

# Pseudo Conformal Geometry and 3 Manifolds

(F.J. Porti)

Motivating Question: is there a universal geometry for 3 manifolds?

$X$  = Smooth manifold

Surfaces: conf,  
proj.

$G$  = group of diffeos of  $X$

$(X, G)$  is a geometry.

## Examples

1)  $X$  = universal cover of  $M$

$G$  = covering transformations

2)  $X$  = Homogeneous Space eg  $H^7$

$G = \text{Isom}(X)$

3)  $X = RP^n$      $G = \text{Aut}(RP^n) \cong \text{PGL}(n+1, \mathbb{R})$

Definition A representation of the geometry  $(X, G)$  into  $(X', G')$  is

(i) local diffeo     $\text{dev} : X \longrightarrow X'$

(ii) homomorphism     $\text{hol} : G \longrightarrow G'$

$\forall g \in G \quad \forall x \in X \quad \text{dev}(g \cdot x) = \text{hol}(g) \text{dev}(x)$

### Examples

- 1) An  $(X, G)$ -structure on manifold  $M$  is a representation  
 $(\tilde{M}, \pi, M) \longrightarrow (X, G)$
- 2) The projective model of  $H^n$   
is a representation Klein model  
 $(H^n, \text{Isom } H^n) \longrightarrow (RP^n, \text{Aut}(RP^n))$

Theorem (v.Thurston, Molnar, Thiel)

let  $(X, G)$  be one of Thurston's eight geometries.

NIL!

- i) If  $X \neq S^2 \times \mathbb{R}, H^2 \times \mathbb{R}$  then there is a projective representation.
- ii) If  $X = S^2 \times \mathbb{R}, H^2 \times \mathbb{R}$  set  $G_0 = \text{Subgroup of } \text{Isom}(X) \text{ preserving } \mathbb{R}\text{-orientation.}$   
Then  $(X, G_0)$  has projective rep.

Corol. If  $M$  has Thurston geometry then  $M$  (or 2-fold cover) has a projective structure.

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Thm (C.)  $\mathbb{R}P^3 \# \mathbb{R}P^3$  has no real projective structure.

proof:

Open Problem (hard?) Are there other connected 3 manifolds with no real proj. struct.?

Conclusion: Projective geometry NOT universal in  $\dim = 3$ .

## Conformal Geometry -

$S^n$ , (anti-)conformal maps  $\equiv \text{Conf}_\pm$ )

mention: charts, "ball model"

Conformal representation

$$(X, G) \rightarrow (S^n, \text{Conf}_\pm)$$

There are conformal reps of

$$\mathbb{H}^3 \quad \mathbb{E}^3 \quad S^3 \quad S^2 \times \mathbb{R} \quad \mathbb{H}^2 \times \mathbb{R}$$

Connect sum of conformal mfd's  
is  $\text{Conf}$  eg  $\mathbb{R}P^3 \# T^3$

Proof:

GOLDMAN: SolV and NIL 3manifolds  
are not conformal

DUONG HWANG: Given closed 3mfds  
 $M \# N$  closed 3mfds  $N$  st

$M \# N$  conformal

M. Kapovich

Conclusion: Conformal geometry NOT  
universal in  $\dim = 3$

## Pseudo Conformal Geometries

$$\Omega \subset \mathbb{R}^{n+1} \subset \mathbb{RP}^{n+1}$$

bounded and strictly convex

every line segment in  $\mathbb{R}^n$   
with endpoints in  $\Omega$   
has interior in  
interior ( $\Omega$ ).



$$PC(\partial\Omega) = \text{Aut}(\mathbb{P}^{n+1}) \cap \text{Stab}(\Omega)$$

$(\partial\Omega, PC(\partial\Omega))$  is a PC geometry

Thm 1 ( $C_+$ , PORTI)

$\text{Sol } V, \text{Isom}(\text{Sol } V)$  is PC

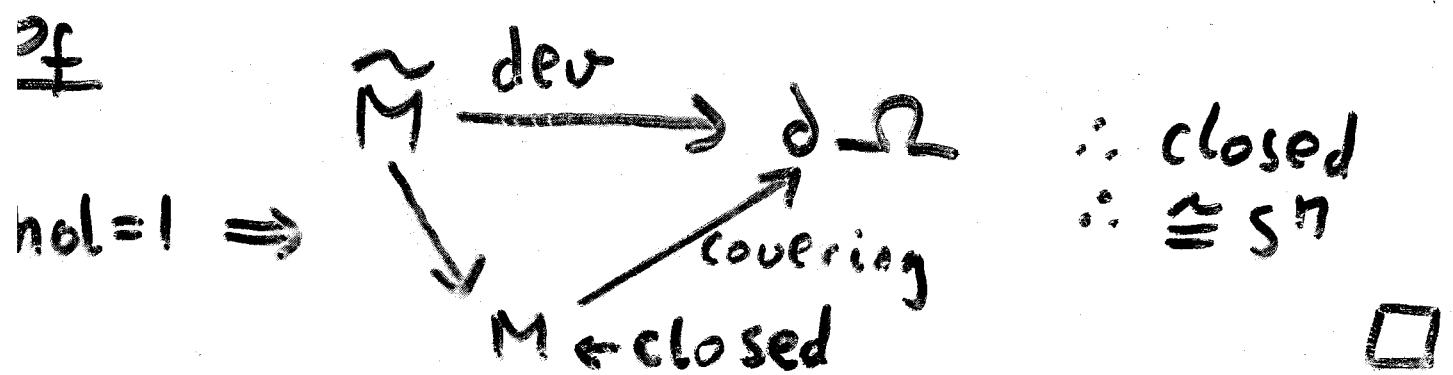
$(\widetilde{\text{PSL}}, \widetilde{\text{SL}})$  is PC

$(\text{NIL}, \text{NIL})$  is not PC

Definition a representation of geometries  $(X, G) \rightarrow (X', G')$  is complete if  $\text{dev}: X \rightarrow X'$  is a covering map. incomplete structs.

Prop 1 If  $M$  and  $N$  are complete PC n-manifolds then so is  $M \# N$ .

Proof: If  $M$  is closed PC n-mfd with trivial holonomy then  $M \cong S^n$



orol Suppose  $M = \text{closed 3mfds}$ ,  
which is compact sum of 3mfds  
each of which admits a Thurston  
geometry st

) no summand is NIL  
i) only summands modelled on  
 $\overline{\text{PSL}_2\mathbb{R}}$  have holonomy in  $\overline{\text{PSL}_2\mathbb{R}}$   
THEN  $M$  has a complete PC  
structure. Thus  $\pi_1 M$  is a  
discrete subgroup of  $\text{SL}(5, \mathbb{R})$ .

Question: does EVERY 3mfds  
have a PC structure?

probably not?

For NIL 3mfds only know  
 $(M, \pi_1 M) \rightarrow (\partial \Sigma, \text{PC}(\partial \Sigma))$   
 $\downarrow \quad \uparrow$  # factoring  
 $(\text{NIL}, \text{Isom}(\text{NIL}))$

PC rep of  $(\overline{PSL}_2 \mathbb{R}, \overline{PSL}_2 \mathbb{R})$

$GL(2, \mathbb{R})$  act on  $\mathbb{R}^2$

: on  $V = \{a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4\}$   
= homog. polys of degree 4  
on  $\mathbb{R}^2$

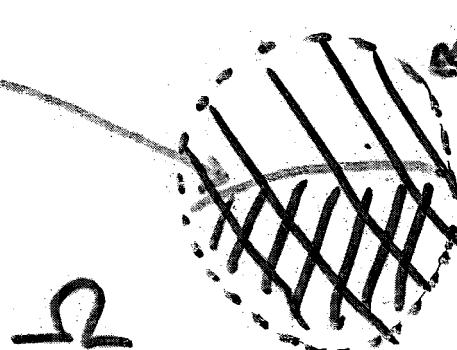
action by isometries of  $J \sim \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

$Pos = \{p \in V : p \text{ is positive definite}\}$   
= convex

$J = \left( \begin{array}{c} \text{orbit of} \\ x^4 - \frac{1}{5}x^2y^2 + y^4 \end{array} \right)$

= convex  
hypersurface

$\Omega \subset \pi(Pos)$



$\subset RP^4$

$\partial \Omega = W$

$\pi : \mathbb{R}^5 - 0 \rightarrow RP^4$

similar to proj model of  $H^3$  !

Apologies 21/01/1993

# PC representation of SOLV

$$SOLV = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & \pm 1/a & c \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ \pm 1/a & 0 & c \\ 0 & 0 & 1 \end{pmatrix} : \right. \\ \left. a \neq 0, b, c \in \mathbb{R} \right\}$$

$\subset \text{Aff}(\mathbb{R}^2)$  prefers  
 $x \& y$  fol's

$$\begin{pmatrix} a & 0 & b \\ 0 & 1/a & c \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow ((x, y) \mapsto (ax + b, \frac{1}{a}y + c))$$

:  $\exists$  5 dim'l irred. rep on

$$= \{x_0X^2 + x_1Y^2 + x_2X + x_3Y + x_4\}$$

$\subset \{\text{quadratic polynomials on } \mathbb{R}^2\}$

Pos = subspace of positive  $\subset V$   
 definite polynomials with  
 unique minimum.

$$L \in \Pi(\text{Pos}) \subset \mathbb{R}\mathbb{P}^4$$

$$\Pi(\text{Pos})$$

