The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1671

Advanced School and Workshop on Singularities in Geometry and Topology

(15 August - 3 September 2005)

Abstracts

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Advanced School and Workshop on Singularities in Geometry and Topology

	9:00-10:00		10:15-11:15		11:30-12:00		14:00-15:00		15:15-16:15		16:30-17:00
Monday 29 August	**David Massey** The Perverse Structure of the Vanishing Cycles and Milnor Equisingularity	BREAK	**James Damon** A resolution for the geometric complexity of regions in R3 and its topological consequences	BREAK	**Ana Claudia Nabarro** Index of a vector field in R2 and the pencil's geometry of its principal part	LUNCH	**Helmut Hamm** Lefschetz theorem for the Picard group of singular quasiprojective varieties	BREAK	**Fouad El Zein** Lefschetz Theorems and Hodge theory	BREAK	**Michael Lönne** Braid Monodromy and Hypersurface singularities
Tuesday 30 August	**Maria Ruas** Topological triviality of families of singular surfaces		**Marcelo Saia** Whitney equisingularity of map germs, polar multiplicities and Euler obstruction of stable types		**Raimundo Nonato Araujo dos Santos** Real Milnor fibration and C-regularity		**Sabir Gusein-Zade** Chern obstructions for collections of 1-forms on singular varieties		**Jörg Schürmann** Indices for 1-forms on singular spaces and characteristic cycles		**Rogerio Mol** Polar classes associated to singular holomorphic foliations
Wednesday 31 August	**Terry Gaffney** Counting singularities, multiplicities, and pairs of modules		**Françoise Michel** The boundary of the Milnor Fiber for germs of surfaces in C3 with non isolated singular locus		**Victor Hugo Jorge Perez** Multigraded rings and Buchsbaum-Rim multiplicities		**Farid Tari** Bifurcations in implicit differential equations		**David Trotman** To be confirmed		**E. Rosales-Gonzalez** Rigidity of Germs of Holomorphic Foliations with Dicritic Singularity
Thursday 1st September	**Claude Weber** Four physicists at the origin of Knot theory in 19th. Century		**Victor Vladimirovich Goryunov** Logarithmic vector fields for discriminants of composite functions		**Meral Tosun** Simple elliptic singularity and the corresponding Lie algebra		**Susumu Tanabe** On Horn-Kapranov uniformisation of the discriminantal loci		**Mutsuo Oka** Tangential Alexander Polynomial of plane curves		**Pho Duc Tai** Plane curve Singularities: from local to global
Friday 2nd September	**Aleksey Alexandrovich Davydov** To be confirmed		**XX** TBA		**Martijn van Manen** Weighted Voronoi diagrams, discrete Morse theory and tropical geometry		**Hans Brodersen** Sufficienty of jets with line singularities		**Bernard Teissier** Valuations		**Closure ceremony**

A Resolution for the Geometric Complexity of Regions in $\mathbf{R}^3$ and its Topological Consequences

James Damon

Abstract

For a compact region $\Omega \subset \mathbf{R}^3$ with smooth generic boundary, we introduce a "resolution type" structure which is defined using the "medial geometry" of Ω. We give a structure theorem for such a structure in terms of a two level graph.

The first stage of the graph represents a decomposition into irreducible components, and the second gives the structure of the components. We derive the topology of the region from the data of the structure and characterize properties of the region in terms of the structure.

Counting Singularities, Multiplicities and Pairs of Modules

Terry Gaffney

Abstract

Using invariants from commutative algebra to count geometric objects is a basic idea in singularities. If $f : \mathbf{C^n}, 0 \to \mathbf{C}, 0$ has an isolated singular point at the origin, then the dimension as a $\mathbf{C}$ vector space of $\mathcal{O}_n/J(f)$ where $J(f)$ is the Jacobian ideal is a good example. This number has been proved to be the Milnor number of the singularity and counts the number of vanishing cycles in the Milnor fiber as well as the number of Morse points hiding at the origin, revealed in a "Morsification".

¿From commutative algebra we recognize the Milnor number as the multiplicity of the Jacobian ideal. This suggests that the multiplicity of ideals and modules should be useful in counting singularities; for examples of applications of the multiplicity of ideals, see [T1], [T2], for modules see [G2], [G-K].

A problem with the multiplicity is that it is only defined for modules and ideals of finite colength, that is, as in the case of the Milnor number and the Jacobian ideal, the vector space obtained by modding out must be finite dimensional. This is not the case for many situations of interest.

The solution to this problem is to use pairs of modules, (M, N), where $M \subset N \subset \mathcal{O}^p_{X,x}$; here M has finite colength in N, N is the "least complicated" module in $\mathcal{O}^p_{X,x}$ consistent with the geometry of X that makes the colength of M infinite. The difference between M and N, which is measured by the multiplicity of the pair, should reflect only the contribution of the geometry at the point x. The multiplicity of a pair, and the basic properties of this multiplicity are developed in [K-T].

The theory of the integral closure of modules plays a decisive role in finding N given M. An introduction to this theory appears in [G-1]. An extension of the construction we use of the pair in this talk appears in [G-3]. This paper indicates how the ideas of this talk could be applied to problems with many strata.

In using invariants to count singularities, it is very helpful to have some control on how the invariants change in a family. The key technical tool is the Multiplicity-Polar Theorem, ([G-4], [G-5]) which we will describe in the talk. This theorem describes the behavior of the multiplicity of a pair of modules in

a family, linking behavior at a general member of the family to behavior at a special member.

We will apply these ideas to the following set-up:

Suppose $X, 0$ is an equidimensional complex analytic set in $\mathbf{C^n}$, and $T^*_X(\mathbf{C^n})$ is the closure of the conormal vectors to the smooth part of X, s a section of $T^*\mathbf{C^n}$, and the intersection of the image of s, denoted $im(s)$, with $T^*_X(\mathbf{C^n})$ is isolated. Then we will calculate the intersection multiplicity $im(s) \cdot T^*_X(\mathbf{C^n})$. It is easy to see that if $X = \mathbf{C^n}$ and $s(x) = df(x)$, where f has an isolated singularity at the origin, then this number is the Milnor number of f.

Variations of this number appear in several different situations, such as:

the theory of differential forms ([E-GS])

the theory of D-modules ([L])

the description of the Milnor fiber of a function with an isolated singularity ([M])

the theory of the Euler invariant, which in turn relates to the theory of characteristic classes on singular spaces ([BMPS]).

References

[BMPS] J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade, *Euler Obstruction and Defects of Functions on Singular Spaces* preprint, 2003.

[E-GZ-S] W. Ebeling, S. M. Gusein-Zade, J. Seade: Homological index for 1-forms and a Milnor number for isolated singularities. Preprint math.AG/0307239.

[G-1] T. Gaffney, *Integral closure of modules and Whitney equisingularity,* Inventiones, 107 1992 301–22.

[G-2] T. Gaffney, *Multiplicities and equisingularity of ICIS germs,* Inventiones, 123 1996 209–220.

[G-3] T. Gaffney, *Generalized Buchsbaum-Rim Multiplicities and a Theorem of Rees,* Communications in Algebra, vol 31 #8 p3811-3828, 2003.

[G-4] T. Gaffney, *Polar methods, invariants of pairs of modules and equisingularity,* Real and Complex Singularities (Sao Carlos, 2002), Ed. T.Gaffney and M.Ruas, Contemp. Math.,#354, Amer. Math. Soc., Providence, RI, June 2004, 113-136.

[G-5] T.Gaffney, *The multiplicity of pairs of modules and hypersurface singularities*, submitted to Real and Complex Singularities, (Sao Carlos-Luminy 2004).

[G-K] T. Gaffney and S. Kleiman, *Specialization of integral dependence for modules*, Inventiones, 137 1999 541-574.

[K-T] S. Kleiman and A. Thorup, *A geometric theory of the Buchsbaum–Rim multiplicity,* Journal of Algebra, 167 1994 168–231.

[L] D. T. Lê, *Morsification of D modules*, Bol. Soc. Mat. Mexicana (3) 4 (1998), no. 2, 229–248.

[M] D.Massey *Hypercohomology of Milnor Fibers*, Topology, v 35, #4,p969-1003,(1996).

[T1] B. Teissier, *Cycles évanescents sections planes et conditions de Whitney,* in "Singularités á Cargèse," Astérisque **7–8**, 1973.

[T2] B. Teissier, *Multiplicités polaires, sections planes, et conditions de Whitney,* in "Proc. La Rábida, 1981." J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle (eds.), Springer lecture notes, 961 1982 314–491.

Logarithmic vector fields for discriminants of composite functions

Victor Goryunov

Abstract

I will be considering composite functions following the approach of our works with Zakalyukin and Mond. The attention with be restricted to the $\mu = \tau$ situation. I will show that the discriminant of a composite function in this case is Saito's free divisor. An explicit algorithm to construct basic logarithmic vector fields will be formulated. 2-parameter families of symmetric matrices will be used as a main illustration.

Chern obstructions for collections of 1-forms on singular varieties.

S.M.Gusein-Zade

Abstract

We introduce a certain index of a collection of germs of 1-forms on a germ of a singular variety which is a generalization of the local Euler obstruction corresponding to Chern numbers different from the top one. This permits to give a definition of Chern numbers of singular varieties. The talk describes a joint work with W.Ebeling.

Lefschetz theorem for the Picard group of singular quasiprojective varieties

Helmut Hamm

Münster University

Let X be a non-singular complex projective variety of dimension n and $Y = X \cap H$ a non-singular hyperplane section. The Lefschetz theorem on hyperplane sections tells that $H^k(X;\mathbf{Z}) \simeq \mathbf{H^k(Y;Z)}$ for $k < n-1$. Let $Pic\, X = H^1(X, \mathcal{O}_X^*)$ be the algebraic Picard group of X which coincides with the analytic one in our case. Using Kodairas vanishing theorem and the exponential sequence we get $Pic\, X \simeq Pic\, Y$ for $n \geq 4$.
The aim of joint work with Lê Dũng Tráng is to pass to the algebraic Picard group of $X \setminus Z$, Z being a Zariski-closed subset, admitting singularities. Here depth conditions come in, for simplicity let us restrict to local complete intersections. We get
Theorem: Suppose that the hyperplane H is transversal to some Whitney stratification of X which is compatible with $Sing\, X$ and Z and that $X \setminus Z$ is locally a complete intersection of dimension ≥ 4, $codim_{X\setminus Z} Sing(X \setminus Z) \geq 3$,$codim_X Z \geq 4$. Then $Pic\, X \simeq Pic\, Y$.
We have two different proofs: one uses a corresponding theorem in the case where $X \setminus Z$ is smooth, the other uses the exponential sequence and a generalized Kodaira vanishing theorem.
In fact the first proof allows to drop the assumption on the codimension of Z while the second one requires only transversality to Z.
Note that the case $Z = \emptyset$ has already been treated algebraically by A.Grothendieck in SGA2. It should be possible to prove the theorem above using his methods, too, replacing the transversality condition by the conditions that $codim_{Y\setminus Z} Sing(Y \setminus Z) \geq 3$, $codim_Y Z \cap H \geq 4$.

A RESOLUTION OF SINGULARITIES OF A TORIC VARIETY AND A NON-DEGENERATE HYPERSURFACE SINGULARITIES

SHIHOKO ISHII

In this lectures the following items will be introduced.

(1) resolutions of the singularities on a toric variety (higher dimensional case)
(2) the canonical divisor on a variety
(3) canonical singularities, log-canonical singularities, terminal singularities, log-terminal singularities
(4) the canonical divisor on a toric variety
(5) non-degenerate hypersurface and Newton polygon
(6) the singularities on a non-degenerate hypersurface

REFERENCES

1. V. I. Danilov *The geometry of toric varieties*, Russian Math. Surveys **33**, (1978) 97–154.
2. W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies**131**, Princeton University Press, (1993).
3. S. Ishii and M. Tomari, *Hypersurface non-rational singularities which look canonical from their Newton boundaries*, Math. Zeitschrift **237**, (2001) 125–147.
4. T. Oda, *Convex bodies and algebraic geometry,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, (1988)
5. A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Inventiones math. **37**, (1976), 253–262.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA, MEGURO, TOKYO, JAPAN
E-MAIL : SHIHOKO@MATH.TITECH.AC.JP

π_1 of discriminant complements

Michael Lönne

Abstract

In the case of Brieskorn Pham polynomials

$$f = x_1^{l_1+1} + \cdots + x_n^{l_n+1}$$

there is a natural presentation of the π_1 of the discriminant complement ot its semi-universal unfolding in terms of its Coxeter Dynkin diagram.

I will present my results and give some flavour of the proof.

The Perverse Structure of the Vanishing Cycles and Milnor Equisingularity

David Massey

Northeastern University

Suppose that f defines a singular, complex affine hypersurface. If the critical locus of f is one-dimensional at the origin, we obtain new general bounds on the ranks of the homology groups of the Milnor fiber, $F_{f,\mathbf{0}}$, of f at the origin, with either integral or $\mathbf{Z}/\mathbf{pZ}$ coefficients. If the critical locus of f has arbitrary dimension, we show that the smallest possibly non-zero reduced Betti number of $F_{f,\mathbf{0}}$ completely determines whether or not f defines a family of isolated singularities, over a smooth base, with constant Milnor number. This result has a nice interpretation in terms of the structure of the vanishing cycles as an object in the perverse category. This is joint work with Lê D. T.

Polar quotients and Jacobian quotients

Françoise Michel

Abstract

I) Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function. Let ℓ be a"general" linear form. D.T. Lê and B. Teissier introduced ($\sim$1974) the notion of **Polar curve and Polar quotients** for the morphism $(\ell, f) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$. We will recall it.

II) **Jacobian quotients**

Let $\varphi = (g, f) : (V, p) \to (\mathbf{C}^2, 0)$be a finite analytic germ on a normal complex surface (V, p). By definition the Jacobian quotients of φ are the first Puiseux exponents of the branches of the discriminant of φ. This is an extension of the notion of polar quotients. We will provide examples.

III) Determination of the Jacobian quotients with the help of any **good resolution** $\pi : \tilde{V} \to V$ of the function $(g, f) : (V, p) \to (\mathbf{C}^2, 0)$.

IV) The **topological invariance** of the Jacobian quotients in relation with the Waldhausen structure of the link M^3 of (V, p).

Polar classes associated to singular holomorphic foliations

Rogerio Mol

Universidade Federal de Minas Gerais
Belo Horizonte - Brasil

Abstract

We define polar classesassociated toa singular holomorphic foliation in a projective complex manifold. We prove that these classes can be calculated in terms of the Chern-Mather classes of the tangent sheaf of the foliation. We use their degrees to establish un upper bound for the degrees of some polar classes associated to an invariant variety.

Index of a vector field in R^2 and the pencil's geometry of its principal part

Ana Claudia Nabarro

ICMC, Universidade de São Paulo, São Carlos

We investigate the local index at isolated zero of vector fields in the plane. If m is the degree of the first non zero jet, X_m, of the vector field X at an isolated zero, we explore the geometry of the pencil associated to X_m when the absolute value of the index of X, $|\mathrm{ind}(X)|$, is m.

This is a join work with M.A.S.Ruas

Plane curve singularities: from local to global.

Pho Duc Tai

Vietnam National University, Hanoi

Abstract

We will review some classical results on plane curves: local and global invariants. Next we will talk about dual of smooth complex curves. Last, we will explain a conjecture by Shioda on unirationality of $K3$ surfaces in positive characteristic. To prove a part of this conjecture, we again use the dual curve technique.

Rigidity and normal forms for generic germs of holomorphic foliations with dicritic singularity in $\mathbf{C}^2$.

Ernesto Rosales-Gonzalez

Abstract

We consider the class of dicritic germs of holomorphic foliations in the complex plane with dicritic singularity and vanishing n-jet at the origin. It is proved, under some genericity assumptions, that the formal equivalence implies their analytic equivalence. Moreover, the formal orbital classification of the class of generic germs is given up to a change of coordinates with identity linear part. A similar result is also established for analytic equivalence.

Joint work with L.Ortiz-Bobadilla and S.Voronin.

Topological Triviality of Families of Singular Surfaces

Maria A. Soares Ruas

We prove that one parameter unfoldings of a $\mathcal{A}$-finitely determined germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ are excellent if and only if Milnor number of the double point locus is constant. For families of corank 1 map germs, we also know that these conditions are equivalent to topological triviality (and even equisingularity) of the family.

It seems to be an open question whether the sane result holds for corank 2 map-germs.

This is a joint work with R. Bedregal and K. Houston.

DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAO, UNIVERSIDADE DE SÃO PAULO - CAMPUS DE SO CARLOS, CAIXA POSTAL 668, 13560-970 SÃO CARLOS SP, BRAZIL

Whitney equisingularity of map germs, polar multiplicities and Euler obstruction of stable types.

M. J. Saia ICMC-USP, São Carlos *

Abstract

For any purely d-dimensional complex analytic space X, Lê and Teissier defined a sequence of d integers associate to the algebra $\mathcal{O}_{X,x}$, called **polar multiplicities**. Gaffney showed that these numbers are part of a list of invariants which control the Whitney equisingularity of any excellent 1-parameter unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p, 0)$, of a finitely determined map germ of discrete stable type $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

Since the number of invariants depends on the dimensions (n, p) and this number can be very big according to n and p are big. Then a natural question is to find the minimum number of invariants in Gaffney's theorem that are necessary and sufficient to guarantee the Whitney equisigularity of the unfolding.

In this talk we deal with the case of corank 1 map germs in $\mathcal{O}(n, p)$, with $n \geq p$. We show how to reduce the number of invariants by finding relations among the polar multiplicities of the stable types which appear in a versal deformation of the germ f.

Lê and Teissier also showed how to describe the Euler obstruction of a nonsingular variety in terms of the polar multiplicities. We use the relations among the polar multiplicities obtained here to show how to compute the Euler obstruction of the stable types in terms of the versal polar multiplicity, defined by Gaffney, and Milnor numbers of ICIS associated to the stable types.

*joint works with Victor Hugo Jorge Perez, Daniel Levcovitz and Eliris Cristina Rizziolli

Indices for 1-forms on singular spaces and characteristic cycles

Jörg Schürmann

In recent years the classical Poincaré-Hopf index of 1-forms with isolated singularities has been generalized in different ways to an embedded (complex analytic) singular space X and 1-forms ω on the ambient oriented manifold M with (stratified) isolated singularities (with respect to X): *radial-*, *obstruction-* and *GSV-index*, and in the case of a holomorphic 1-form also a *homological* index.

We explain a new and simple approach to these indices based on the theory of *characteristic cycles* of constructible functions α. Here one gets a *micro-local* index $i(\alpha, \omega)$ as a local intersection number of the characteristic cycle $CC(\alpha)$ of α and the fundamental class $[\omega]$ of the graph of ω in T^*M. For suitable constructible functions this specializes to the radial-, obstruction- or GSV-index. By the functorial behaviour of characteristic cycles one also gets a *proportionality theorem* in the case of a *normally radial* 1-form.

The most natural description of the *homological* index is a K-theoretical description in Grothendieck groups of coherent sheaves based on the (algebraic) *De Rham* complex of X. This can be compared with a similar index based on the socalled *Du Bois* complex of X, which is defined by a suitable *cubical hypperresolution* of X by smooth manifolds. Using the theory of *filtered holonomic D-modules* and the *singular Riemann-Roch theorem*, this last index can be identified with the radial index. Finally all these results work in the more general context of a *compact connected component* of the (stratified) singular locus of ω (with respect to X).

On Horn-Kapranov uniformisation of the discriminantal loci

Susumu Tanabé

Abstract

In this talk we give a rational uniformisation equation of the discriminant loci associated to a non-degenerate affine complete intersection variety depending on deformation parameters. To show this formula we establish a relation of the fibre-integral of the variety with the hypergeometric function of Horn and that of Gelfand-Kapranov-Zelevinski

Bifurcations in implicit differential equations

Farid Tari

August 17, 2005

Abstract

Implicit differential equations (IDEs) appear in many areas, such as differential geometry, partial differential equations and control theory. For example, lines of curvature, asymptotic and characteristic lines on a smooth surface in $\mathbb{R}^3$ are given by IDEs and the characteristic lines of a general linear second-order differential equation are also given by an IDE.

In this talk, I will present a framewrok for studing the local bifurcations of singularities of IDEs in generic families of equations. The main interest is in binary/quadratic differential equations (BDEs), written locally in the form

$$a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 = 0,$$

where the coefficients a, b, c are smooth (i.e. C^∞) functions. A BDE defines two directions in the region where $\delta = (b^2 - ac)(x,y) > 0$, a double direction on the set $\delta = 0$, and no direction where $\delta < 0$. The set $\delta = 0$ is the *discriminant* of the BDE.

The discriminant plays a key role in the study of BDEs (away from it we have a pair of transverse foliations or nothing). The singularities of the discriminant are best studied when it is viewed as the determinant of the family of symmetric matrices $\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}$ associated to the equation. One can obtain the generic bifurcations of the discriminant curve using Bruce's work on families of symmetric matrices. However, in order to obtain the configurations of the integral curves of the members of the family of BDEs, one has also to take into consideration the appearance of local and non-local phenomena in direction fields.

In the talk, I will highlight the methods for studying IDEs/BDEs and present results on bifurcations of their codimension 2 singularities (and show pictures!).

Simple elliptic singularity and the corresponding Lie algebra

Meral Tosun

August 17, 2005

The famous connection between simple singularities of surfaces and the nilpotent varieties of simple Lie algebras was conjectured by A. Grothendieck and solved by E. Brieskorn. A very nice consequence of that connection is to construct semi-universal deformations of simple singularities by using the corresponding Lie algebras. We construct semi-universal deformation spaces of some simple elliptic singularities by using the Lie algebra $sl(2, \mathbf{C}) \oplus sl(2, \mathbf{C})$.

Weighted Voronoi diagrams, discrete Morse theory and tropical geometry

Martijn van Manen

Hokkaido University

The spines of amoebas, or the tropical hypersurfaces, are shown to be weighted Voronoi diagrams (also called power diagrams). We recall from computer science how the latter ones are constructed, and we compare this to the techniques of Passare and Rullgard. Then we show that the distance function used to define a weighted Voronoi diagram yields a discrete Morse function on the triangulation. Finally we use Maslov dequantization to compute the medial axis.