Computation of Hyperbolic Structures on 3-Manifolds and 3-Orbifolds

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with Damian Heard

Outline

1. Description of **Orb**: a new computer program by Damian Heard for computing and studying hyperbolic structures on very general 3-manifolds and 3-orbifolds.

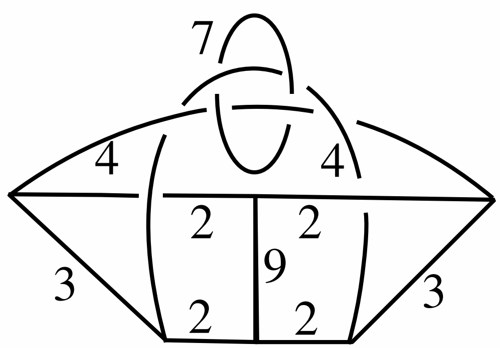
2. Some applications to classification of knotted graphs and low volume hyperbolic 3-orbifolds.

3. A brief demonstration of **Orb**.

Orbifolds:

A **3-orbifold** is a space locally modelled on $\mathbb{R}^3$ modulo finite groups of diffeomorphisms.

An orientable 3-orbifold is determined by its *underlying space* Q which is an orientable 3-manifold and *singular locus* Σ which is a trivalent graph (possibly disconnected or empty) with each edge or circle labelled by an integer $n \geq 2$. For example:



A **hyperbolic structure** on such an orbifold is a singular hyperbolic metric with **cone angles** $2\pi/n$ along each edge labelled n.

At a trivalent vertex we allow:

angle sum $> 2\pi$ giving a finite vertex,

angle sum $= 2\pi$ giving a cusp,

angle sum $< 2\pi$ giving a totally geodesic boundary component.

Basic method for computing hyperbolic structures

• Decompose the manifold or orbifold into tetrahedra.

• Find geometric shapes for tetrahedra in hyperbolic space (dihedral angles, edge lengths) so that:

1. faces are glued by isometries
2. sum of dihedral angles around each edge is 2π (or the desired cone angle).

(Also need **completeness** conditions if the space is non-compact.)

Generalized hyperbolic tetrahedra

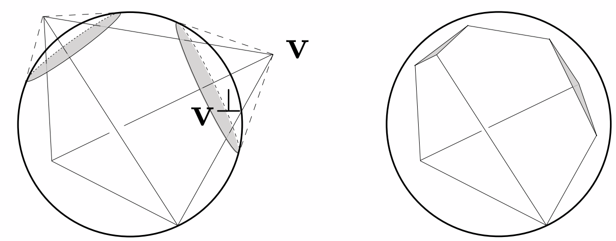
In hyperbolic geometry can use tetrahedra with

• finite vertices (inside $\mathbb{H}^3$),

• ideal vertices (on the sphere at infinity), or

• hyperinfinite vertices (beyond the sphere at infinity)!

This is easiest to see in the projective model for $\mathbb{H}^3$:



Hyperinfinite vertices are truncated as shown. Interiors of edges must meet $\mathbb{H}^3$.

Some existing programs

SnapPea by Jeff Weeks

Uses **ideal triangulations** to find hyperbolic structures on **cusped** hyperbolic 3-manifolds (finite volume, non-compact) and closed manifolds obtained from these by Dehn filling.

Can start by drawing a projection of a knot or link, and find hyperbolic structures on the link complement and on manifolds obtained by Dehn surgery.

(See preprint of Weeks: math.GT/0309407)

Geo by Andrew Casson

Uses **finite triangulations** to find hyperbolic and spherical structures on **closed** manifolds.

ographs by B. Martelli, R. Frigerio, C.Petronio

Finds hyperbolic structures with **totally geodesic boundary** using triangulations by truncated tetrahedra.

Orb by Damian Heard

Uses **generalized hyperbolic tetrahedra** with finite, ideal and hyperinfinite vertices. (Can pass continuously between these and allow flat and negatively oriented tetrahedra.)

Can deal with **orbifolds** and **cone-manifolds** where the cone angle around an edge is not necessarily 2π.

Can start with a projection of a graph in S^3 and try to find hyperbolic structures with prescribed cone angles around all the edges

How Orb works

Suppose we have an orbifold in S^3 whose singular locus is a graph Σ with integer labels on the edges. (For this talk, I'll generally assume all vertices are finite.)

Step 1. Finding triangulations

Given a projection of Σ, find a triangulation of S^3 with Σ contained in the 1-skeleton by extending the approach of W. Thurston and J. Weeks. Can also retriangulate to change and simplify the triangulation, using 2-3 and 3-2 moves etc.

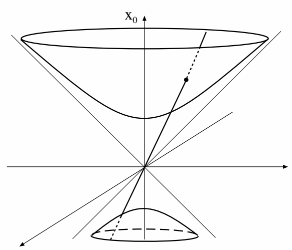
Step 2. Finding hyperbolic structures

We work in Minkowski space $\mathbb{E}^{3,1}$, i.e. $\mathbb{R}^4$ with the indefinite inner product

$$\langle x, y\rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_3 y_3$$

and use the **hyperboloid model**:

$$\mathbb{H}^3 = \{x \in \mathbb{E}^{3,1} | \langle x, x\rangle = -1, x_0 > 0\}.$$

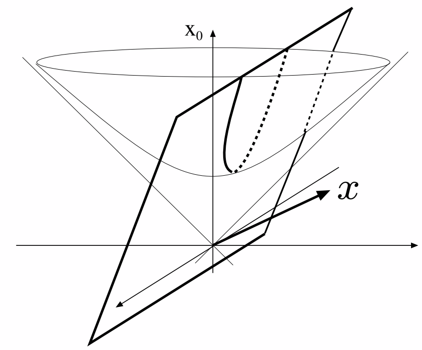


The distance d between $x, y \in \mathbb{H}^3$ is given by

$$\langle x, y\rangle = -\cosh d.$$

Each point $x \in \mathbb{E}^{3,1}$ with $\langle x, x\rangle > 0$ represents a **normal** to the **geodesic plane** in $\mathbb{H}^3$

$$\Pi_x = \{w \in \mathbb{H}^3 | \langle x, w\rangle = 0\}.$$

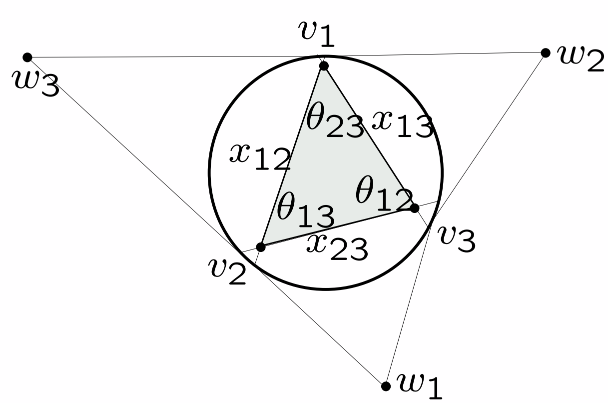


If Π_x intersects Π_y inside $\mathbb{H}^3$ then the angle θ_{xy} between the two planes is given by

$$\cos\theta_{xy} = \frac{\langle x, y\rangle}{\|x\|\|y\|}.$$

Let $\triangle$ be a generalized tetrahedron with vertices v_1, v_2, v_3, v_4 in $\mathbb{E}^{3,1}$. This gives a basis for $\mathbb{E}^{3,1}$ with dual basis w_1, w_2, w_3, w_4 such that $\langle v_i, w_j \rangle = \delta_{ij}$. Geometrically, w_i represents a choice of normal to the face of $\triangle$ opposite vertex v_i.

2-dimensional picture in projective model:



Let G be the **vertex Gram matrix** of $\triangle$:

$$G = [\langle v_i, v_j \rangle] = [v_{ij}].$$

Then $\triangle$ is determined (up to isometry) by G since the length x_{ij} of edge ij is

$$\cosh x_{ij} = -\frac{v_{ij}}{\sqrt{v_{ii}v_{jj}}}.$$

Let G^* be the **normal Gram matrix** of $\triangle$:

$$G^* = [\langle w_i, w_j \rangle] = [w_{ij}].$$

Then the dihedral angle of edge ij is

$$\cos \theta_{ij} = \frac{w_{ij}}{\sqrt{w_{ii}w_{jj}}}.$$

Further it is easy to check that $G^* = G^{-1}$.

Parameters and equations

Given a triangulation of a 3-orbifold we have

- one parameter v_{ij} per edge,
- one parameter v_{ii} per vertex.

From these we can calculate the dihedral angles of each tetrahedron. Moreover, faces paired by gluing maps will be automatically isometric.

So the only equations we have to satisfy are the **edge equations**, i.e.

- the sum of dihedral angles around each edge is the desired cone angle.

These can be solved using Newton's method, starting with suitable regular generalized tetrahedra as the initial guess.

By Mostow-Prasad rigidity the hyperbolic structure on the 3-orbifold is unique if it exists. Hence geometric invariants are actually *topological invariants*.

Using Orb we can find: volume (using formulas of A. Ushijima), matrix generators, Dirichlet domains, lengths of closed geodesics, presentations of π_1, homology groups, covering spaces, ...

For hyperbolic manifolds with geodesic boundary we can also compute the **canonical cell decomposition** (defined by Kojima). This allows us to decide if such manifolds are homeomorphic and compute their symmetry groups.

Features to be added:

• Dehn filling

• addition of 2-handles or handlebodies to manifolds with boundary of genus ≥ 2,

• computation of spherical structures.

By combining Orb with the program Snap (developed by Oliver Goodman) we will also be able to find

• exact solutions,

• arithmetic invariants.

Application 1: Enumeration and classification of knotted graphs in S^3

(Hodgson, Heard; J. Saunderson, N. Sheridan, M. Chiodo)

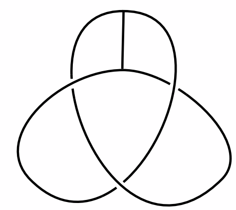
Much work in knot theory has been motivated by attempts to build up knot tables (e.g. Tait, Conway, Hoste-Thistlethwaite-Weeks). A very natural generalization is to study **knotted graphs** in S^3, say up to isotopy. There has been much less work on the tabulation of knotted graphs. In 1989, Rick Litherland produced a table of 90 prime knotted **theta curves** up to 7 crossings, using an Alexander polynomial invariant to distinguish graphs.

H. Moriuchi has recently verified these tables by using Conway's approach and the Yamada polynomial invariant.

We have shown that these knotted graphs can be distinguished by hyperbolic invariants computed using Orb. In fact there is a complete invariant: We compute the hyperbolic structure with geodesic boundary consisting of 3-punctured spheres, such that all meridian curves are parabolic. (This is a limit of hyperbolic orbifolds where all labels $\to \infty$, i.e. all cone angles $\to 0$). Kojima's canonical decomposition then determines the graph completely. This also allows us to determine the symmetry group of all these graphs.

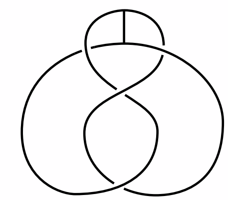
Start of Litherland's table of θ graphs

For each graph we give volume of hyperbolic structure with meridians parabolic, symmetry group, reversibility.



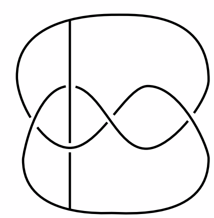
5.333489566898

C_4 r



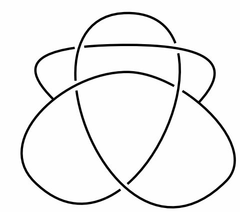
7.706911802810

C_4 r



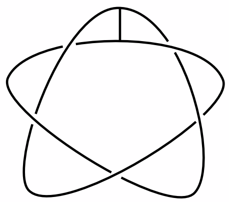
10.396867320885

D_3 n



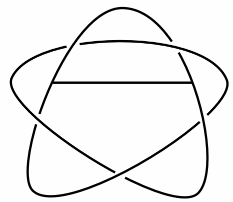
8.929317823097

C_4 r



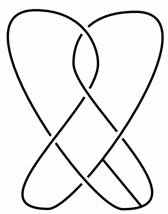
6.551743287888

C_4 r



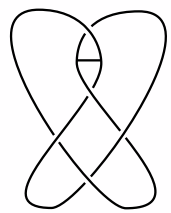
8.355502146380

C_2 r



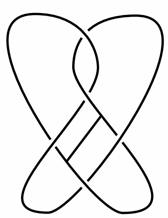
8.967360848788

C_4 r



8.793345603865

C_4 r

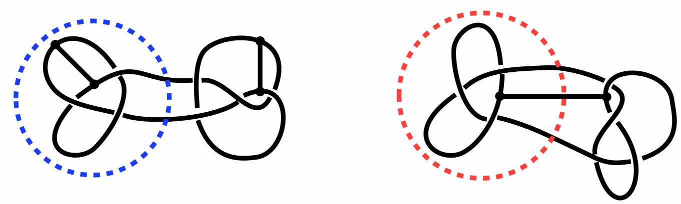


9.966511883698

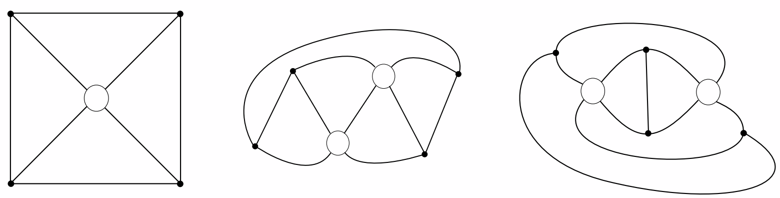
C_2 r

Building up tables of knotted graphs

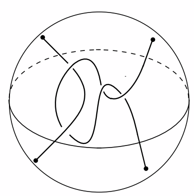
We have also extended these tables to enumerate and classify all prime knotted trivalent graphs in S^3 with 2 or 4 trivalent vertices, and up to 7 crossings. Here **prime** means there is no 2-sphere meeting the graph in at most 3 points dividing the graph into non-trivial pieces.



Our method is based on Conway's approach: First we enumerate **basic prime polyhedra** with vertices of degree 3 and 4, using the program **plantri** of B. McKay and G. Brinkmann.



Then replace degree 4 vertices by **algebraic tangles** to obtain projections of knotted graphs.

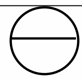
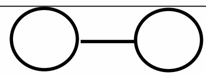
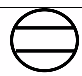
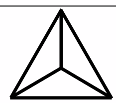
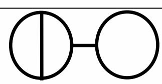
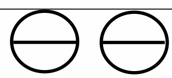
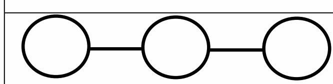
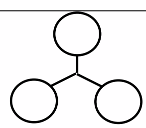
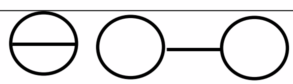


We then remove repeated projections by finding a canonical description for each one using the ideas behind plantri.

Finally we distinguish the graphs using hyperbolic invariants computed using Orb, e.g. volumes of associated orbifolds and Kojima's canonical decomposition.

The following table summarizes the knotted graphs produced.

Prime trivalent graphs:
up to 4 vertices and 7 crossings

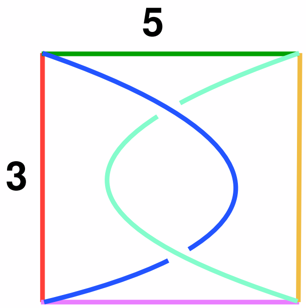
basic graph	no. of circle components 0	1	2
	90	50	4
	48	9	0
	810	143	3
	554	121	3
	529	29	0
	60	3	0
	57	0	0
	8	0	0
	8	0	0

Application 2: enumeration of low volume hyperbolic 3-orbifolds

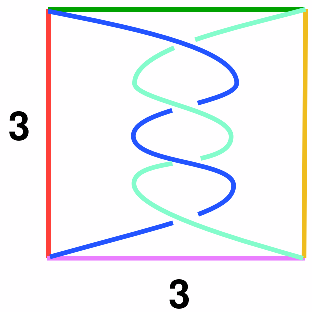
By varying the labels on the knotted graphs obtained above we can start generating hyperbolic orbifolds with underlying space S^3. This work is just beginning; currently we are looking at orbifolds with connected graphs as singular locus.

The following table shows a few of the lowest volume orbifolds. The first 14 orbifolds on our list were already known (and included in a paper of Zimmermann). After that some new low volume hyperbolic 3-orbifolds start to appear.

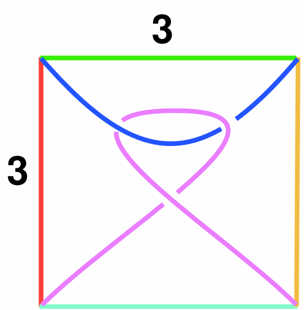
Some low volume hyperbolic 3-orbifolds



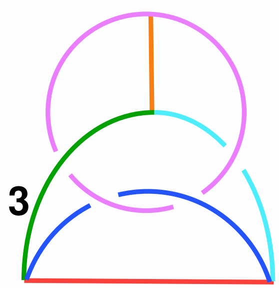
Vol: 0.03905



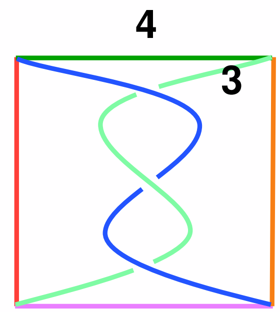
Vol: 0.04089



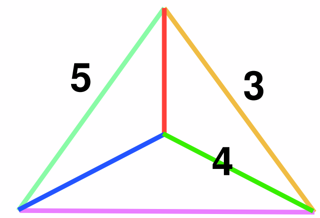
Vol: 0.05265



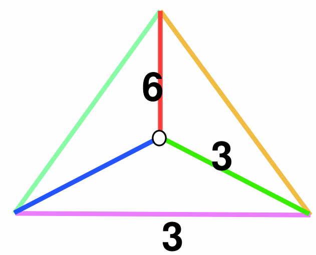
Vol: 0.065965



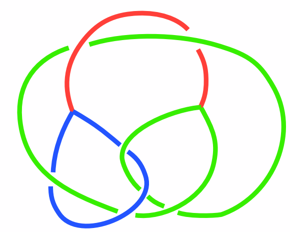
Vol: 0.06619



Vol: 0.071770



Vol: 0.0845785
smallest cusped
orbifold



Vol: 0.117838
smallest 2 vertex
orbifold

(All edges labelled 2 except where otherwise indicated.)

Availability of Orb

Orb uses Qt for its user interface and should run on any unix system. We've been using it on Macs running OS X and linux machines.

Orb should be available for distribution by the end of July.

I will add a link to Orb on my webpage: www.ms.unimelb.edu.au/$\sim$cdh