Part 0. Terminology

All 3-manifolds are compact, connected, and oriented, all knots are in S^3, all maps are proper, and all surfaces are embedded.

A surface in a 3-manifold is *incompressible* if the inclusion induces an injective map on π_1; A 3-manifold M is: *irreducible* if every embedded 2-sphere in M bounds a ball M; *∂-irreducible* if every proper disc in M separates a ball from M; *atoroidal* if every $\mathbb{Z} \oplus \mathbb{Z}$ subgroup in $\pi_1 M$ is conjugate into $\pi_1 \partial M$, is a *Seifert manifold*, if it is finitely covered by a circle bundle over a surface.

A closed orientable 3-manifold is called *geometric* if it admits one of the following geometries: H^3 (hyperbolic), $\widetilde{PSL_2}(\mathbb{R})$, $H^2 \times E^1$, Sol, Nil, E^3 (Euclidean), $S^2 \times E^1$, S^3 (spherical).

The JSJ-decomposition of a irreducible 3-manifold M is the canonical splitting of M along a finite (possibly empty) collection $\mathcal{T}$ of disjoint and non-parallel, incompressible, tori into maximal Seifert fibered or atoroidal compact sub-manifolds. We call the components of $M \setminus \mathcal{T}$ the *JSJ-pieces* of M.

Thurston's geometrization conjecture claims that each JSJ-piece of any closed, irreducible 3-manifold is geometric. A compact irreducible 3-manifold is called *geometrizable* if it verifies Thurston's geometrization conjecture.

Say a 3-manifold M *dominates* (*1-dominates*) a 3-manifold N if there is a non-zero degree (degree one) proper map $f : M \to N$.

Let k_1 and k_2 be two knot. Say $k_1 \geq k_2$, or equivalently say that k_1 1-dominates k_2, if $E(k_1)$ 1-dominates $E(k_2)$, where $E(k_i)$ is the knot exterior of k_i. If $k_1 \geq k_2$ but $k_1 \neq k_2$, we often write $k_1 > k_2$. Then

(1) $k \geq O$ for each knots.

(2) The relation $\geq$ on knots is a partial order.

Say a knot is *small* if each incompressible surfaces in $E(k)$ is boundary parallel.

Part M. On finiteness on domination of 3-manifolds

Boileau-Rubinstein-Wang

With Thurston's conjectural picture of 3-manifolds, the following simple and natural question was raised in the 1980's (and formally appeared in the 1990's, see [Ki, 3.100 (Y.Rong)]).

Question 1. *Does every closed orientable* 3-*manifold 1-dominates at most finitely many closed geometrizable* 3-*manifolds.*

If we allow any degree, 3-manifolds supporting one of the geometries $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R}), Nil$ can dominate infinitely many 3-manifolds. The following generalization of Question 1 makes sense:

Question 2. *Let M be a closed 3-manifold. Does M dominate at most finitely many closed geometrizable 3-manifolds N not supporting the geometries of $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R})$, Nil.?*

In this setting, known results are

Theorem 0. [Soma, Porti-Reznikov, Zhou-W, Hayat-Zieschang-W]

(1) Any closed 3-manifold 1-dominates at most finitely many geometric 3-manifolds.

(2) A compact 3-manifold dominates at most finitely many geometric 3-manifolds supporting geometries of either H^3 or $\mathbb{H}^2 \times \mathbb{E}^1$.

By Theorem 0, positive answer to Question 2 implies positive answer to Question 1, and Question 2 reduces to the following:

Question 3 *Let M be a closed 3-manifold. Does M dominate at most finitely many, closed, irreducible 3-manifolds N with non-trivial JSJ decomposition?*

Question 3 is divided into 2 steps:

1. *Finiteness of JSJ-pieces*: show that there is a finite set $\mathcal{HS}(M)$ of compact orientable 3-manifolds such that each JSJ-piece of a 3-manifold N dominated by M belongs to $\mathcal{HS}(M)$.

2. *Finiteness of gluing*: For a given finite set $\mathcal{HS}(M)$ of Seifert manifolds and of complete hyperbolic 3-manifolds with finite volume, there are only finitely many ways of gluing elements in $\mathcal{HS}(M)$ to get closed 3-manifolds dominated by M.

Remark. (1) Derbez Show that every graph manifold 1-dominates at most finitely many geometrizable 3-manifolds.

irreducible are not essential for both M, N.

(2) By degree one map produced by null-homotopy surgery (Boileau-W), we may assume that M is irreducible in the questions.

Soma proved the finiteness of hyperbolic JSJ-pieces. Now we complete the first Step:

Theorem 1. [Finiteness of JSJ pieces] *Let M be a closed, orientable, 3-manifold. Then there is a finite set $\mathcal{HS}(M)$ of compact 3-manifolds, such that the JSJ-pieces of any geometrizable 3-manifold N dominated by M belong to $\mathcal{HS}(M)$, provided that N is not supporting the geometries of $\mathbb{S}^3$, $\widetilde{PSL_2}(\mathbb{R}), Nil$.*

Theorem 1 is derived from a finiteness result for the Thurston norm.

Let X be a compact, orientable 3-manifold and $Y \subset \partial X$ be a subsurface.

For an oriented surface $(F, \partial F) \hookrightarrow (X, Y)$. Set $\chi_-(F) = \max\{0, -\chi(F)\}$ if F is connected, otherwise let $\chi_-(F) = \sum \chi_-(F_i)$, where F_i are the components of F.

Then for $z \in H_2(X, Y; \mathbb{Z})$ the *Thurston norm* $\|z\|$ of z is defined as minimum of $\chi_-(F)$, where F runs over all surfaces representing z in $H_2(X, Y; \mathbb{Z})$.

Then extends it to $H_2(X, Y; \mathbb{R})$.

Definition. *For a finite set of elements* $\alpha = \{a_1, ..., a_k\}$ *of* $H_2(X, Y; \mathbb{Z})$ *define*

$$TN(\alpha) = max\{\|a_i\|, i = 1, ..., k\}.$$

Then define $TN(X, Y)$, Thurston norm of the pair (X, Y), to be the minimum of $TN(\alpha)$, where α runs over all finite generating set of $H_2(X, Y; \mathbb{Z})\}$.

Theorem 2. [Finiteness of the Thurston norm]*Let M be an irreducible, closed, orientable 3-manifold. Then $TN(M_S, \partial M_S)$ picks only finitely many values when S runs over all closed, incompressible surfaces embedded in M.*

Theorem 2 is derived from the finiteness of a version of "patterned guts".

In 3-manifold topology, the term "guts" has several different interpretations. However, finiteness of guts is a basic principle, which originated from Kneser's work. For some recent applications related to guts in 3-manifold theory, see [A], [Ga2], [JR]. We now discuss the precise definition of patterned guts needed for our study of non-zero degree maps.

Suppose X is a ∂-irreducible and irreducible, compact, orientable 3-manifold. According to Jaco-Shalen-Johannson theory, there is a unique decomposition, up to proper isotopy:

$$X = (X \setminus \text{Seifert pairs}) \cup \text{Seifert pairs}.$$

Furthermore the Seifert pairs have unique decompositions, up to proper isotopy:

$$\text{Seifert pairs} = (\text{Seifert pairs} \setminus IB_X^-) \cup IB_X^-,$$

where IB_X^- is formed by the components of the Seifert pairs which are I-bundles over surfaces F with negative Euler characteristic $\chi(F)$. Hence we have a decomposition

$$X = (X \setminus IB_X^-) \cup_{A_X} IB_X^- = G_X \cup_{A_X} IB_X^-,$$

where A_X is the collection of frontier annuli of IB_X^- in X. We call $G_X = X \setminus IB^-$ the *guts* of X, and the decomposition above the *GI- decomposition* for X.

Suppose S is a closed, incompressible surface in an irreducible 3-manifold M. For such a surface S, we write the GI decomposition of M_S as

$$M_S = G_S \cup_{A_S} IB_S^-.$$

Definition. Suppose X is a 3-manifold. A ∂-pattern for X is a finite collection of disjoint annuli $A \subset \partial X$, and given A we say that X is ∂-patterned.

Theorem 3. *Let M be a closed, orientable, irreducible 3-manifold. Then there is a finite set $\mathcal{G}(M)$ of connected, compact, orientable, ∂-patterned 3-manifolds such that for each closed, incompressible (not necessarily connected) surface $S \subset M$, all patterned guts components of $(G_S, G_S \cap A_S)$ belong to $\mathcal{G}(M)$.*

We also prove the finiteness of gluing when the targets are integral homology 3-spheres.

Theorem 4. *Any closed orientable* 3-*manifold dominates only finitely many geometrizable integral homology* 3-*spheres.*

By Haken's finiteness theorem, there is a maximum number $h(M)$ of pairwise disjoint, non-parallel, closed, connected, incompressible surfaces embedded in M.

Lemma 1. *Let M and N be two closed, irreducible and orientable 3-manifolds. If M dominates N, then $h(M) \geq h(N)$.*

The dual graph $\Gamma(N)$ to the JSJ-decomposition of an irreducible homology sphere N is a tree. By Lemma 1, the number of edges of $\Gamma(N)$ is $\leq h(M)$, the Haken number of M.

Lemma 2. Only finitely many Seifert fibered integral homology 3-spheres are dominated M.

For a given graph Γ, let $\mathcal{D}(M, \Gamma)$ be the set of geometrizable closed integer homology 3-spheres N such that:

1. N is dominated by M.
2. The JSJ-graph $\Gamma(N)$ is isomorphic to Γ.
3. Each vertex manifold has a fixed topological type.

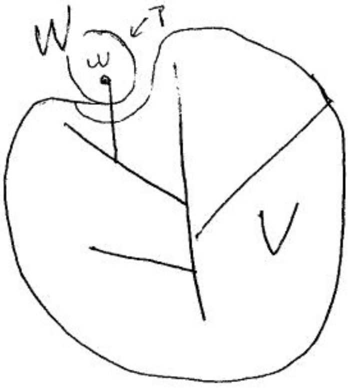
The Finiteness of JSJ pieces, and Lemmas 1,2 reduce the proof of Theorem 4 to the following proposition:

Prop 1. *The set $\mathcal{D}(M, \Gamma)$ is finite.*

The proof of Proposition 1 is by induction on the number n_Γ of edges of Γ. If $n_\Gamma = 0$, Proposition 1 is true by Theorem 0. We assume the result to be true for $n_\Gamma \leq n-1$ and prove it for $n_\Gamma = n$.

Let $N \in \mathcal{D}(M,\Gamma)$. Let w be a **leaf** of Γ and let e be the attached edge. Denote by W the geometric submanifold in $\mathcal{HS}(M)$ corresponding to w and let $V = M \setminus W$. The compact 3-manifolds V and W are both integral homology solid tori with boundary an incompressible torus corresponding to the edge e. Notice that the topological type of W is fixed by definition of $\mathcal{D}(M,\Gamma)$, while the topological type of V may depend on N.

Since V and W are **integral homology solid tori**, one can fix on each torus ∂V and ∂W a basis for the first homology group: $\{\mu_V, \lambda_V\}$ and $\{\mu_W, \lambda_W\}$ such that:



1. $\mu_V \subset \partial V$ and $\mu_W \subset \partial W$ each bounds a properly embedded surface F_V and F_W respectively in V and W.

2. Intersection $\mu_V \cdot \lambda_V = \mu_W \cdot \lambda_W = 1$

Lemma 3. *The gluing map $\phi : \partial V \to \partial W$ satisfies the following equations, where $\varepsilon = \pm 1$, $p, q \in \mathbb{Z}$:*

(1) $\phi(\mu_V) = p\mu_W + \epsilon\lambda_W$,

(2) $\phi(\lambda_V) = \varepsilon(pq+1)\mu_W + q\lambda_W$.

By pinching V to a solid torus to gets a degree-one map $f_V : N \to W(p/\varepsilon)$, where the homology sphere $W(p/\varepsilon)$ is obtained by Dehn filling W with a solid torus. Hence $W(p/\varepsilon)$ is dominated by M and we can show **The integer p takes only finitely many values**.

By pinching W to a solid torus, one gets a degree-one map $f_W : N \to V(-q/\varepsilon)$. Hence $V(-q/\varepsilon)$ is dominated by M and we can show **The manifold $V = M \setminus int(W)$ takes only finitely many topological types and the integer q takes only finitely many values.**

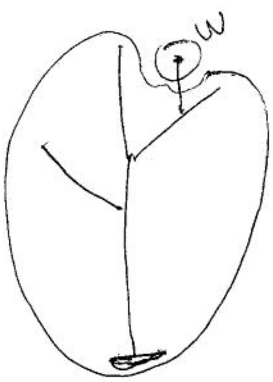
The argument for integral homology spheres can be modified to prove the following

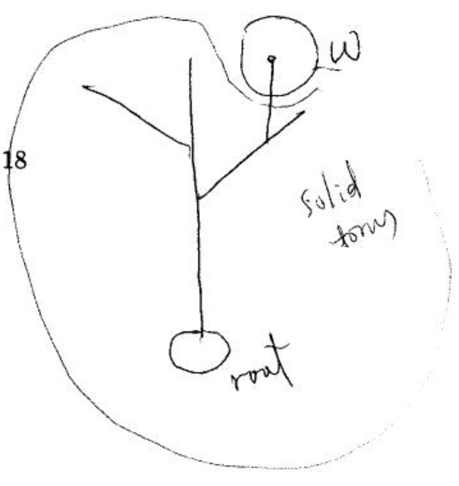
Corollary 1. Any compact orientable 3-manifold dominates at most finitely many knot complements in $\mathbb{S}^3$.

Let $E(k)$ be the exterior of a knot k in $\mathbb{S}^3$. The dual graph $\Gamma(k)$ to the JSJ-decomposition of $E(k)$ is a rooted tree, where the root corresponds to the unique vertex manifold containing $\partial E(k)$.

Let w be a leaf of Γ which is not the root. Recall that $S^3 = E(k) \cup N(k)$. Let W be the JSJ-piece of $E(k)$ corresponding to w and let $V = \mathbb{S}^3 \setminus int(W)$. Then V is a solid torus such that $V \setminus int(N(k)) = E(k) \setminus int(W)$, which we will denote by U. Then we have $E(k) = U \cup_\phi W$, where $\phi : \partial V \to \partial W$ is the gluing map.

In the proof of Theorem 4, we proved the finiteness of both integers p and q by pinching first V, then W. In the case of a knot complement $E(k)$ we can only pinch W. However in this case, only one integer is involved in determining the gluing due to the fact that W is the exterior of a non-trivial knot k_W in $\mathbb{S}^3$ which is determined by its exterior [GL].





Conjecture [Ki, Problem 1.12 (J. Simon)] *Given a knot* $k \subset \mathbb{S}^3$*, there are only finitely many knots* $k_i \in \mathbb{S}^3$ *for which there is an epimorphism* $\phi_i : \pi_1(E(k)) \to \pi_1(E(k_i))$.

The Conjecture is true if k is small and each epimorphism ϕ_i is ∂-preserving [Reid-W].

Corollary 1 gives another answer to Simon's Conjecture.

Corollary 2. *There are only finitely many knots* $k_i \subset \mathbb{S}^3$ *for which there is an epimorphism* $\phi_i : \pi_1(E(k)) \to \pi_1(E(k_i))$ *such that the image of the longitude is non-trivial.*

Part K. 1-domination on knots

Boilaeu-Boyer-Rolfsen-Wang -Lackenby

Let $g(k)$, $\Lambda(k)$, Δ_k, $V(k)$ be the genus, Alexander module, Alexander polynomial, and Gromov volume of k respectively.

If $k_1 \geq k_2$, then

(1) $g(k_1) \geq g(k_2)$ (Gabai);

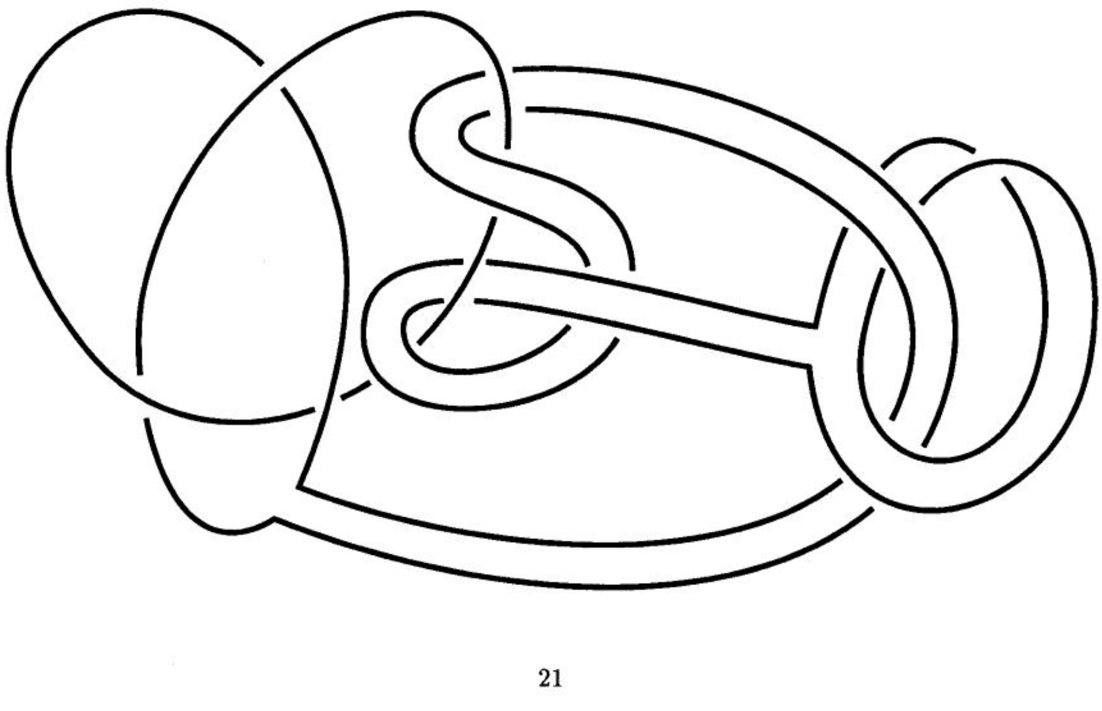
(2) $V(k_1) \geq V(k_2)$ (Gromov);

(3) $\Lambda_{k_1} = \Lambda_{k_2} \oplus \Lambda$, in particular $\Delta_{k_2} | \Delta_{k_1}$;

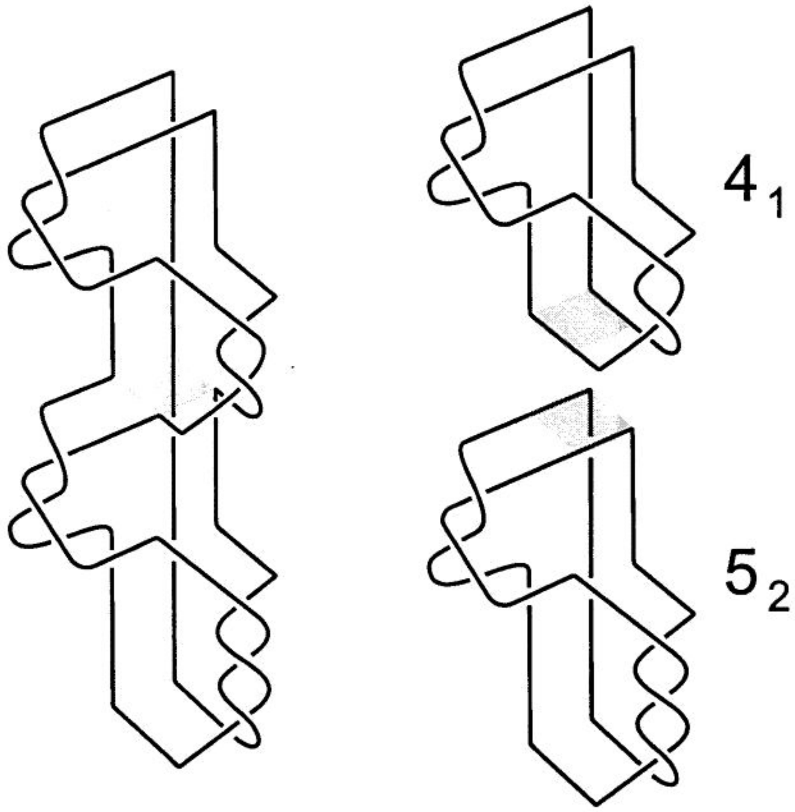
Alexander polynomial is easy to calculate and (3) above already gives some interesting applications.

Example 1.

(1) Figure 1 is a band connected sum k of the trefoil knot 3_1 and the trivial knot with $\Delta_k(t) = 1-t^2+t^4$, which contains no $\Delta_{3_1}(t) = 1-t+t^2$ as a factor. It follows that band connected sum does not 1-dominates its factors in general.



(2) Figure 2 is a Murasugi sum k of 5_2 and 4_1 with $\Delta_k(t) = 2 - 3t + 3t^2 - 3t^3 + 2t^4$, which contain no either $\Delta_{4_1}(t) = 1 - 3t + t^2$ or $\Delta_{5_2}(t) = 2 - 3t + 2t^2$ as a factor. It follows that Murasugi sum does not 1-dominates its factors in general.



Corollary 1. *(1) Any non-fiber knot with $\Delta_k(t)$ leading coefficient $\neq 1$ does not 1-dominate any fiber knots of the same genus.*

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(2) Suppose k_1 and k_2 of the same genus, k_1 is an alternating knot and k_2 is a fiber knot. Then $k_1 \geq k_2$ implies $k_1 = k_2$.

Rigidity results about 1-dominations on knots is "$k \geq k'$ implies that $k = k'$, if ....."

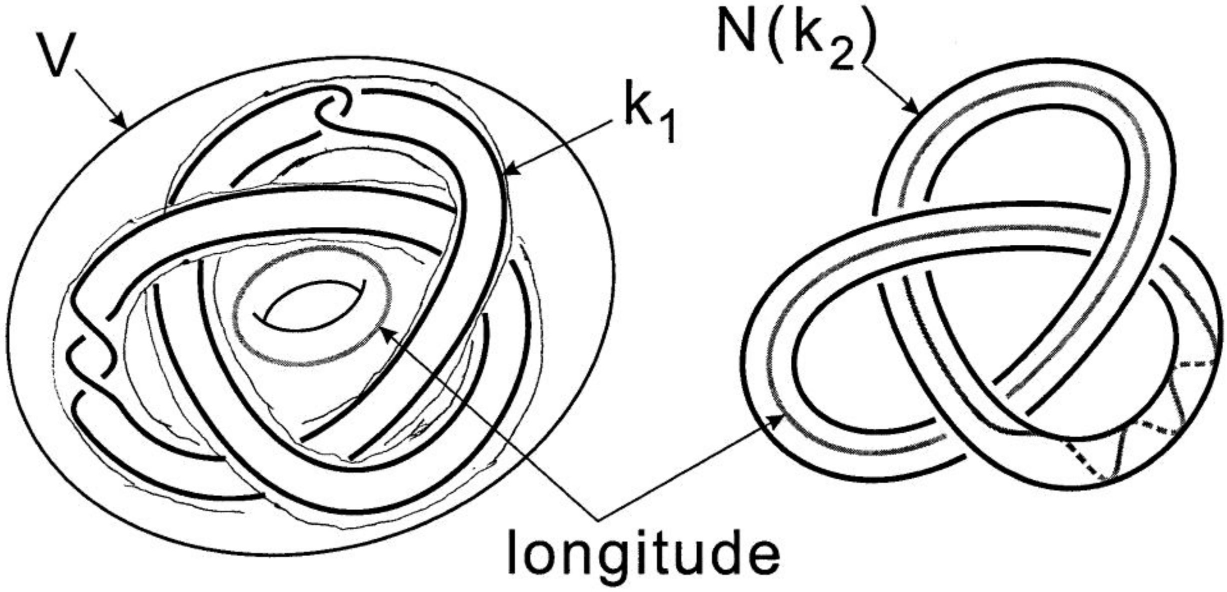
Some previous rigidity results are: $k \geq k'$ implies that $k = k'$,

(1) if k and k' have the same Gromov volume and k is hyperbolic [Gromov-Thurston]; or

(2) if k and k' have the same Alexander polynomial and k is fibred; or

(3) if k and k' have the same genus and k is Seifert [Rong].

Example 2. Non-trivial 1-dominations $k \to k_1$ of the same genus, the same Alexander polynomial, and the same Gromov volume. Moreover all those invariants are non-vanishing.



Let $k = h(k_1)$ be a satellite of k_2 indicated by Figure below. Then we have 1-domination $k \to k_1$ given by dis-satellization.

The JSJ-piece of $E(k)$ consists of three components: two Seifert pieces and one hyperbolic piece H, which is homeomorphic to the Hopf link complement; and the JSJ-piece of $E(k_1)$ consists of two components: one Seifert piece and one hyperbolic piece H. It become clear that both k and k_1 are of genus 1, and have the same Gromov volume which equal to the hyperbolic volume of the Hopf link complement. They also have the same Alexander polynomials, since the h is longitude preserving.

There are arbitrary long 1-domination sequences of knots with the genus, Alexander polynomials and Gromov volumes are all same.

In Example 2, the fact that the winding number of k with k_2 is zero is essential in order to construct non-trivial 1-domination $k > k_1$ with many invariants the same.

Indeed we have the following rigidity result.

Theorem. *Suppose that any companion of k has non-zero winding number. If $k \geq k'$ with the same Gromov volume and the same genus, then $k = k'$.*

Closely related rigidity results, we will study the bound of the length n of 1-domination sequences of knots $k_0 > k_1 > k_2 > .... > k_n$ with given k_0.

Theorem. [Rong, Soma] *Any 1-domination sequence $M_0 > M_1 > ... > M_i > ....$ of 3-manifolds in Thurston's picture has a bounded length for given M_0.*

Definition.

(1) Say a Seifert surface S of a knot k is *free* if $E(k) \setminus S$ is a handlebody. Say a knot k is *free*, if all its incompressible Seifert surfaces are free.

(2) Define $\hat{g}(k)$ be the maximum $g(S)$ for all incompressible Seifert surfaces S of k.

(1) There are examples of $\hat{g}(k) = \infty$,

(2) $\hat{g}(k) = g(k)$ for fiber knots and 2-bridge knots (Hatcher-Thurston).

(3) $\hat{g}(k)$ is bounded for alternating knots (Menasco-Thistlethwaite) and for small knots (Lackenby).

(4) alternating knots (Menasco), Montesinos knots (Oertel) Small knots, fiber knots are free knots;

(5) If a knot k has a companion of winding number zero, then k is not free.

Proposition. *Suppose k_0 is a free knot with bounded $\hat{g}(k_0)$. Then any 1-domination sequence $k_0 > k_1 > .... > k_n$ of knots has $n \leq \hat{g}(k_0)$.*

Proof. The core of the proof is the following simple fact.

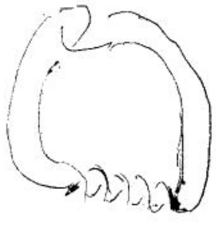
Let k be a free knot, $f : E(k) \to E(k')$ be a degree one map, and S' be a Seifert surface of k' with genus $g(k')$. By classical argument in 3-manifold topology, f can be properly homotoped so that $S = f^{-1}(S')$ is a connected incompressible Seifert surface of k.

Then f induces a proper degree one

$$f^* : H = E(k) \setminus S \to E(k') \setminus S' = H'$$

Since k is free, H is a handlebody, then H' is a handlebody.

One can easy to argue that if $g(S) = g(S')$, then $k = k'$.



minimal

2-bridge knot

Cor. 1. If k_0 is 2-bridge knot,

then the length

$$k_0 > k_1 > k_2 > \cdots > k_n$$

is bounded by $f(k_0)$

Cor 2. twisted knots are

Minimal