CYCLIC BRANCHED COVERS OF PRIME KNOTS

joint with M. BOILEAU

ICTP TRIESTE – 23rd JUNE 2005

Nakanishi - Sakuma.

$M \longrightarrow$ n-fold cyclic branched cover of K and K'
$\mathbb{Z}_n \oplus \mathbb{Z}_n$-fold branched cover of L.

$$\begin{tikzcd}[column sep=small] & M \arrow[dl, "n"'] \arrow[dr, "n"] & \\ (S^3, K) \arrow[dr, "n"'] & & (S^3, K') \arrow[dl, "n"] \\ & (S^3, L = \bar{K} \cup \bar{K}') & \end{tikzcd}$$

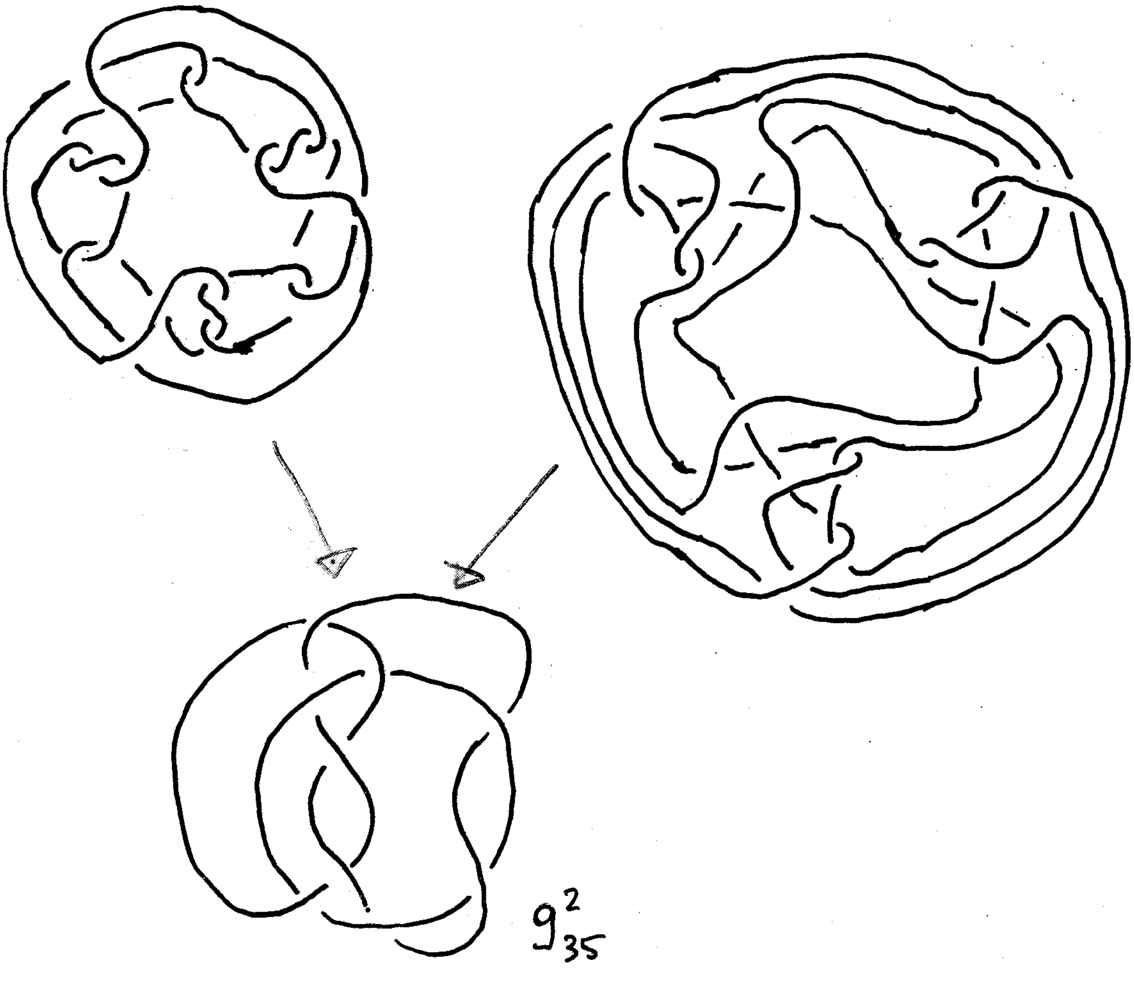
$\bar{K}, \bar{K}'$ trivial
$(lk(\bar{K}, \bar{K}'), n) = 1$

N.B. * $\mathbb{Z}_n \oplus \mathbb{Z}_n = \langle h, h' \rangle$ acts on M

h deck transformation for K
h' deck transformation for K'

* h' (resp. h) induces an <u>n-axial symmetry</u> of K (resp. K') <u>with trivial quotient</u>

e.g.
$n=3$



9^2_{35}

$$\begin{tikzcd}
 & {}^{h}\circlearrowleft M \circlearrowright^{h'} \arrow[dl, "n"'] \arrow[dr, "n"] & \\
(S^3, K) & & (S^3, K')
\end{tikzcd}$$

<u>Theorem</u> (Zimmermann)

If K hyperbolic, $n \geq 3$ then K and K' are related by Nakanishi and Sakuma's standard abelian construction

i.e.

$$\begin{tikzcd}
(S^3, K) \circlearrowright^{\bar{h}'} \arrow[dr, "n"'] & & (S^3, K') \circlearrowright^{\bar{h}} \arrow[dl, "n"] \\
 & (S^3, L = \bar{K} \cup \bar{K}') &
\end{tikzcd}$$

Consequences :

1. Three cyclic branched covers of orders ≥ 3 determine a hyperbolic knot

2. a hyperbolic knot has at most one n-twin if $n \geq 3$.

Definition : K' is an n-twin of K if K and K' have the same n-fold cyclic branched cover.

Proof:

1. K hyperbolic $\Rightarrow$ the symmetries of K <u>commute</u> $(n \geq 3)$

   $+$

   A axis of symmetry for k = trivial knot

   $\Rightarrow$ $A \cup k$ = Hopf link.

2. K hyperbolic, Smith's conjecture $\Rightarrow$

   symmetries of K of order ≥ 3 are <u>unique</u>

N.B. * hyperbolicity is only used to prove

   1. commutativity of symmetries of order

   2. uniqueness of symmetries of order ≥ 3.

* In 1. reason by contradiction and show that K is trivial.

Q: Is the standard abelian situation the only possible one?

<u>Rk</u> Need to require:

* K is prime

  e.g. k non invertible then $K = k \# k$ and $K' = k \# rk$ have the same n-fold cyclic branched cover for all $n \geq 2$.

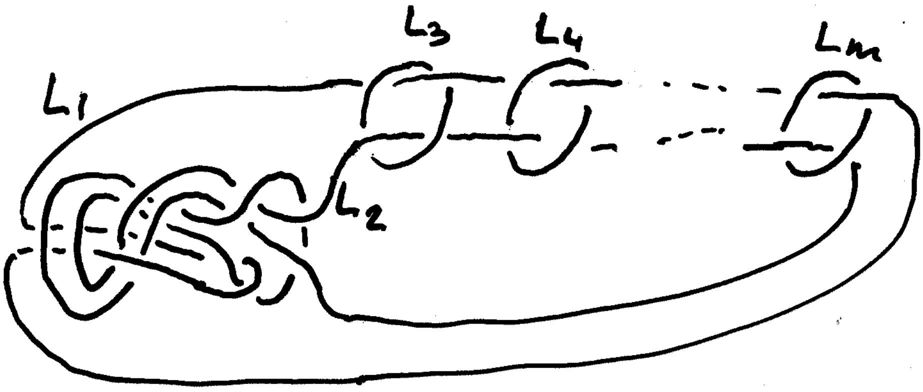
* $n \geq 3$

  e.g. for $n = 2$ one has Conway mutations, ...

New Construction (Boileau – P.)

* $L = L_1 \cup L_2 \cup \dots \cup L_m$ $\quad m \geq 3 \quad L$ hyperbolic
* $L_1 \cup L_k$ and $L_2 \cup L_k$ Hopf for all $k \geq 3$
* $L_3 \cup \dots \cup L_m$ trivial link
* L_1 and L_2 non exchangeable
* $(lk(L_1, L_2), n) = 1$ $\quad$ e.g. $lk(L_1, L_2) = 1$.

e.g.



$$N = (S^3, k) \xrightarrow{n} \left(S^3, L = (L_1 = \bar{K}) \cup (L_2 = \bar{K}') \cup L_3 \dots \cup L_m\right) \xleftarrow{n} (S^3, k') = N'$$

$n \geq 3$

$$(S^3, K) = \left(N \setminus \widetilde{L_3 \cup \dots \cup L_m}\right) \cup \bigcup_{i=3}^{m} E_i \qquad K \text{ image of } k$$

$$(S^3, K') = \left(N' \setminus \widetilde{L_3 \cup \dots \cup L_m}\right) \cup \bigcup_{i=3}^{m} E_i \qquad K' \text{ image of } k'$$

E_i = non trivial knot exterior.

Analysis of the behaviour of deck transformations
⟹ understanding of all possible situations

Theorem 1. (Boileau, P.)

Let K be a prime knot, p an odd prime.
Assume that K' is a p-twin of K. Then
either K and K' arise from the standard abelian construction
or the deck transformation for K' induces a <u>partial axial symmetry</u> of K

i.e. $\exists\, P \subset E(K) = S^3 \setminus \mathcal{U}(K)$, P union of geometric pieces of the JSJ decomposition of E(K)

such that

$*$ $P \supset O_p$: $E(K) \setminus \bigcup_{\substack{T \in JSJ \\ \omega(T) = p \\ T = \text{torus}}} T \supset O_p$ connected component containing $\partial E(K)$

$*$ the deck transformation for K' induces a symmetry ψ of P such that

- $\psi(O_p) = O_p$
- $\mathrm{Fix}(\psi) \subset O_p$, $\mathrm{Fix}(\psi) \neq \emptyset$

Idea of Proof:

Let M be the p-fold cyclic branched cover for K and K'. Let $M = \bigcup_i V_i$ the JSJ-decomposition of M into geometric pieces.

* the dual JSJ-graph is a tree

Let h (resp h') be the deck transformation for K (resp K')

* If $h(V_i) = h'(V_i) = V_i$, then, up to conjugation, $[h|_{V_i}, h'|_{V_i}] = 1$.

* If $\mathrm{Fix}(h) \subset V_i$ then, up to conjugation, $h'(V_i) = V_i$

Lack of commutativity appears in a well-specified case

Theorem 2 (Boileau, P.)

Let K admit three rotational symmetries with pairwise distinct odd prime orders and with trivial quotient.

Then K is the trivial knot.

idea of proof:

Sakuma described all possible axial symmetries for a composite knot

$\Rightarrow$ Composite knots cannot admit symmetries with trivial quotient.

$\Rightarrow$ K must be prime

Theorem (Sakuma)

If K is totally prime (i.e. all its companions are prime) and pedigreed (i.e. no companion has winding # 0) then, up to conjugation, its symmetries of order ≥ 3 commute.

$\Rightarrow$ If K is totally prime and pedigreed then K is trivial

Else: Use Dehn surgery to construct a new non trivial knot, which is totally prime and pedigreed and satisfies the hypotheses of Theorem 2.

Properties of partial symmetries and their behaviour with respect to axial symmetries with trivial quotient.

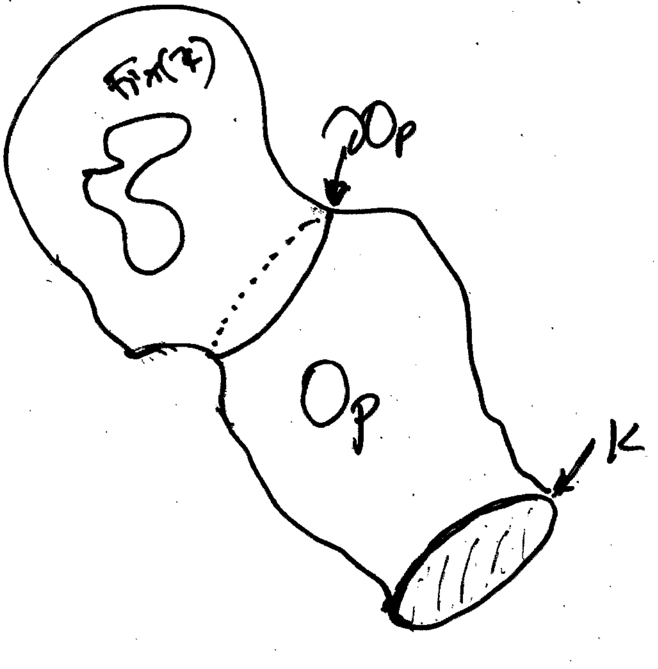
Lemma A (Boileau, P.)

K a prime knot. Then there is at most one odd prime p such that K has a p-twin inducing a partial symmetry.
Moreover, if $q \neq p$ is an odd prime which is the order of an axial symmetry τ of K with trivial quotient then

∂O_p consists of precisely two components (one being $\partial E(K)$)

and $Fix(\tau) \subset E(K) \setminus O_p$.

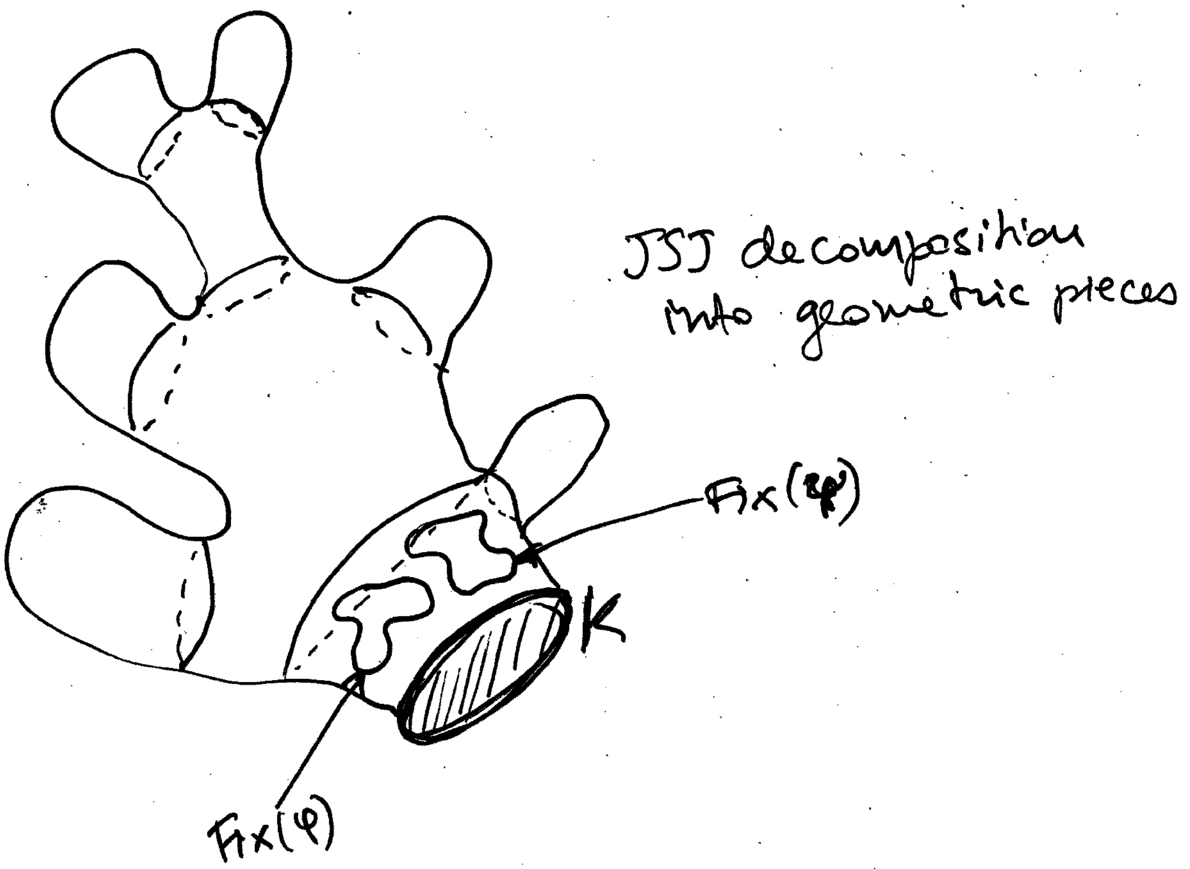
N.B. The situation of figure can indeed happen



Behaviour of axial symmetries with trivial quotien

Lemma B (Boileau, P.)

Let φ and ψ be two axial symmetries of distinct odd prime orders and trivial quotient for a prime knot K.
Then both $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\psi)$ are contained in the geometric component of the JSJ-decomposition for $E(K)$ adjacent to $\partial E(K)$.



Main result: determination of prime knots

Theorem 3 (Boileau, P.)

Let K be a prime knot.

(a) there are at most two odd prime numbers for which K admits a p-twin.

(b) For any given odd prime p, K admits at most one p-twin.

$\Longrightarrow$ three cyclic branched covers of odd prime orders suffice to determine a prime knot.

COMPARE WITH:

Theorem (Kojima)

a prime knot is determined by a cyclic branched cover provided its order is sufficiently large

Proof

(a) By contradiction, assume there are three add primes.

* If the corresponding p-twins all arise from the standard abelian construction then Theorem 2 tells us that K is trivial $\Rightarrow$ it is determined by each of its covers (by Smith's conjecture)

* One can thus assume that one (and, by Lemma A, at most 1) p-twin induces a partial symmetry.

Let ψ and φ be the axial symmetries with trivial quotient induced by the other twins.

- By Lemma A $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\psi)$ are not in the JSJ component containing $\partial E(K)$
- On the other hand, by Lemma B, $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\psi)$ belong to the component containing $\partial E(K)$

Contradiction!

idea of Proof:

(b)

Theorem (Sakuma)

Up to conjugation, the symmetries of a prime knot are unique provided that their order is odd.

$\Rightarrow$ At most one p-twin inducing an axial symmetry of K.

More technical in the other case (follows from the fact that if two deck transformations coincide on a geometric piece of the cover, then they coincide everywhere)