Uncountably many Kleinian groups and mapping class groups

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Main results

A Kleinian group is a discrete torsion-free subgroup of $\text{Iso}^+(\mathbb{H}^3)$.

M is hyperbolic if $M = \mathbb{H}^3/\Gamma$, where Γ is Kleinian.

$\mathcal{MCG}(M)$ is the mapping class group of M.

Theorem 1. *For every countable group G there is a hyperbolic 3-manifold M such that:*

$$G \cong \text{Out}(\pi_1(M)) \cong \text{Iso}(M) \cong \mathcal{MCG}(M).$$

Corollary 2. *There are uncountably many pairwise non-isomorphic Kleinian groups.*

Corollary 3. *Any countable group is the group of outer automorphisms of a countable group.*

Related results

- Kojima ('88): any finite group is the isometry group of a compact hyperbolic 3-manifold.

- Mostow ('68) and Gabai-Meyerhoff-Thurston ('03): if M is compact hyperbolic, then $\mathrm{Out}(\pi_1(M))$, $\mathrm{Iso}(M)$ and $\mathcal{MCG}(M)$ are isomorphic finite groups.

- In dimension 2, there are uncountably many surfaces but countably many fundamental groups.

- Glaser ('65): there exist uncountably many fundamental groups of 3-manifolds.

- Myers ('00): there exist uncountably many indecomposable fundamental groups of 3-manifolds.

- Matumoto ('89): Any group is the group of outer automorphisms of a group.

Today

Theorem. *For every countable group G there is a hyperbolic 3-manifold M such that:*

$$G \times \mathbb{Z}/_2 \cong \mathrm{Out}(\pi_1(M)) \cong \mathrm{Iso}(M) \cong \mathcal{MCG}(M),$$

$$G \cong \mathrm{Iso}^+(M) \cong \mathcal{MCG}^+(M).$$

The strategy

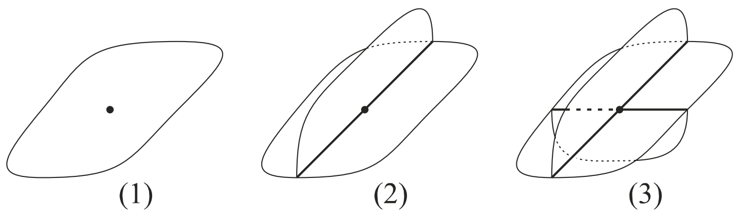
Use the shadow construction (F. Costantino, D. Thurston) to associate a hyperbolic manifold $M(P)$ to any special polyhedron P.

Prove that

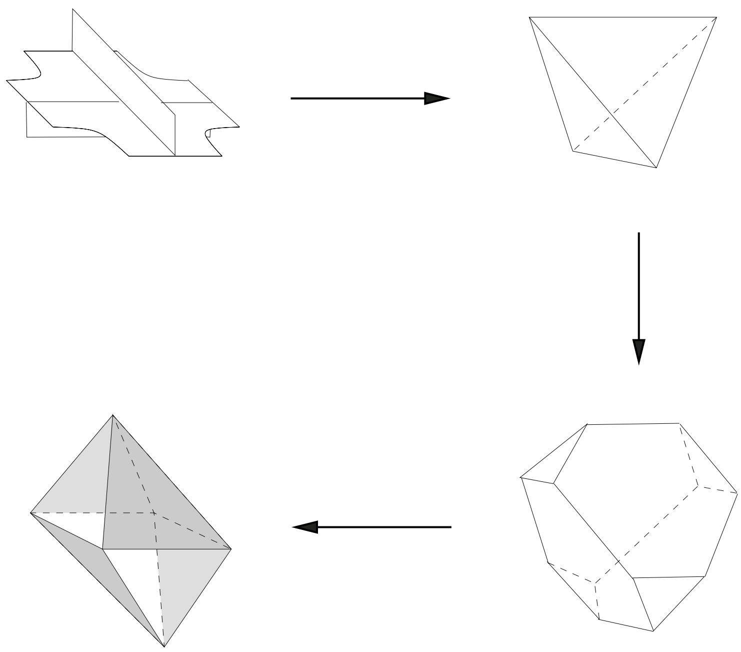
$$\begin{array}{rcl} \mathrm{Aut}(P) & \cong & \mathrm{Iso}(M(P)) \\ \cong\ \mathrm{Out}(\pi_1(M(P))) & \cong & \mathcal{MCG}(M(P)). \end{array}$$

In a joint work with F. Costantino, B. Martelli, and C. Petronio we implemented a similar strategy to give a new proof of Kojima's result.

From special polyhedra to manifolds



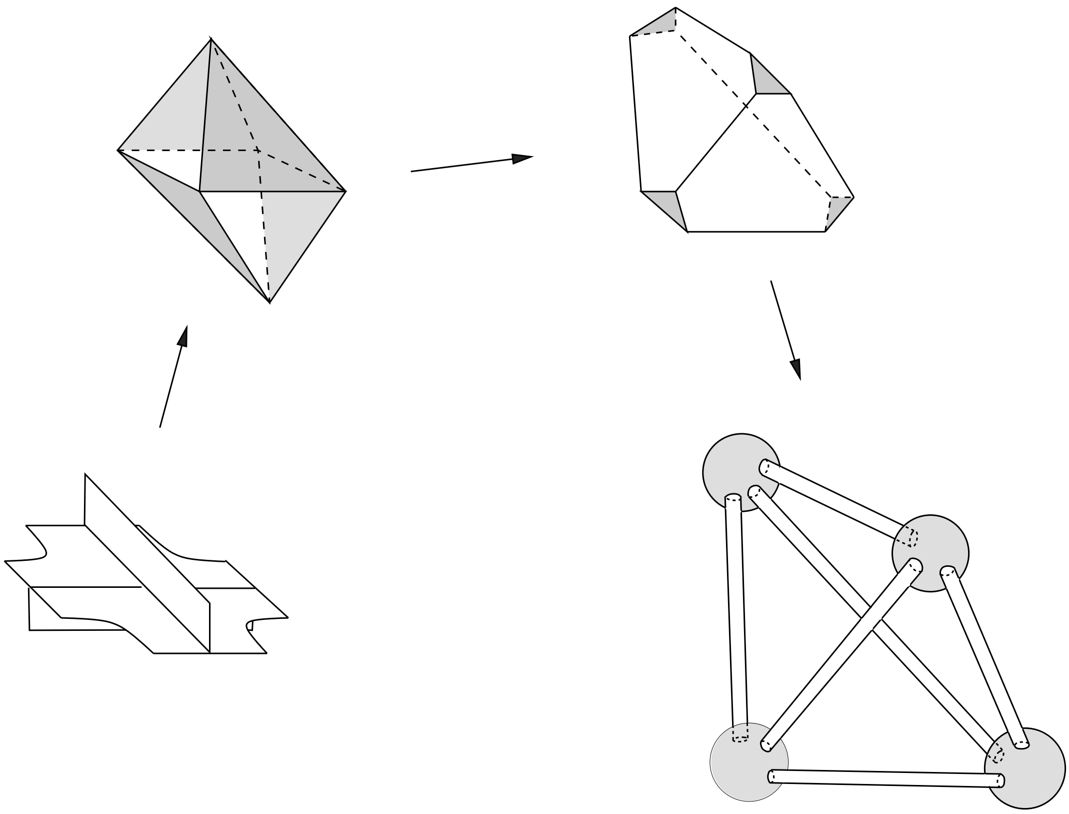
The local structure of a special polyhedron.



Associating to any vertex of a special polyhedron P an ideal octahedron, we get the hyperbolic manifold with geodesic boundary $N(P)$.

Let $\widetilde{N}(P)$ be the orientation cover of N with automorphism τ. We set

$$M(P) = \widetilde{N}(P) \Big/ \left(\tau|_{\partial \widetilde{N}(P)} \right).$$



$$\mathrm{Iso}(K) = \mathrm{Iso}(O) \times \mathbb{Z}/_2 = \mathrm{Iso}^{+}(K) \times \mathbb{Z}/_2.$$

$$
\begin{array}{ccc}
\text{vertices of } P & \longleftrightarrow & \text{blocks of } M(P) \\
\text{edges of } P & \longleftrightarrow & \text{punct. spheres of } M(P) \\
\text{faces of } P & \longleftrightarrow & \text{toric cusps of } M(P)
\end{array}
$$

There are a natural injection

$$\mathrm{Aut}(P) \to \mathrm{Iso}^+(M(P)),$$

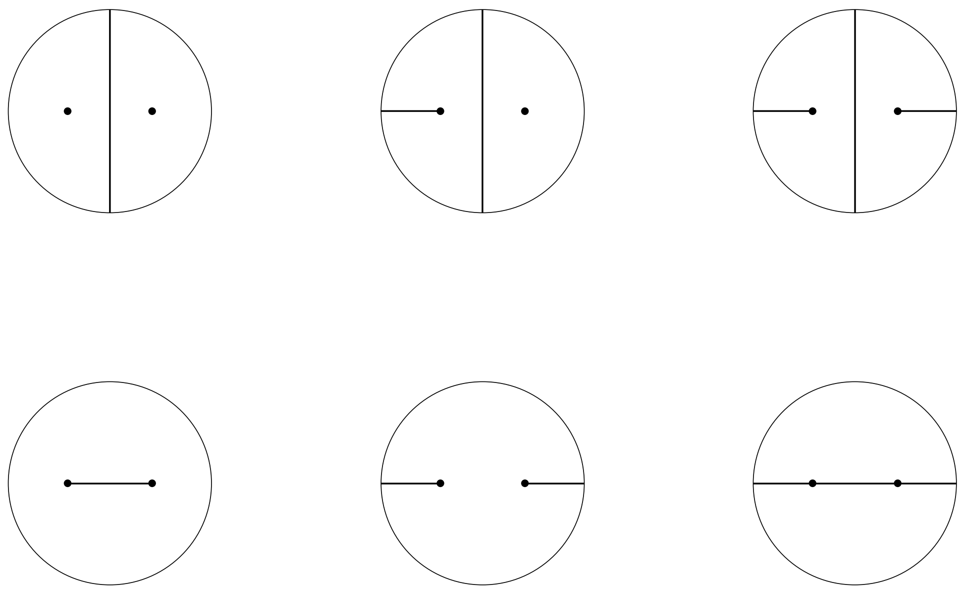
and an orientation-reversing involutive isometry σ commuting with Aut(P). Thus

$$\mathrm{Aut}(P) \times \mathbb{Z}/_2 \ < \ \mathrm{Iso}(M(P)).$$

Proposition. *Suppose every face of P has at least two edges. Then any geodesic 3-punctured sphere in $M(P)$ is the boundary of a block in the decomposition arising from P. Thus,*

$$
\begin{aligned}
\mathrm{Iso}(M(P)) &= \mathrm{Aut}(P) \times \mathbb{Z}/_2, \\
\mathrm{Iso}^+(M(P)) &= \mathrm{Aut}(P).
\end{aligned}
$$

Let $S \subset M(P)$ be a geodesic 3-punctured sphere. Since $K \setminus \partial K$ does not contain 3-punctured spheres, we can suppose that the intersection of S with the boundaries of the blocks is non-empty. This intersection is given by disjoint simple geodesics, so it falls in one of the following cases:



But at every puncture of S there should appear either 0 or $\geqslant 3$ geodesics, a contradiction.

Proposition. *Every countable group is the automorphism group of a special polyhedron.*

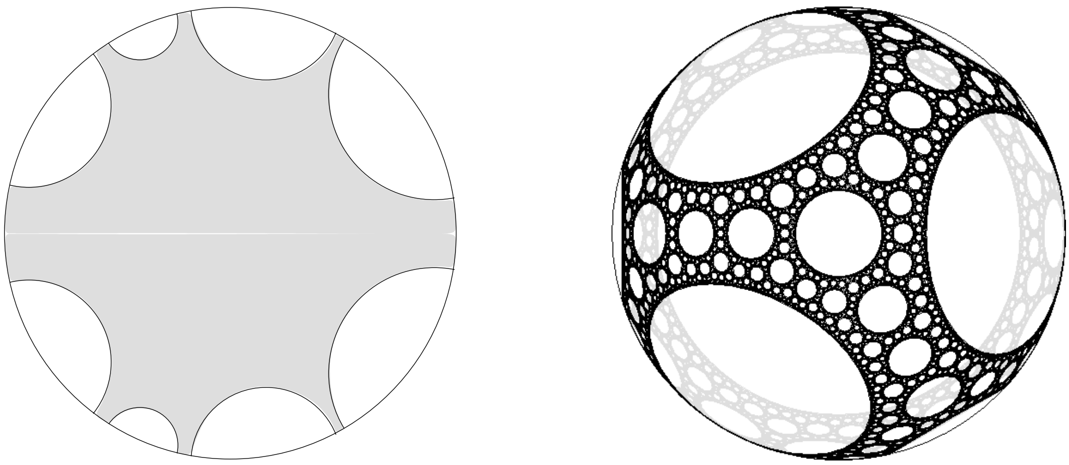
Proof:

- Construct P with $\pi_1(P) = G$, and observe that $G = \mathrm{Deck}(\widetilde{P}) < \mathrm{Aut}(\widetilde{P})$;

- Modify P in order to ensure $\mathrm{Deck}(\widetilde{P}) = \mathrm{Aut}(\widetilde{P})$.

$$
\begin{tikzcd}
\mathrm{Aut}(P) \times \mathbb{Z}/_2 \arrow[d, shift right] & \mathcal{MCG}(M(P)) \arrow[d, shift right] \\
\mathrm{Iso}(M(P)) \arrow[u, dashed, shift right] \arrow[r, shift left] \arrow[d, shift right] & \dfrac{\{\mathrm{Homeo}(M(P))\}}{\text{homotopy}} \arrow[u, dashed, shift right] \arrow[l, dashed, shift left] \\
\mathrm{Out}(\pi_1(M(P))) \arrow[u, dashed, shift right] &
\end{tikzcd}
$$

Let V by complete finite-volume hyperbolic with non-empty geodesic boundary. Then $\widetilde{V}$ is a convex polyhedron in $\mathbb{H}^3$ bounded by a countable number of disjoint geodesic planes.



$$\pi_1(V) \cong \Gamma < \mathrm{Iso}(\widetilde{V}) < \mathrm{Iso}(\mathbb{H}^3).$$

V is the convex core of Γ.

Maximally parabolic groups

If H is a finitely generated Kleinian group, then $\mathbb{H}^3/H$ has a compact core with Euler characteristic $\chi(H)$. The number $b(H) = -3\chi(H)$ only depends on the isomorphism class of H.

Every Kleinian group isomorphic to H contains at most $b(H)$ non-conjugated rank-1 maximal parabolic subgroups. H is *maximally parabolic* if it contains $b(H)$ non-conjugated rank-1 maximal parabolic subgroups.

Theorem (Keen-Maskit-Series). *H is maximally parabolic if and only if it is the π_1 of a finite-volume hyperbolic manifold with geodesic boundary given by 3-punctured spheres.*

Theorem (Keen-Maskit-Series). *If H is maximally parabolic and $\varphi : H \to H'$ is a type-preserving isomorphism, then H' is maximally parabolic and φ is induced by conjugation by an element of* $\mathrm{Iso}(\mathbb{H}^3)$.

Proposition. *Any isomorphism $\varphi : \pi_1(M(P)) \to \pi_1(M(P'))$ is induced by an isometry.*

- φ is type-preserving;
- If $M_n = \bigcup_{i=0}^n K_i$ is connected, $\Gamma_n = \pi_1(M_n)$ is maximally parabolic, so there exists $g_n \in \mathrm{Iso}(\mathbb{H}^3)$ with $\varphi(\gamma) = g_n \gamma g_n^{-1}$ for all $\gamma \in \Gamma_n$;
- Since $g_n g_0^{-1}$ commutes with all the elements in Γ_0, we have $g_n = g_0$ for all n.

Corollary. *We have*

$$\mathrm{Iso}(M) \cong \mathrm{Out}(\pi_1(M)) \cong \frac{\mathrm{Homeo}(M(P))}{\text{homotopy}}.$$

From homotopy to isotopy

A 3-manifold is:

- irreducible if every embedded 2-sphere bounds an embedded 3-disc;

- ∂-irreducible if for any component S of ∂M the natural map $\pi_1(S) \to \pi_1(M)$ is injective;

- end-reducible if there exist a compact set $L \subset M$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of simple loops in $M \setminus L$ with the following properties: any compact subset of M intersects only a finite number of λ_i's, and each λ_i is homotopically trivial in M and homotopically non-trivial in $M \setminus L$.

- end-irreducible if it is not end-reducible.

Suppose M is compact with non-empty boundary. Then int(M) is end-irreducible if and only if M is ∂-irreducible.

Theorem (Waldhausen). *Suppose M is compact with non-empty boundary, irreducible and ∂-irreducible, and not an I-bundle over a closed surface. If $f, f' : M \to M$ are homotopic homeomorphisms, then f and f' are isotopic.*

Theorem (M.Brown). *Suppose M is non-compact without boundary, irreducible, and end-irreducible. Also suppose that M is not an $\mathbb{R}$-bundle over a closed surface, and let $f, f' : M \to M$ be homotopic homeomorphisms. Then f and f' are isotopic.*

Proposition. *Each $M(P)$ is irreducible and end-irreducible.*

Suppose $M(P)$ is end-reducible, and let L and λ_i be as above. We can suppose $L \subset \cup_{i=0}^{n} K_i = L'$. Up to passing to a subsequence, one of the following conditions hold:

1. either the λ_i's lie outside L',

2. or the λ_i's tend to a toric C cusp touched by L'.

Being non-trivial in $M(P) \setminus L$, the λ_i's are non-trivial either in $M(P) \setminus L'$, or in C. Since the boundary of L' is incompressible in $M(P)$, in both cases the λ_i's should be non-trivial in $M(P)$, a contradiction.

How to get rid of the $\mathbb{Z}/_2$-factor

Just put between any pair of glued 3-punctured spheres (*i.e.* in correspondence with any edge of P) a block K' with the following properties:

- K' is hyperbolic with geodesic boundary given by two 3-punctured spheres with punctures p_1, p_2, p_3, q_1, q_2, q_3.

- There exists an isometry $\varepsilon : K' \to K'$ with $\varepsilon(p_i) = q_i$, $i = 1, 2, 3$.

- The symmetric group on 3 elements $\mathfrak{S}_3$ acts isometrically on K' in such a way that $\sigma(p_i) = p_{\sigma(i)}$, $\sigma(q_i) = q_{\sigma(i)}$, $\sigma \in \mathfrak{S}_3$, $i = 1, 2, 3$.

- K' admits no orientation-reversing isometries.