .

Let $\varepsilon : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}$, sending Γ on 1.
and $\overline{(\)} : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma]$ defined
by $g \in \Gamma \longrightarrow g^{-1}$

§ 1) The homotopic intersection form

Proposition 1 : $\exists \omega : \Gamma \times \Gamma \longrightarrow \mathbb{Z}[\Gamma]$

defined geometrically such that

$$1) \omega : \Gamma \times \Gamma \longrightarrow \mathbb{Z}[\Gamma]$$

$$\downarrow \text{ab.} \qquad \downarrow \varepsilon$$

$$H \times H \xrightarrow{<, >} \mathbb{Z}$$

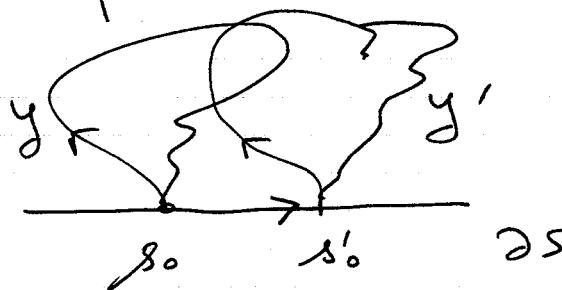
$$2) \underline{\omega(y, x) = -\overline{\omega(x, y)} + (y^{-1})(x^{-1})}$$

$$3) \underline{\omega(x y, z) = \omega(x, z) + x \omega(y, z)}$$

$$4) \underline{\omega(x, yz) = \omega(x, y) + \omega(x, z)y^{-1}}$$

Remark : ω can be extended by bilinearly to $\mathbb{Z}[\Gamma] \times \mathbb{Z}[\Gamma]$.

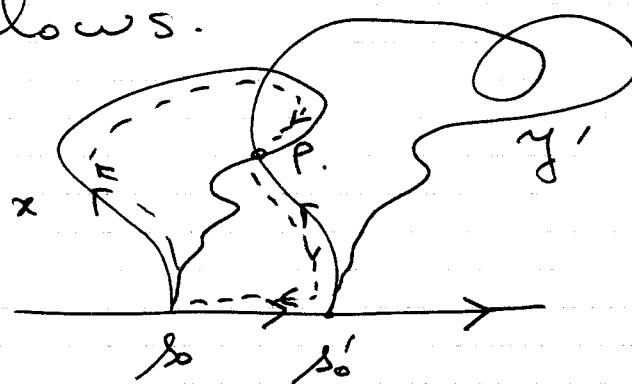
Proof : Simple definition of ω . Let $x, y \in \Gamma$ represented by loops based at s_0 . Push y into y' in such a way that y' is a loop based at s'_0 .



Make x, y' transversal and set

$$\omega(x, y) = \sum_{P \in x \cap y'} \epsilon_P g_P$$

ϵ_P as before, g_P defined as follows.

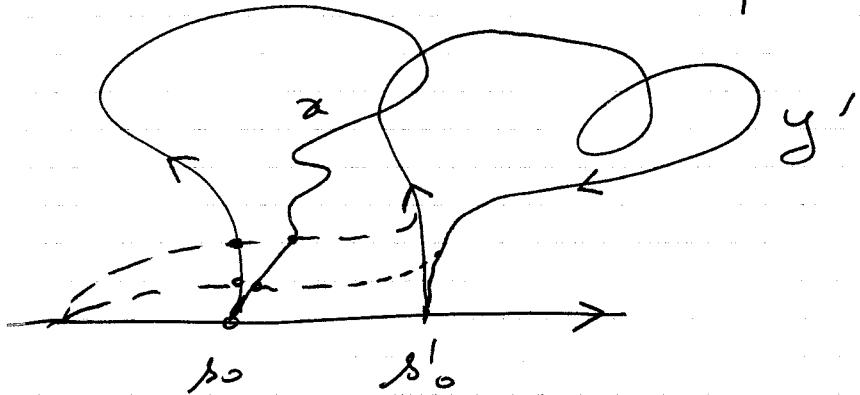


ω depends only on the homotopy class of x and y . Easy.

Properties 3) and 4) follow easily from the definition.

Property 2 seems bizarre: to define $\omega(y, x)$ we have to push x to the right, or equivalently

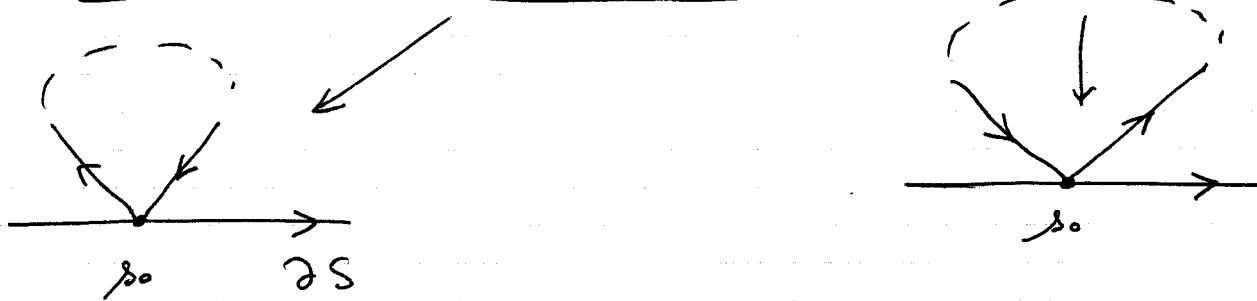
4/ push y' to the left. Doing this we introduce 4 exlia points:



We later we will need:

Lemma 2 : If $x \in \Gamma$ is represented by an embedded based loop then

$$\omega(x), x = 1 - x \quad \text{or} \quad 1 - \bar{x}$$



Comment: I have wr. this paper more than a year ago. Last week, after a stay at Paris, I learned that V. Turaev has proved a Proposition 1 in a paper: in

Math USSR Sbornik (1979).

Fortunately (for me) the remainder of my talk is disjoint from Turaev's paper.

5/82) The form ω and Fox's free differential calculus.

Definition: A derivation (resp. an anti-derivation) on Γ is a map

$D: \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma]$ such that:

1) D is linear for $+$

2) $D(uv) = \epsilon(v) D(u) + u D(v)$

(Leibniz formula)

(resp $D(uv) = \epsilon(v) D(u) + D(v)u^{-1}$)

Def A biderivation is a map

$\Theta: \mathbb{Z}[\Gamma] \times \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma]$ which
is a - derivation relative to the first coordinate.
- anti-derivation " " second .

Ex: Let Γ be a free group
and $(z_1, z_2 \dots z_m)$ a free basis.

$\frac{\partial}{\partial z_i}: \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[\Gamma]$ is the

derivation defined by:

$$\frac{\partial z_j}{\partial z_i} = \delta_{ij}.$$

Notation: For $u \in \Gamma$, set $\partial u = \begin{pmatrix} \frac{\partial u}{\partial z_1} \\ \vdots \\ \frac{\partial u}{\partial z_m} \end{pmatrix}$

REMARK : properties 3), 4) of

Proposition 1 say that ω is a biderivation.

For a biderivation Θ on Γ with free basis (z_1, \dots, z_m) we associate a matrix $(m, m) \in \text{M}_m(\mathbb{Z}[\Gamma])$:

$$\Lambda = \begin{pmatrix} & & & & s \\ & & & & | \\ & & -\Theta(z_i, z_j) & & | \\ & & & & | \\ & & & & | \end{pmatrix}$$

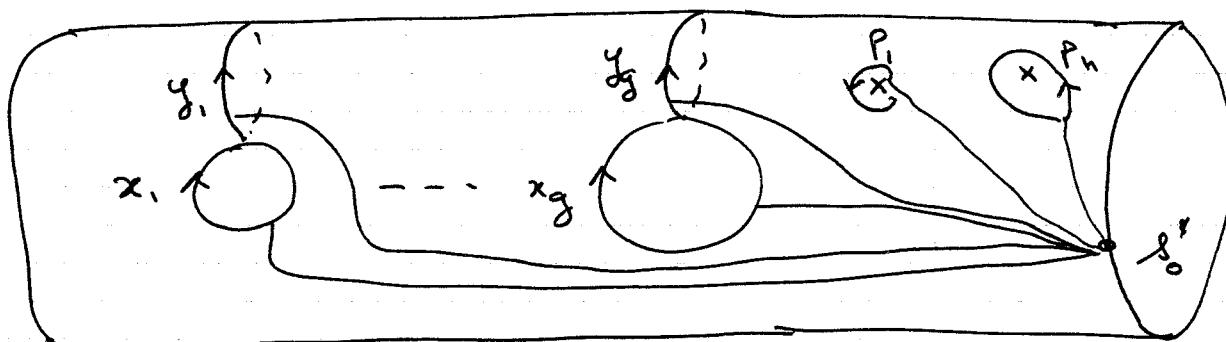
Proposition 3 For a biderivation Θ on Γ with basis (z_1, \dots, z_m) we have:

$$\Theta(x, y) = \partial x^t \times \Lambda \times \overline{\partial y}$$

Proof : True for $x = z_i^\pm, y = z_j^\pm$

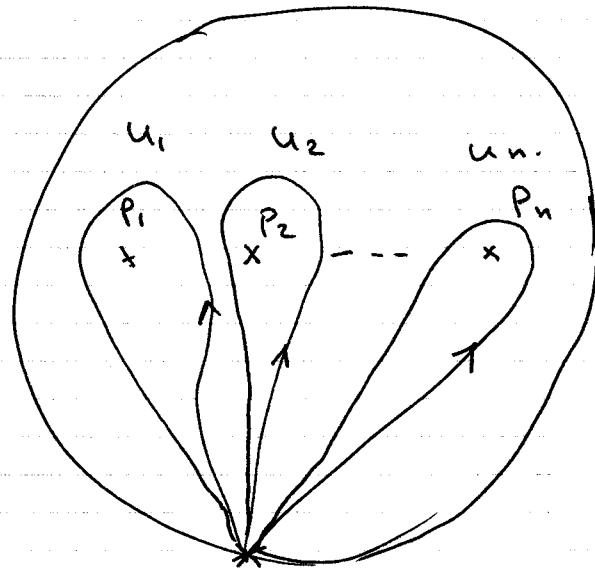
Next proceed by induction on the length of x and y written as words in z_i 's.

First fundamental example : $S_{g, b, n}$



In such a basis $\rightarrow \Lambda_{g, n}$

$$X/2 \stackrel{?}{=} \text{example} \quad S_{0,1,\dots,n} = D_n = D^2 - \{P_1, \dots, P_n\}$$



The matrix \$x\$ is

A

$$\Omega_n = \begin{pmatrix} 1-u_1^{-1} & & & & \\ (1-u_2)(1-u_1^{-1}) & 1-u_2^{-1} & & & \\ \vdots & & \ddots & & \\ (1-u_n)(1-u_1^{-1}) & \cdots & & \ddots & 1-u_n^{-1} \end{pmatrix}$$

§3) Some applications.

a) \$w\$ is related to Reidemeister pairing. I can prove very quickly a result of Papakyria - Kopoulos on the planarity of coverings of surface.

b) Application to mapping class group of surface.

$\mathcal{M}_{g,b,n}$ = mapping class group of $S_{g,b,n}$

= { isotopy classes of homeomorphisms
of $S_{g,b,n}$, equal to identity on ∂S . }

Given a free basis for Γ , \rightarrow

3/ we define the Fox-matrix of f

$$F\mathbb{B}(f) = \left(\frac{\partial f(z_i)}{\partial z_j} \in \Gamma \right)$$

where f denote also the so. $\Gamma \hookrightarrow$
Elementary calc: $\mathbb{B}(f \circ g) = \mathbb{B}(f) \times {}^f\mathbb{B}(g)$

Obvious but fund remark:

$$\omega(f(x), f(y)) = f \omega(x, y) \quad \forall x, y \in \Gamma \\ \forall f \in \mathcal{M}_{g, b, n}$$

~~defining~~

$$+ \begin{pmatrix} a_{ij} \\ \pi \\ \mathbb{Z} \Gamma \end{pmatrix} = \left(f(a_{ij}) \in \mathbb{Z} \Gamma \right)$$

$$F: \mathcal{M}_{g, b, n} \xrightarrow{\text{twisted homo.}} GL_m(\mathbb{Z}[\Gamma]) \xrightarrow{ab} GL_m(\mathbb{Z}H)$$

$F^{ab}: F^{ab}(fg) = \mathbb{B}^{ab}(f) \times {}^f\mathbb{B}^{ab}(g)$

~~From Prop 3 and 2 the fund remark~~

~~$\mathbb{B}(f) \times \mathbb{D} & \mathbb{B}$~~

~~$\mathbb{B}(f) \times \mathbb{D} & \mathbb{B}$~~

$$\mathbb{B}(f) \times \mathbb{D} = \{f / f_*\} \text{ the so called}$$

Torelli group of $S_{g, b, n}$

$$J_{g, b, n} \xrightarrow{F^{ab}} GL_m(\mathbb{Z}H)$$

true h-mo.

Question of Norita: F^{ab} injective?

Remark $g=0 \quad b=1$

$$\mathcal{M}_{0, 1, n} = B_n \quad J_{0, 1, n} = P_n$$

10% Lemma 4 $\Gamma = \langle z, -z_n \rangle$ $f \in \mathcal{M}_{g, b, n}$

Λ the matrix of ω in $\langle z_i \rangle$:

$$\Rightarrow \overline{\mathbb{B}(f)}^t \times \Lambda \times F(f) = \Lambda$$

$$\overline{F(f)}^t \times \Lambda^{ab} \times F^{ab}(f) = \Lambda$$

Just combination of Prop 3 + remark.

↓ This formula was written by Monta using lengthy calculation, when Λ was found in Papaky, without reference to geometry.

In some sense this says that $F(f)$ is a symplectic matrix, because F respect Λ and that Λ in case $n=0$, in the basis mentioned above $\Lambda \iff \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

~~check basis~~

Thm 5 for $n=0$ $g \geq 2$ $b=1$

$$J_{g,1,0} = J_{g,1} \xrightarrow{f \in J} GL_m(\mathbb{Z}/\mathbb{H})$$

is not injective. Remark $F^{ab}(f) \times \Lambda \times F^{ab}(f) = \Lambda$

Suzuki obt decrire using a lot of calculation an el ∞ of the kernel, without any mention how he found it.

There is a lot of examples explaining why Suzuki example works, without any calculation.

Prop 6 Let α be an embedded circle separating $S_{g,b}$

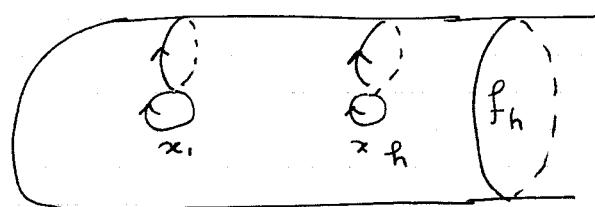
D(a) The Dehn twist. Then

$$F^{ab}(D(a)) = I + \overline{\partial_H \alpha} \times \partial_H^t \alpha \times \Lambda^{ab}$$

where $\alpha \in \Gamma$ is any representative of a (so defined up to conjugation with respect to the free basis of Γ of example 1).

Proof : Prove the formula for f_h

Just computation



Then if c bounds a surface of genus \mathbb{R}

$$\exists \varphi \in M_{g,b} \text{ / s.t. } c = \varphi(f_h)$$

Use the fact that $\partial_H c = B^{ab}(\varphi) * \partial f_h$
and use lemma 4

Corollary a, b as in Prop 6.

If $\alpha, \beta \in \Gamma$ repres. If $\omega_H(\alpha, \beta) = 0$

$\Rightarrow F^{ab}(D(a))$ and $F^{ab}(D(b))$ commute.

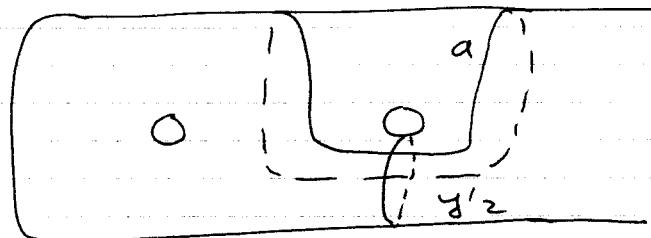
$$\text{If } F^{ab}(a), F^{ab}(b)$$

$$= I + \overline{\partial_H \alpha} \times \partial_H^t \alpha \times \Lambda^{ab} + \overline{\partial_H \beta} \times \partial_H^t \beta \times \Lambda^{ab} \\ + \overline{\partial_H \alpha} \times (\partial_H^t \alpha \times \Lambda \times \overline{\partial_H \beta}) \times \partial_H^t \beta \Lambda$$

$$\text{Prop 3} \\ \omega_H(\alpha, \beta)$$

$$\Rightarrow [D(a), D(b)] \in \ker F^{ab} \text{ if } \#_1 \quad \omega(a, b) = 0 \text{ and } a \cap b \neq \emptyset$$

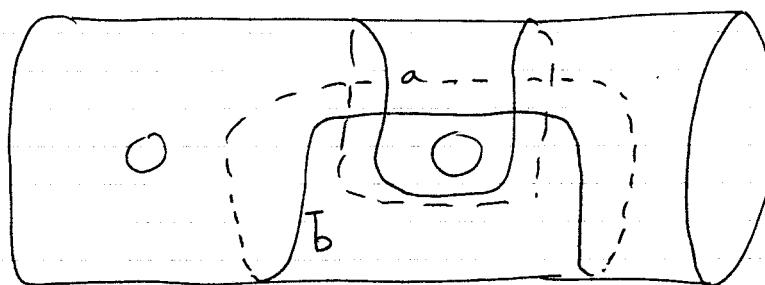
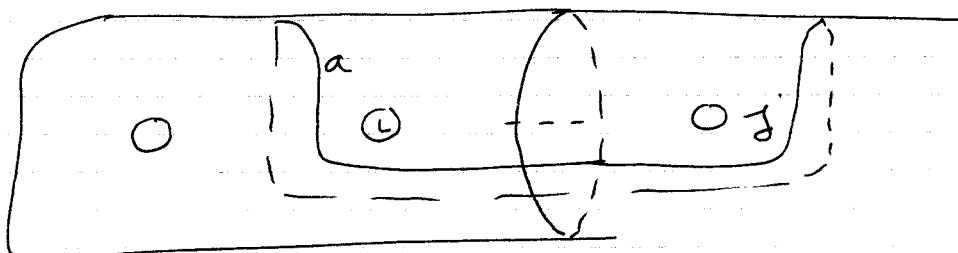
2) The example of Suzuki



$$b = \text{Deh}(y_2)(a)$$

Other examples

b.



III App to braids.

Recall $S_{0,1,n} = D_n = D - \{P_1, \dots, P_n\}$.

$$\mathcal{M}_{0,1,n} = B_n.$$

$$\mathcal{I}_{0,1,n} = P_n.$$

$$F^{ab} : \mathcal{I}_{0,1,n} \longrightarrow GL_n(\mathbb{Z} H)$$

\hookrightarrow | is known as the Gauss representation

$$\frac{F^{ab}}{F^{ab}} \times \mathcal{I}_n^{ab} \times F^{ab} = S_n^{ab}.$$

where S_n is triangular ~~shape~~

3/ Impose strong restriction on F^{ab} to be
a Gaußier matrix

F^{ab} preserve $\omega^{ab} + \overline{\omega}^b$. By sending
 $x_i \rightarrow$ appropriate $t_i \in S^1 \subset \mathbb{C}$, then
prove that $\{\text{Gaußier matrices}\}$ is conjugate
to a subgroup of the Unitary group

$$B_n \xrightarrow[\text{not homo.}]{{F^{ab}}} GL_n(\mathbb{Z} H) \longrightarrow GL_n(\mathbb{Z}[z])$$

$\mathbb{Z}[t, t^{-1}]$

homo. called

the Bureau representation. and we

have $\forall \sigma \in B_n \quad \overline{Bu(\sigma)}^t \times \widetilde{\Omega}_n \times Bu(\sigma) = \widetilde{\Omega}_n$

when $\widetilde{\Omega}_n = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1-t & \\ & & & 1 \end{pmatrix}$

Again the fact that $\widetilde{\Omega}_n$ is
triangular impose stronger restriction

Spirer prove a relation

$$\overline{Bu(\sigma)}^t \times M \times Bu(\sigma) = n$$

but for a much more complicated
matrix.

(i) $M \in GL(\mathbb{Z}[t^{\pm 1/2}])$

(ii) hermitian.

than Spirer one

As an example I have the
following cor., which cannot be

M obtained by Sgnr

Cor If $M = \begin{pmatrix} x & x & x \\ 0 & x_p & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a

Bman matrix then

$$M = \begin{pmatrix} 1 & 0 \\ 0 & A_{n-p} \end{pmatrix}$$