A Combination Theorem for Convex Hyperbolic Manifolds

Mark Baker (Univ Rennes)

Daryl Cooper (UCSB)

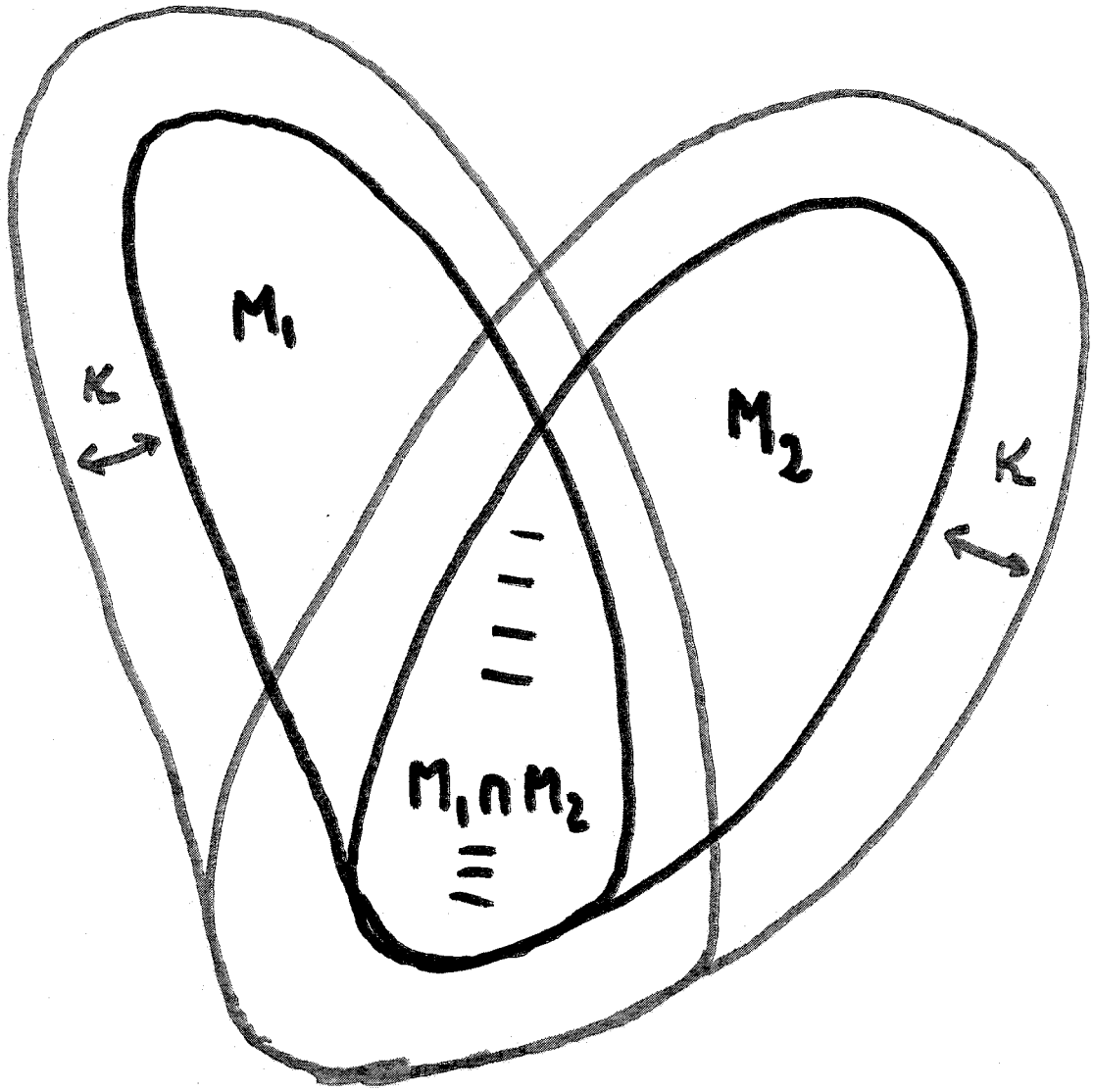
1. Idea of Convex Combination Thm

Let M be connected hyperbolic n-manifold that is the union of two convex hyperbolic n-submanifolds

If M has a K thickening with the same topology

then M can be thickened to be convex

(Can take $K = 6$)


M_1
M_2
K
K
$M_1 \cap M_2$

2. Properties of convex hyp manifolds

Let M be hyperbolic n-manifold.

Then $\exists$ two maps :

$\mathrm{dev} : \tilde{M} \to \mathbb{H}^n$ developing map

$\mathrm{hol} : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ holonomy

s.t.

$$\mathrm{dev}(g \cdot x) = \mathrm{hol}(g) \cdot \mathrm{dev}(x)$$

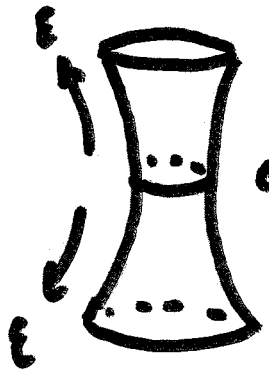
<u>Definition</u> : M is convex if dev is 1-1 and $\mathrm{dev}(\tilde{M}) \subseteq \mathbb{H}^n$ is convex.

$\Longleftrightarrow$ Any two points of $\tilde{M}$ can be joined by a geodesic arc

$\Longleftrightarrow$ Every path in M is homotopic rel endpts to a geodesic in M

M convex $\Rightarrow$ $M = \mathrm{dev}(\tilde{M}) / \mathrm{hol}(\pi_1(M))$

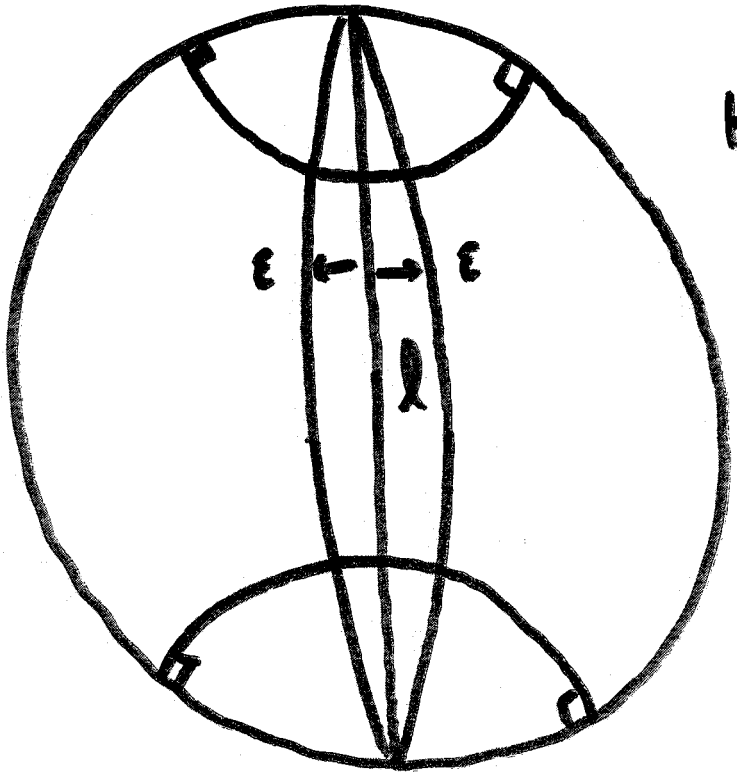
<u>Example</u> : hyperbolic annulus

$A =$ 

← geodesic length ℓ

ε-thickening of geodesic loop

$\mathrm{dev}(\tilde{A})$



$\mathbb{H}^2$

$A = \mathrm{dev}(\tilde{A}) / \mathrm{hol}(\pi_1(A))$

3. Manifolds with convex thickenings

M has convex thickening if $\exists\, N$ convex s.t. $M \subset N$ and $\mathrm{incl}_* : \pi_1(M) \to \pi_1(N)$ is an isomorphism

$\iff$ $\mathrm{dev} : \tilde{M} \to \mathbb{H}^n$ is injective

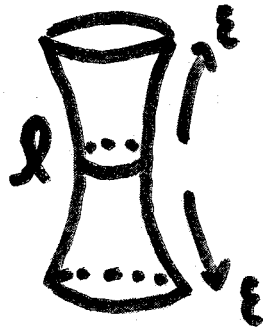
$\iff$ $\mathrm{hol} : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ is discrete and 1-1

Then can take $N = \mathbb{H}^n / \mathrm{hol}(\pi_1(M))$

$\Rightarrow$ M 'corresponds' to a discrete subgp of $\mathrm{Isom}(\mathbb{H}^n)$.

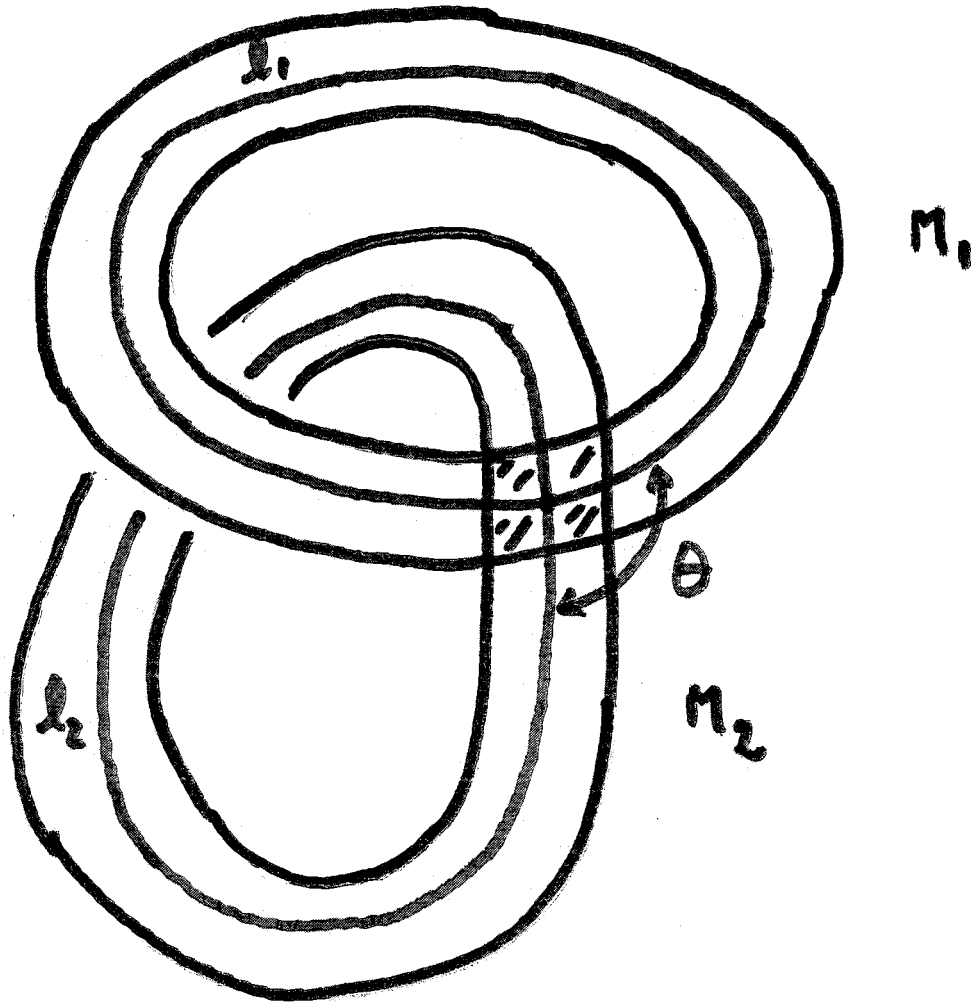
4. Two dimensional example

$M_1 = M_2 =$ hyperbolic annulus

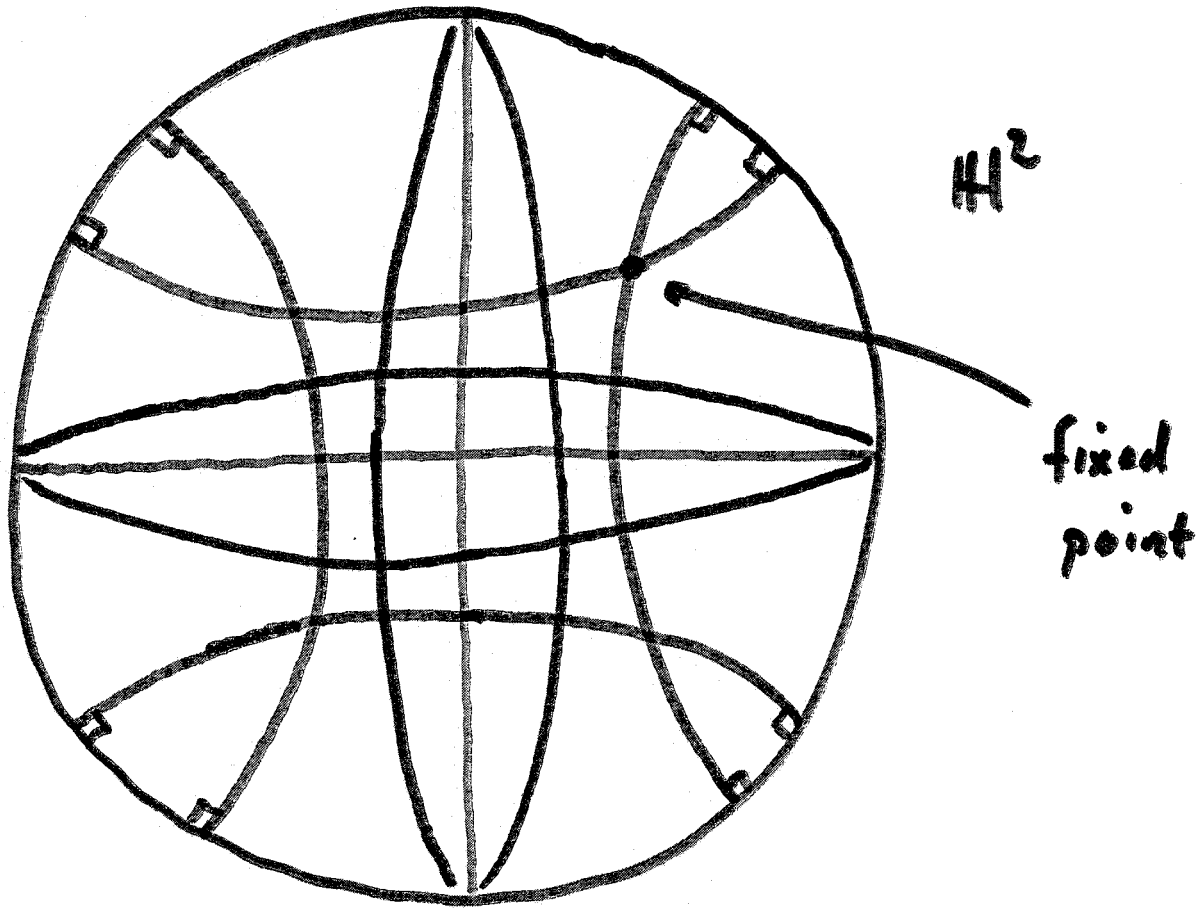


$M = M_1 \cup M_2$

$=$ punctured torus $T(\ell, \theta, \varepsilon)$

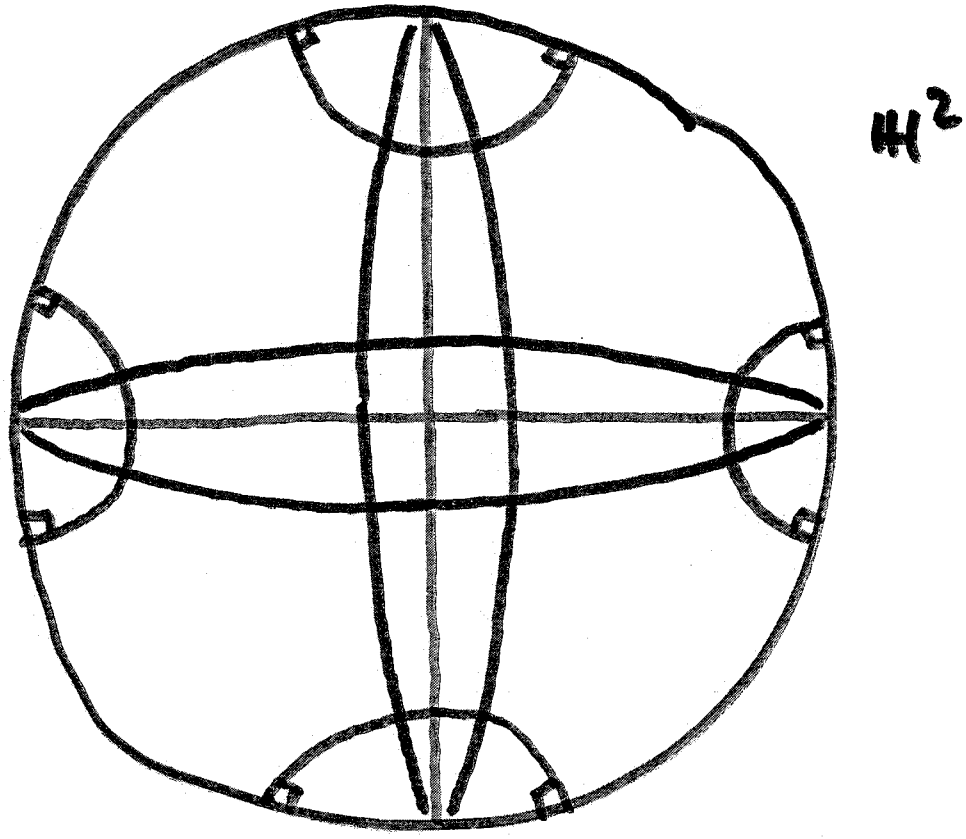


<u>Case 1</u> : $M = M_1 \cup M_2$ has no convex thickening if ℓ is too short



$\mathrm{hol} : \pi_1(M) \to \mathbb{H}^2$ is not discrete and 1-1

<u>Case 2</u> : If ℓ is long enough, then $M = M_1 \cup M_2$ has a convex thickening :



$\mathrm{hol} : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^2)$ is 1-1

$\mathrm{hol}(\pi_1(M))$ is discrete

* <u>NOTE</u> : If ℓ is too short (case 1), we can lengthen ℓ by gluing covers M_1', M_2' to obtain case 2.

5. Convex Combination Theorem

There is a constant K (≥ 6) such that if:

1) $Y = Y_1 \cup Y_2$ is connected hyp. n-manifold where Y_1, Y_2 are convex n-submanifolds

2) $M = M_1 \cup M_2$ is connected hyp. n-manifold where M_1, M_2 are convex n-submanifolds

3) $M \subset Y$ and Y_i is a thickening of M_i ($i=1,2$)

4) $\forall p \in M$, $\exp_p : T_p^K M \to Y$ is defined ($T_p^K M$ = tangent vectors of length $\leq K$)

5) No bumping: For $p \in \overline{M_i} \setminus \mathrm{int}(M_1 \cap M_2)$ we have $\exp_p(T_p^K M) \subset Y_i$ ($i=1,2$)

6) Every component of $Y_1 \cap Y_2$ contains a point of $M_1 \cap M_2$

Then M has a convex thickening. Also if Y finite vol $\Rightarrow$ M geom. finite

6. Corollary : Virtual Amalgam Theorem

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ be a discrete subgp

$A, B \subset \Gamma$ geom. finite subgps

(+ technical condition if parabolics)

Then

$\exists\ A' < A\ ,\ B' < B$ finite index

such that

$$G' \equiv \langle A', B' \rangle = A' \underset{A' \cap B'}{*} B'$$

7. Multiple immersed boundary slopes

M compact 3-manifold, T = torus $\subset \partial M$

Def: A slope α on T is a multiple immersed boundary slope, MIBS, if $\exists$ compact surface S and $f: S \looparrowright M$ such that

i) $f_*: \pi_1(S) \to \pi_1(M)$ is injective

ii) f is not homotopic rel ∂S into ∂M

iii) $\forall \beta \in \partial S$, $f_*([\beta]) = n[\alpha]$

(for $n > 0$)

Theorem: If $\mathrm{int}(M)$ has complete hyp. structure, then every slope on T is a MIBS

Note: False for SFS

Example:

MIBS:

$(0,1)$, $(6,1)$

Idea of Proof:

① Culler - Shalen $\Rightarrow \exists$ Q.F. surfaces

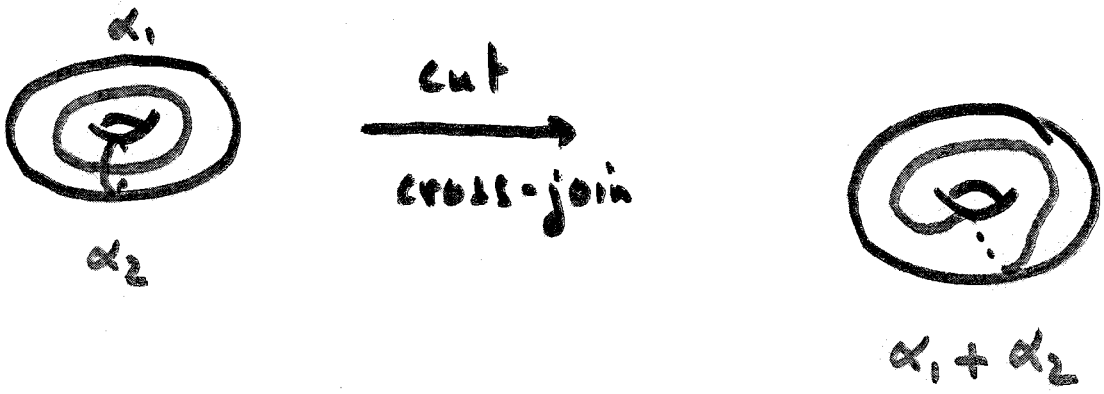
$F_1, F_2 \hookrightarrow M$ s.t.

$\partial F_1 =$ parallel copies of $\pm \alpha_1$

$\partial F_2 =$ parallel copies of $\pm \alpha_2$

$[\alpha_1] \neq \pm[\alpha_2]$ in $H_1(T)$

(2) Intuitively: cut and cross-join copies of F_1, F_2 to get slopes on T:



Problem: $F_1 \cup F_2$ will probably not be π_1-injective

(3) Taking covers $\tilde{F}_1, \tilde{F}_2$ and using convex combination theorem, we obtain convex 3-manifold X and isometric immersion

$$f : X \hookrightarrow M$$

that times desired slopes.

π_1 injectivity follows from

<u>Lemma</u>: $f : \pi_1(X) \to \pi_1(M)$ is 1-1