Deformations of reducible representations of 3-manifold groups into $\mathrm{PSL}_2(\mathbb{C})$

Summer School and Conference on Geometry and Topology of 3-Manifolds

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Introduction

- Let M be a connected, compact, orientable, 3-manifold such that $\beta_1(M) = 1$ and such that $\partial M \cong S^1 \times S^1$ is a torus.
- Given a homomorphism $\alpha\colon \pi_1(M) \to \mathbf{C}^*$, we define an abelian representation $\rho_\alpha\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$:

$$\rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha^{\frac{1}{2}}(\gamma) & 0 \\ 0 & \alpha^{-\frac{1}{2}}(\gamma) \end{pmatrix} \qquad \forall \gamma \in \pi_1(M)$$

  where $\alpha^{\frac{1}{2}}\colon \pi_1(M) \to \mathbf{C}^*$ is a map (not necessarily a homomorphism) such that $(\alpha^{\frac{1}{2}}(\gamma))^2 = \alpha(\gamma)\ \forall \gamma \in \pi_1(M)$.
- The representation ρ_α is reducible, ie $\rho_\alpha(\pi_1(M))$ has global fixed points in $P^1(\mathbf{C})$.

Question

- When can ρ_α be deformed into irreducible representations (ie representations whose image has no fixed point in $P^1(\mathbf{C})$)?
- Different versions of this question have been studied by
  - Frohman–Klassen, Herald and H–Kroll for $\mathrm{SU}(2)$;
  - H–Porti–Suarez and Shors for $\mathrm{SL}_2(\mathbf{C})$;
  - Ben Abdelghani and Ben Abdelghani–Lines for semi-simple Lie groups.
- It was always assumed that $\alpha\mid_{\mathrm{tors}(H_1(M))}$ is trivial.

Notations

Let M be given as in the introduction.

- $R(M) := \hom(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ denotes the variety of representations of $\pi_1(M)$ in $\mathrm{PSL}_2(\mathbf{C})$.

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- If $\rho \in R(M)$ then $\chi_\rho \colon \pi_1(M) \to \mathbf{C}$ given by

$$\chi_\rho(\gamma) = (\operatorname{tr} \rho(\gamma))^2 \quad \forall \gamma \in \pi_1(M)$$

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- The set of all characters $X(M)$ can be identified with the algebraic quotient $R(M)//\mathrm{PSL}_2(\mathbf{C})$ and is called the *character variety* of M.

Theorems

Let α be a *simple* zero of the Alexander polynomial;

- then ρ_α is contained in precisely two irreducible components of $R(M)$, one of dimension 4 containing irreducible representations and another of dimension 3 containing only abelian ones. In addition ρ_α is a smooth point of both varieties and the intersection at the orbit of ρ_α is transverse.

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- then ρ_α is contained in precisely two irreducible components of $R(M)$, one of dimension 4 containing irreducible representations and another of dimension 3 containing only abelian ones. In addition ρ_α is a smooth point of both varieties and the intersection at the orbit of ρ_α is transverse.
- then χ_α is contained in precisely two irreducible components of $X(M)$, which are curves and are the quotients of the components of $R(M)$. In addition χ_α is a smooth point of both curves and the intersection at χ_α is transverse.

The abelian component

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- We define $\alpha\varphi\colon \pi_1(M) \to \mathbf{C}^*$ by

$$\alpha\varphi(\gamma) = \alpha(\gamma)\varphi(\phi(\gamma)) \quad \gamma \in \pi_1(M),$$

where $\phi \in H^1(M;\mathbf{Z}) \cong \hom(H_1(M),\mathbf{Z})$ is a generator.

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- The $\mathrm{PSL}_2(\mathbf{C})$ orbit of $\rho_{\alpha\varphi}$ is two dimensional if $\alpha\varphi$ is nontrivial.
- Hence V_α is contained in an at least three dimensional component. We denote by $S_\alpha(M) \subset R(M)$ the closure of the $\mathrm{PSL}_2(\mathbf{C})$–orbit of V_α ($\Rightarrow \dim S_\alpha(M) \geq 3$).

The Alexander polynomial: setup

- Fix a projection $p\colon H_1(M) \to \mathrm{tors}(H_1(M))$ and a generator $\phi \in H^1(M) \cong \hom(H_1(M), \mathbf{Z})$:

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- Define $\phi_\sigma\colon \pi_1(M) \to \mathbf{C}[t^{\pm 1}]^*$ by $\phi_\sigma(\gamma) = \sigma(p(\gamma))t^{\phi(\gamma)}$.

The twisted Alexander invariant

In the sequel we shall write $R := \mathbf{C}[t^{\pm 1}]$.

- The twisted Alexander module $H_1^{\phi_\sigma}(M)$ is defined by

$$H_1^{\phi_\sigma}(M) = H_1(M, \mathbf{C}[t^{\pm 1}]_{\phi_\sigma}).$$

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- Note that $\Delta_k^{\phi_\sigma}$ and $H_1^{\phi_\sigma}(M)$ depend on the projection p and on the generator $\phi \in H^1(M)$.

The symmetry property

- Consider the following involution on $\mathbf{C}[t^{\pm 1}]$:

$$\overline{\sum_i a_i\, t^i} = \sum_i \overline{a_i} t^{-i},$$

where $\overline{a_i}$ denotes the complex conjugate of $a_i \in \mathbf{C}$.

- An element $\eta \in \mathbf{C}[t^{\pm 1}]$ is called *symmetric* if there exists a unit $\epsilon \in \mathbf{C}[t^{\pm 1}]^*$ such that $\bar{\eta} = \epsilon\eta$.

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Theorem: ***(Blanchfield)*** Let M be as in the introduction. Then for each homomorphism $\sigma\colon \mathrm{tors}(H_1(M;\mathbf{Z})) \to \mathrm{U}(1)$ and for each projection $p\colon H_1(M) \to \mathrm{tors}(H_1(M))$ the polynomial $\Delta_k^{\phi_\sigma}$ is symmetric. $\square$

A zero of the Alexander polynomial

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- This definition does not depend on the choice of the projection p and the generator ϕ.
- Moreover we have $\alpha = \beta \circ \phi_\sigma$ where $\beta\colon \mathbf{C}[t^{\pm 1}] \to \mathbf{C}$ is simply the evaluation $\beta(\eta(t)) = \eta(a)$ at $a = \alpha(s_p(1))$.

André Weil's construction

Let $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ be a representation. The Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ turns into a $\pi_1(M)$-module via $\mathrm{Ad} \circ \rho$ which will be denoted by $\mathfrak{sl}_2(\mathbf{C})_\rho$:

$$\gamma \circ x = \rho(\gamma) x \rho(\gamma^{-1}) \quad \forall x \in \mathfrak{sl}_2(\mathbf{C}).$$

There is a natural inclusion of the Zariski tangent space into the space of 1-cocycles $T_\rho R(\pi_1(M)) \hookrightarrow Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_\rho)$:

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- Let ρ_ϵ be a path of representations through $\rho_0 = \rho$
- We define $d\colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})_\rho$ by

$$d(\gamma) := \left.\frac{d\,\rho_\epsilon(\gamma)}{d\,\epsilon}\right|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma\,.$$

Cocycles and coboundaries

- Then $d\colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})_\rho$ is 1-cocycle ie, d satisfies:

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \gamma_1 \circ d(\gamma_2),\ \forall \gamma_1, \gamma_2 \in \pi_1(M)\,.$$

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- If $\rho_\epsilon = A_\epsilon \rho A_\epsilon^{-1}$ such that A_0 is the identity then d is a 1-coboundary ie, $\exists y \in \mathfrak{sl}_2(\mathbf{C})$ such that $d(\gamma) = (1-\gamma) \circ y$ $\forall \gamma \in \pi_1(M)$.

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- The $\pi_1(M)$-module $\mathfrak{sl}_2(\mathbf{C})_{\rho_\alpha}$ decomposes as $\mathfrak{sl}_2(\mathbf{C})_{\rho_\alpha} = \mathbf{C}_+ \oplus \mathbf{C}_0 \oplus \mathbf{C}_-$ where

$$\mathbf{C}_+ = \mathbf{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad \mathbf{C}_0 = \mathbf{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \mathbf{C}_- = \mathbf{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here $\mathbf{C}_0$ is a trivial $\pi_1(M)$-module and $\gamma \in \pi_1(M)$ acts on $z_\pm \in \mathbf{C}_\pm$ via $\gamma \cdot z_\pm := \alpha(\gamma)^{\pm 1} z_\pm$.

Abelian representations

Note that $\alpha = \beta \circ \phi_\sigma$ implies that $\dim H^1(\pi_1(M); \mathbf{C}_\pm) = k$ iff α is a root of $\Delta_{k-1}^{\phi_\sigma}$ but not a root of $\Delta_k^{\phi_\sigma}$.

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Lemma: Let $\alpha\colon \pi_1(M) \to \mathbf{C}^*$ be a representation. If α is not a zero of Δ^{ϕ_σ} then ρ_α in $R(M)$ is a smooth point of the component $S_\alpha(M)$.

Moreover, $S_\alpha(M)$ is the unique component through ρ_α and $\dim S_\alpha(M) = 3$.

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Proof: We have

$$3 \le \dim S_\alpha(M) \le \dim_{\rho_\alpha} R(M) \le \dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_\alpha).$$

Hence $\dim H^1(\pi_1(M); \mathbf{C}_\pm) = 0 \Rightarrow \dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_\alpha) = 3$. $\square$

Metabelian representations

- Let $\alpha\colon \pi_1(M) \to \mathbf{C}^*$ be a homomorphism and let $d\colon \pi_1(M) \to \mathbf{C}$ be a map. Then $\rho_\alpha^d\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ given by

$$\rho_\alpha^d(\gamma) = \begin{pmatrix} 1 & d(\gamma) \\ 0 & 1 \end{pmatrix} \rho_\alpha(\gamma) = \pm \begin{pmatrix} \alpha^{\frac{1}{2}}(\gamma) & \alpha^{-\frac{1}{2}}(\gamma) d(\gamma) \\ 0 & \alpha^{-\frac{1}{2}}(\gamma) \end{pmatrix}$$

is a homomorphism if and only if d satisfies $d(\gamma_1\gamma_2) = d(\gamma_1) + \alpha(\gamma_1) d(\gamma_2)$ ie, $d \in Z^1(\pi_1(M), \mathbf{C}_+)$. Moreover, ρ_α^d is non abelian if and only if d is not a coboundary.

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- *(Burde, DeRham)* There exists a reducible, non abelian representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ such that $\chi_\rho = \chi_{\rho_\alpha}$ if and only if α is a zero of the Alexander invariant.

Cohomology of metabelian reps

From now on we'll suppose that α is a simple zero of the Alexander polynomial ($\Rightarrow H^1(\pi_1(M);\mathbf{C}_\pm) \cong \mathbf{C}$).

- Suppose $d_\pm \in Z^1(\pi_1(M);\mathbf{C}_\pm)$ represents a generator of $H^1(\pi_1(M);\mathbf{C}_\pm)$. Denote by $\rho^\pm$ the metabelian representations into the upper/lower triangular group given

$$\rho^+(\gamma) = \begin{pmatrix} 1 & d_+(\gamma) \\ 0 & 1 \end{pmatrix} \rho_\alpha(\gamma) \qquad \text{and} \qquad \rho^-(\gamma) = \begin{pmatrix} 1 & 0 \\ d_-(\gamma) & 1 \end{pmatrix} \rho_\alpha(\gamma).$$

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- Let $\mathfrak{b}_+ \subset \mathfrak{sl}_2(\mathbf{C})$ denote the Borel subalgebra of upper triangular matrices. It is a $\pi_1(M)$-module via $\mathrm{Ad} \circ \rho^+$.

- There is a short exact sequence of $\pi_1(M)$-modules:

$$0 \longrightarrow \mathfrak{b}_+ \longrightarrow \mathfrak{sl}_2(\mathbf{C})_{\rho^+} \longrightarrow \mathbf{C}_- \longrightarrow 0$$

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- **Proposition:** If α is a simple zero of the Alexander invariant then $H^1(\pi_1(M); \mathfrak{b}_+) = 0$ and the projection to the quotient $\mathfrak{sl}_2(\mathbf{C})_{\rho^+} \longrightarrow \mathfrak{sl}_2(\mathbf{C})_{\rho^+}/\mathfrak{b}_+ \cong \mathbf{C}_-$ induces an isomorphism

$$H^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_{\rho^+}) \cong H^1(\pi_1(M); \mathbf{C}_-) \cong \mathbf{C}.$$

Deformations

Let Γ be a group and let $\rho \in R(\Gamma)$.

- A *formal deformation* of ρ is a homomorphism $\rho_\infty : \Gamma \to \mathrm{PSL}_2(\mathbf{C}[[t]])$

$$\rho_\infty(\gamma) = \pm \exp(\sum_{i=1}^{\infty} t^i u_i(\gamma))\rho(\gamma)$$

where $u_i : \Gamma \to \mathfrak{sl}_2(\mathbf{C})$ are maps such that $\rho_\infty \mid_{t=0} = \rho$. Here $\mathbf{C}[[t]]$ denotes the ring of formal power series.

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Let Γ be a group and let $\rho \in R(\Gamma)$.

- A *formal deformation* of ρ is a homomorphism $\rho_\infty\colon \Gamma \to \mathrm{PSL}_2(\mathbf{C}[[t]])$

$$\rho_\infty(\gamma) = \pm \exp\Big(\sum_{i=1}^{\infty} t^i u_i(\gamma)\Big)\rho(\gamma)$$

  where $u_i\colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})$ are maps such that $\rho_\infty\mid_{t=0} = \rho$. Here $\mathbf{C}[[t]]$ denotes the ring of formal power series.

- ρ_∞ is a homomorphism up to first order iff $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2(\mathbf{C})_\rho)$ is a cocycle. We call a cocycle $u_1 \in Z^1(\Gamma, \mathfrak{sl}_2(\mathbf{C})_\rho)$ *integrable* if there is a formal deformation of ρ with leading term u_1.

Obstructions

Let $u_1, \dots, u_k \colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})_\rho$ be given such that $\rho_k(\gamma) = \exp(\sum_{i=1}^k t^i u_i(\gamma))\rho(\gamma)$ is a homomorphism modulo t^{k+1}. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1,\dots,u_k)} \in H^2(\Gamma; \mathfrak{sl}_2(\mathbf{C})_\rho)$ with the following properties:

- $\exists u_{k+1} \colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})$ such that $\rho_{k+1}(\gamma) = \exp(\sum_{i=1}^{k+1} t^i u_i(\gamma))\rho(\gamma)$ is a morphism modulo t^{k+2} if and only if $\zeta_{k+1} = 0$.

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- $\exists u_{k+1} \colon \Gamma \to \mathfrak{sl}_2(\mathbf{C})$ such that $\rho_{k+1}(\gamma) = \exp(\sum_{i=1}^{k+1} t^i u_i(\gamma))\rho(\gamma)$ is a morphism modulo t^{k+2} if and only if $\zeta_{k+1} = 0$.
- Naturality: if $f \colon \Gamma' \to \Gamma$ is a morphism then $f^*\rho_k := \rho_k \circ f$ is also a morphism modulo t^{k+1} and $f^*(\zeta_{k+1}^{(u_1,\ldots,u_k)}) = \zeta_{k+1}^{(f^*u_1,\ldots,f^*u_k)}$.

La Fin

Lemma: Let $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ be a reducible, non abelian representation such that $\dim Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_\rho) = 4$. If $\rho \circ i_\#\colon \pi_1(\partial M) \to \mathrm{PSL}_2(\mathbf{C})$ is neither trivial nor a representation onto a Klein group then every cocycle in $Z^1(\pi_1(M); \mathfrak{sl}_2(\mathbf{C})_\rho)$ is integrable.

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Corollary: The representation ρ^+ is a smooth point of $R(M)$ with local dimension four.