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Topology of the complements to curves and hypersurfaces (Fundamental groups and Alexander Polynomials)

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TOPOLOGY OF THE COMPLEMENTS TO CURVES AND HYPERSURFACES (FUNDAMENTAL GROUPS AND ALEXANDER POLYNOMIALS)

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1. Introduction

For a given hypersurface $V \subset \mathbb{P}^n$, the fundamental group $\pi_1(\mathbb{P}^n - V)$ plays a crucial role when we study geometrical objects over $\mathbb{P}^n$ which are branched over V. By the hyperplane section theorem of Zariski [56], Hamm-Lê [17], the fundamental group $\pi_1(\mathbb{P}^n - V)$ can be isomorphically reduced to the fundamental group $\pi_1(\mathbb{P}^2 - C)$ where $\mathbb{P}^2$ is a generic projective subspace of dimension 2 and $C = V \cap \mathbb{P}^2$. A systematic study of the fundamental group was started by Zariski [55] and further developments have been made by many authors. See for example Zariski [55], Oka [35] $\sim$ [37], Libgober [23]. For a given plane curve, the fundamental group $\pi_1(\mathbb{P}^2 - C)$ is a strong invariant but it is not easy to compute. Another invariant which is weaker but easier to compute is the Alexander polynomial $\Delta_C(t)$. This is related to a certain infinite cyclic covering space branched over C. Important contributions are done by Libgober, Randell, Artal, Loeser-Vaquié, and so on. See for example [21, 14, 47, 27, 51, 11, 10, 46, 31, 2, 15, 25, 48, 3, 50, 8] The main purpose of this note is to give an introduction to the study of the fundamental group and the Alexander polynomial (§§2,3).

The fundamental group $\pi_1(\mathbb{P}^2 - C)$ is a strong invariant of the curve but it is not strong enough to distinguish certain singularities. We will show that the fundamental groups $\pi_1(\mathbb{C}^2_L - C)$ for various tangent line L carry more information, where $\mathbb{C}^2_L = \mathbb{P}^2 - L$. As for the Alexander polynomial $\Delta(t)$, it is also not enough to consider only the generic line at infinity. We define the tangential Alexander polynomials $\Delta_C(t, P)$, $P \in C$ and study their properties.

Most of the description of this note follows that of [42] except the non-generic affine fundamental group.

2. Fundamental groups

2.1. **van Kampen Theorem.** Let $C \subset \mathbb{P}^2$ be a projective curve which is defined by $C = \{[X, Y, Z] \in \mathbb{P}^2 \mid F(X, Y, Z) = 0\}$ where $F(X, Y, Z)$ is a reduced homogeneous polynomial $F(X, Y, Z)$ of degree d. The first systematic studies of the fundamental group $\pi_1(\mathbb{P}^2 - C)$ were done by Zariski [55, 54, 56] and van Kampen [52]. They used so called *pencil section method* to compute the fundamental group. This is still one of the most effective method to compute the fundamental group $\pi_1(\mathbb{P}^2 - C)$ when C has many singularities.

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Let $\ell(X,Y,Z)$, $\ell'(X,Y,Z)$ be two independent linear forms. For any $\tau = (S,T) \in \mathbb{P}^1$, let $L_\tau = \{[X,Y,Z] \in \mathbb{P}^2 \,|\, T\ell(X,Y,Z) - S\ell'(X,Y,Z) = 0\}$. The family of lines $\mathcal{L} = \{L_\tau \,|\, \tau \in \mathbb{P}^1\}$ is called the *pencil* generated by $L = \{\ell = 0\}$ and $L' = \{\ell' = 0\}$. Let $\{B_0\} = L \cap L'$. Then $B_0 \in L_\tau$ for any τ and it is called *the base point* of the pencil. We assume that $B_0 \notin C$. L_τ is called a *generic line* (resp. *non-generic line*) of the pencil for C if L_τ and C meet transversally (resp. non-transversally). If L_τ is not generic, either L_τ passes through a singular point of C or L_τ is tangent to C at some smooth point. We fix two generic lines L_{τ_0} and L_{τ_∞}. Hereafter we assume that τ_∞ is the point at infinity ∞ of $\mathbb{P}^1$ (so $\tau_\infty = \infty$) and we identify $\mathbb{P}^2 - L_\infty$ with the affine space $\mathbb{C}^2$. We denote the affine line $L_\tau - \{B_0\}$ by L_τ^a. Note that $L_\tau^a \cong \mathbb{C}$. The complement $L_{\tau_0} - L_{\tau_0} \cap C$ (resp. $L_{\tau_0}^a - L_{\tau_0}^a \cap C$) is topologically S^2 minus d points (resp. $(d+1)$ points). We usually take $b_0 = B_0$ as the base point in the case of $\pi_1(\mathbb{P}^2 - C)$. In the affine case $\pi_1(\mathbb{C}^2 - C)$, we take the base point b_0 on L_{τ_0} which is sufficiently near to B_0 but $b_0 \neq B_0$. Let us consider two free groups

$$F_1 = \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \quad \text{and} \quad F_2 = \pi_1(L_{\tau_0}^a - L_{\tau_0}^a \cap C, b_0).$$

of rank $d-1$ and d respectively. We consider the set $\Sigma := \{\tau \in \mathbb{P}^1 |\, L_\tau \text{ is a non-generic line}\} \cup \{\infty\}$. We put ∞ in Σ so that we can treat the affine fundamental group simultaneously.

There exists canonical action of $\pi_1(\mathbb{P}^1 - \Sigma, \tau_0)$ on F_1 and F_2 (see [42]). We call this action *the monodromy action* of $\pi_1(\mathbb{P}^1 - \Sigma, \tau_0)$. For $\sigma \in \pi_1(\mathbb{P}^1 - \Sigma, \tau_0)$ and $g \in F_1$ or F_2, we denote the action of σ on g by g^σ. The relations in the group F_ν

$$(R_1) \qquad \langle g^{-1} g^\sigma = e \mid g \in F_\nu, \sigma \in \pi_1(\mathbb{P}^1 - \Sigma, \tau_0) \rangle, \quad \nu = 1, 2$$

are called *the monodromy relations*. The normal subgroup of F_ν, $\nu = 1, 2$ which are normally generated by the elements $\{g^{-1}g^\sigma, |\, g \in F_\nu\}$ are called *the groups of the monodromy relations* and we denote them by N_ν for $\nu = 1, 2$ respectively. The original van Kampen Theorem can be stated as follows. See also [7, 6].

Theorem 1. ([52]) *The following canonical sequences are exact.*

$$1 \to N_1 \to \pi_1(L_{\tau_0} - L_{\tau_0} \cap C, b_0) \to \pi_1(\mathbb{P}^2 - C, b_0) \to 1$$

$$1 \to N_2 \to \pi_1(L_{\tau_0}^a - L_{\tau_0}^a \cap C, b_0) \to \pi_1(\mathbb{C}^2 - C, b_0) \to 1$$

Here 1 *is the trivial group. Thus the fundamental groups* $\pi_1(\mathbb{P}^2 - C, b_0)$ *and* $\pi_1(\mathbb{C}^2 - C, b_0)$ *are isomorphic to the quotient groups* F_1/N_1 *and* F_2/N_2 *respectively.*

We denote the commutator subgroup of G by $D(G)$. Let L be a line and we denote its complement $\mathbb{P}^2 - L$ by $\mathbb{C}_L^2$. The most important case is when L and C inetrsect transversely. In such a case, the topology of $\mathbb{C}_L^2 - C$ does not depend on L and we call it *the generic affine complement* and we often write as $\mathbb{C}^2 - C$ instead of $\mathbb{C}_L^2 - C$. The relation of the fundamental groups $\pi_1(\mathbb{P}^2 - C, b_0)$ and $\pi_1(\mathbb{C}_L^2 - C, b_0)$ are described by the following. Let $\iota : \mathbb{C}_L^2 - C \to \mathbb{P}^2 - C$ be the inclusion map.

Lemma 2. ([34])

(1) *We have the following extension*

$$1 \to \mathbf{N}(\omega) \xrightarrow{\gamma} \pi_1(\mathbb{C}^2 - C, b_0) \xrightarrow{\iota_\sharp} \pi_1(\mathbb{P}^2 - C, b_0) \to 1$$

where $\mathbf{N}(\omega)$ *is the normal subgroup which is normally generated by a lasso* ω *for* L.

(2) *Assume further* L *is generic. Then*

(2-i) ω *is in the center of* $\pi_1(\mathbb{C}^2_L - C)$ *and* $\mathbf{N}(\omega)$ *is an infinite cyclic group.*

(2-ii) *Two commutator subgroups coincide i.e.,* $D(\pi_1(\mathbb{C}^2_L - C)) = D(\pi_1(\mathbb{P}^2 - C))$. *Thus* $\pi_1(\mathbb{P}^2 - C)$ *is abelian if and only if* $\pi_1(\mathbb{C}^2_L - C)$ *is abelian.*

Remark 3. *For non-generic line* L, $\pi_1(\mathbb{C}_L - C)$ *may be non-abelian even if* $\pi_1(\mathbb{P}^2 - C)$ *is abelian. For example, let* $C = \{Y^2 Z - X^3 = 0\}$ *and take* $L = \{Z = 0\}$. *Then* $\pi_1(\mathbb{C}^2_L - C) \cong B_3$ *where* $B_3 = \langle a, b; aba = bab\rangle$.

2.2. **Examples of monodromy relations.** We recall several basic examples of the monodromy relations. Let C be a reduced plane curve of degree d.

We consider a model curve $C_{p,q}$ which is defined by $y^p - x^q = 0$ and we study $\pi_1(\mathbb{C}^2 - C_{p,q})$. For this purpose, we consider the pencil lines $x = t$, $t \in \mathbb{C}$. We consider the local monodromy relations for σ, which is represented by the loop $x = \varepsilon(2\pi i t)$, $0 \le t \le 1$. We take local generators $\xi_0, \xi_1, \ldots, \xi_{p-1}$ of $\pi_1(L_\varepsilon, b_0))$ as in Figure 1. Loops are counter-clockwise oriented. It is easy to see that each point of $C_{p,q} \cap L_\varepsilon$ are rotated by the angle $2\pi \times q/p$. Let $q = mp + q'$, $0 \le q' < p$. Then the monodromy relations are:

$$(R_1) \qquad \xi_j (= \xi_j^\sigma) = \begin{cases} \omega^m \xi_{j+q'} \omega^{-m}, & 0 \le j < p - q' \\ \omega^{m+1} \xi_{j+q'-p} \omega^{-(m+1)}, & p - q' \le j \le p - 1 \end{cases}$$

$$(R_2) \qquad \omega = \xi_{p-1} \cdots \xi_0.$$

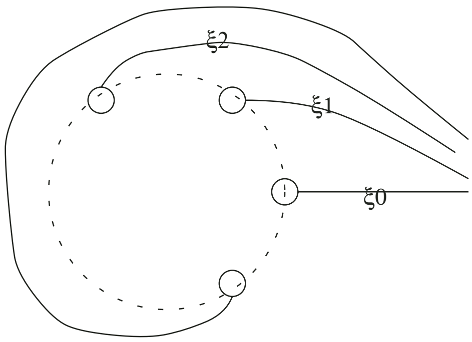

FIGURE 1. Generators

For the convenience, we introduce two groups $G(p,q)$ and $G(p,q,r)$.

$$G(p,q) := \langle \xi_1, \ldots, \xi_p, \omega \mid R_1, R_2 \rangle, \quad G(p,q,r) := \langle \xi_1, \ldots, \xi_p, \omega \mid R_1, R_2, R_3 \rangle$$

where R_3 is the vanishing relation of the big circle $\partial D_R = \{|y| = R\}$:

$$(R_3) \qquad \omega^r = e.$$

Now the above computation gives the following.

Lemma 4. *We have* $\pi_1(\mathbb{C}^2 - C_{p,q}, b_0) \cong G(p,q)$ *and* $\pi_1(\mathbb{P}^2 - C_{p,q}, b_0) \cong G(p,q,1)$.

The groups of $G(p,q)$ and $G(p,q,r)$ are studied in [36, 13]. For instance, we have

Theorem 5. ([36]) (i) *Let* $s = \gcd(p,q)$, $p_1 = p/s$, $q_1 = q/s$. *Then* ω^{q_1} *is the center of* $G(p,q)$.
(ii) *Put* $a = \gcd(q_1, r)$. *Then* ω^a *is in the center of* $G(p,q,r)$ *and has order* r/a *and the quotient group* $G(p,q,r)/ < \omega^a >$ *is isomorphic to* $\mathbb{Z}_{p/s} * \mathbb{Z}_a * F(s-1)$.

Corollary 6. ([36]) *Assume that* $r = q$. *Then* $G(p,q,q) = \mathbb{Z}_{p_1} * \mathbb{Z}_{q_1} * F(s-1)$. *In particular, if* $\gcd(p,q) = 1$, $G(p,q,q) \cong \mathbb{Z}_p * \mathbb{Z}_q$.

Let us recall some useful relations which follow from the above model.
(I) **Tangent relation.** Assume that C and L_0 intersect at a simple point P with intersection multiplicity p. Such a point is called a *flex* point of order $p-2$ if $p \geq 3$ ([55]). This corresponds to the case $q = 1$. Then the monodromy relation gives $\xi_0 = \xi_1 = \cdots = \xi_{p-1}$ and thus $G(p,1) \cong \mathbb{Z}$. As a corollary, Zariski proves that *the fundamental group* $\pi_1(\mathbb{P}^2 - C)$ *is abelian if* C *has a flex of order* $\geq d-3$. In fact, if C has a flex of order at least $d-3$, the monodromy relation is given by $\xi_0 = \cdots = \xi_{d-2}$. (Recall that a smooth point $P \in C$ is called a flex of order k if the intersection multiplicity of the tangent line T_PC and C at P is $k+2$. Thus we also say a flex of intersection multiplicity $k+2$.) On the other hand, we have one more relation $\xi_{d-1} \ldots \xi_0 = e$. In particular, considering the smooth curve defined by $C_0 = \{X^d - Y^d = Z^d\}$, we get that $\pi_1(\mathbb{P}^2 - C)$ is abelian for a smooth plane curve C, as C can be joined to C_0 by a path in the space of smooth curves of degree d.

Example 7. *Let* C *be an irreducible quartic and assume that the singularity configuration* $\Sigma(C)$ *is not* $3A_2$. *Then by the classification of the singularity configuration and the flex number formula in §3.2,* C *has at least one flex point. Thus* $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/4\mathbb{Z}$.

(II) **Nodal relation.** Assume that C has an ordinary double point (i.e., a node) at the origin and assume that C is defined by $x^2 - y^2 = 0$ near the origin. This is the case when $p = q = 2$. Then as the monodromy relation, we get the commuting relation: $\xi_1\xi_2 = \xi_2\xi_1$. Assume that C has only nodes as singularities. The commutativity of $\pi_1(\mathbb{P}^2 - C)$ was first asserted by Zariski [55] and is proved by Fulton-Deligne [12, 16]. See also [30, 46, 45].
(III) **Cuspidal relation.** Assume that C has a cusp at the origin which is locally defined by $y^2 - x^3 = 0$ ($p = 2, q = 3$). Then monodromy relation is: $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$. This relation is known as the generating relation of the braid group B_3 (Artin [4]). Similarly in the case $p = 3$, $q = 2$, we get the relation $\xi_1 = \xi_3$, $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2$.

2.3. **First Homology group** $H_1(\mathbb{P}^2 - C)$**.** Assume that C is a projective curve with r irreducible components $C_1, \ldots, C_r$ of degree $d_1, \ldots, d_r$ respectively. By Lefschetz duality, we have the following.

Proposition 8. $H_1(\mathbb{P}^2 - C, \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}^{r-1} \times (\mathbb{Z}/d_0\mathbb{Z})$ *where* $d_0 = \gcd(d_1, \ldots, d_r)$. *In particular, if* C *is irreducible* ($r = 1$)*, the fundamental group is a cyclic group of order* d_1.

2.4. **Relation with Milnor Fibration.** Let $F(X,Y,Z)$ be a reduced homogeneous polynomial of degree d which defines $C \subset \mathbb{P}^2$. We consider the Milnor fibration of F [26] $F : \mathbb{C}^3 - F^{-1}(0) \to \mathbb{C}^*$ and let $M = F^{-1}(1)$ be the Milnor fiber. By the theorem of Kato-Matsumoto [19], M is path-connected. We consider the following diagram where the vertical map is the restriction of the Hopf fibration.

$$\begin{tikzcd}
& \mathbb{C}^* \arrow[d, "i"] \arrow[dr, "j"] & \\
M \arrow[r, hook, "\iota"] \arrow[dr, "p"'] & \mathbb{C}^3 - F^{-1}(0) \arrow[r, "F"] \arrow[d, "q"] & \mathbb{C}^* \\
& \mathbb{P}^2 - C &
\end{tikzcd}$$

Proposition 9. ([34])(I) *The following conditions are equivalent.*
(i) $\pi_1(\mathbb{P}^2 - C)$ *is abelian.*
(ii) $\pi_1(\mathbb{C}^3 - F^{-1}(0))$ *is abelian.*
(iii) $\pi_1(M)$ *is abelian and the first monodromy of the Milnor fibration* $h_* : H_1(M) \to H_1(M)$ *is trivial.*
(II) *Assume that C is irreducible. Then $\pi_1(M)$ is isomorphic to the commutator subgroup of $\pi_1(\mathbb{P}^2 - C)$ ([38]). In particular, $\pi_1(\mathbb{P}^2 - C)$ is abelian if and only if M is simply connected.*

2.5. **Degenerations and fundamental groups.** Let C be a reduced plane curve. The total Milnor number $\mu(C)$ is defined by the sum of the local Milnor numbers $\mu(C,P)$ at the singular points P of C. We consider an analytic family of reduced projective curves $C_t = \{F_t(X,Y,Z) = 0\}$, $t \in U$ where U is a connected open set with $0 \in \mathbb{C}$ and $F_t(X,Y,Z)$ is a homogeneous polynomial of degree d for any t. We assume that C_t, $t \neq 0$ have the same configuration of singularities so that they are topologically equivalent but C_0 obtain more singularities, i.e., $\mu(C_t) < \mu(C_0)$. We call such a family *a degeneration of C_t at $t = 0$* and we denote this, for brevity, as $C_t \to C_0$. Then we have the following property about the fundamental groups.

Theorem 10. *There is a canonical surjective homomorphism for $t \neq 0$: $\varphi : \pi_1(\mathbb{P}^2 - C_0) \to \pi_1(\mathbb{P}^2 - C_t)$. In particular, if $\pi_1(\mathbb{P}^2 - C_0)$ is abelian, so is $\pi_1(\mathbb{P}^2 - C_t)$.*

Thus if $C_t \cup L \to C_0 \cup L$ is a degeneration, we get

Corollary 11. *There is a a surjective homomorphism: $\pi_1(\mathbb{C}^2_L - C_0) \to \pi_1(\mathbb{C}^2_L - C_t)$.*

Corollary 12. *Let C_t, $t \in \mathbb{C}$ be a degeneration family. Assume that we have a presentation*

$$\pi_1(\mathbb{P}^2 - C_0) \cong \langle g_1, \dots, g_d \,|\, R_1, \dots, R_s \rangle$$

Then $\pi_1(\mathbb{P}^2 - C_t)$, $t \neq 0$ can be presented by adding a finite number of other relations.

2.6. **Product formula.** Assume that C_1 and C_2 are reduced curves of degree d_1 and d_2 respectively which intersect transversely and let $C := C_1 \cup C_2$. We take a line at infinity L_∞ such that $L_\infty \cap C_1 \cap C_2 = \emptyset$ and we consider the the corresponding affine space $\mathbb{C}^2 = \mathbb{P}^2 - L_\infty$.

Theorem 13. (Oka-Sakamoto [44]) *Let $\varphi_k : \mathbb{C}^2 - C \to \mathbb{C}^2 - C_i,\ k = 1,2$ be the inclusion maps. Then the homomorphism $\varphi_{1\#} \times \varphi_{2\#} : \pi_1(\mathbb{C}^2 - C) \to \pi_1(\mathbb{C}^2 - C_1) \times \pi_1(\mathbb{C}^2 - C_2)$ is isomorphic.*

Corollary 14. *Assume that $C_1, \dots, C_r$ are the irreducible components of C and $\pi_1(\mathbb{P}^2 - C_j)$ is abelian for each j and they intersect transversely so that $C_i \cap C_j \cap C_k = \emptyset$ for any distinct three i, j, k. Then $\pi_1(\mathbb{P}^2 - C)$ is abelian.*

2.7. **Covering transformation.** Assume that C is a reduced curve defined by $f(x, y) = 0$ in the affine space $\mathbb{C}^2 := \mathbb{P}^2 - L_\infty$. Take positive integers $n \geq m \geq 1$. We assume that the origin O is not on C and the coordinate axes $x = 0$ and $y = 0$ intersect C transversely and $C \cap \{x = 0\}$ and $C \cap \{y = 0\}$ has no point on L_∞. Consider the doubly branched cyclic covering

$$\Phi_{m,n} : \mathbb{C}^2 \to \mathbb{C}^2,\ (x, y) \mapsto (x^m, y^n).$$

Put $f_{m,n}(x, y) := f(x^m, y^n)$ and put $\mathcal{C}_{m,n} = \{f_{m,n}(x, y) = 0\} = \Phi_{m,n}^{-1}(C)$.

When the line at infinity L_∞ is generic, the topology of the complement of $\mathcal{C}_{m,n}(C)$ depends only on C and m, n. We will call $\mathcal{C}_{m,n}(C)$ as *a generic (m, n)-fold covering transform* of C.

We denote the canonical homomorphism $(\Phi_{m,n})_\sharp : \pi_1(\mathbb{C}^2 - \mathcal{C}_{m,n}(C)) \to \pi_1(\mathbb{C}^2 - C)$ by $\phi_{m,n}$ for simplicity.

Theorem 15. ([38]) *Assume that $n \geq m \geq 1$ and let $\mathcal{C}_{m,n}(C)$ be as above. Then the canonical homomorphism*

$$\phi_{m,n} : \pi_1(\mathbb{C}^2 - \mathcal{C}_{m,n}(C)) \to \pi_1(\mathbb{C}^2 - C)$$

is an isomorphism.

Furthermore if the line at infinity L_∞ is generic, it induces a central extension of groups

$$1 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\iota} \pi_1(\mathbb{P}^2 - \mathcal{C}_{m,n}(C)) \xrightarrow{\widetilde{\phi_{m,n}}} \pi_1(\mathbb{P}^2 - C) \to 1$$

The kernel of $\widetilde{\phi_{m,n}}$ is generated by an element ω' in the center and $\widetilde{\phi_{m,n}}(\omega')$ is homotopic to a lasso ω for L_∞ in the target space. The restriction of $\widetilde{\phi_{m,n}}$ gives an isomorphism of the respective commutator groups $\widetilde{\phi_{m,n\sharp}} : D(\pi_1(\mathbb{P}^2 - \mathcal{C}_{m,n}(C))) \to D(\pi_1(\mathbb{P}^2 - C))$. We have also the exact sequence for the first homology groups:

$$1 \to \mathbb{Z}/n\mathbb{Z} \to H_1(\mathbb{P}^2 - \mathcal{C}_{m,n}(C)) \xrightarrow{\overline{\Phi}_{m,n}} H_1(\mathbb{P}^2 - C) \to 1$$

Corollary 16. *([38]) Assume that the line at infinity L_∞ is generic for C. Then*
(1) $\pi_1(\mathbb{P}^2 - \mathcal{C}_{m,n}(C))$ is abelian if and only if $\pi_1(\mathbb{P}^2 - C)$ is abelian.
(2) Assume that C is irreducible. Put

$$F(x, y, z) = z^d f(x/z, y/z), \quad F_{m,n}(x, y, z) = z^{dn} f_{m,n}(x/z, y/z)$$

Let $M_{m,n}$ and M be the Minor fibers of $F_{m,n}$ and F respectively. Then we have an isomorphism of the respective fundamental groups: $\pi_1(M_{m,n}) \cong \pi_1(M)$.

For a group G, we consider the following condition : $Z(G) \cap D(G) = \{e\}$ where $Z(G)$ is the center of G. This is equivalent to the injectivity of the composition: $Z(G) \to G \to H_1(G)$.

When this condition is satisfied, we say that G satisfies *homological injectivity condition of the center* (or (H.I.C)-condition in short).

3. Examples of curves with easy fundamental groups

For a better understanding, we will give some examples.

3.1. **Abelian fundamental groups.** A curve C with small singularities has often commutative fundamental group $\pi_1(\mathbb{P}^2 - C)$. Some xamples are here:

(0) C is a smooth irreducible curve.

(1) Irreducible curves with only A_1-singularities (i.e., nodes) by [55, 16, 12, 18, 30, 46]. This was a conjecture and proved by Fulton.

(2) irreducible curve of degree d with a nodes and b cusps (i.e., A_2) with $6b+2a < d^2$ ([30]).

(3) $\pi_1(\mathbb{P}^2 - C)$ (respectively $\pi_1(\mathbb{C}^2_L - C)$) is abelian for any irreducible curve of degree d which has a flex of order $\geq d-3$ (resp. of order $d-2$) ([55]).

Example 17. *For example, the following curve is a rational curve of degree n with $\frac{(n-1)(n-2)}{2}$ nodes and three flexes of order $n-2$ and $\pi_1(\mathbb{P}^2 - D_n) \cong \mathbb{Z}/n\mathbb{Z}$. See* [32].

$$D_n: \quad u(t) = t^n, \qquad v(t) = (-1-t)^n.$$

Now we consider $\pi_1(\mathbb{C}^2_L - C)$. If $\pi_1(\mathbb{P}^2 - C)$ is abelian and L is generic for C, $\pi_1(\mathbb{C}^2_L - C)$ is also abelian by Lemma 2. However for non-generic L, this is not always true. Let us recall

Proposition 18. *(*[38]*) Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping and assume that 0 is not an atypical value at infinity and $C = f^{-1}(0)$ is smooth in $\mathbb{C}^2$. Then $\pi_1(\mathbb{C}^2 - C) \cong \mathbb{Z}$.*

Recall that α is a atypical value at infinity if the topological triviality at infinity fails at $t = \alpha$ for the family $f^{-1}(t)$ (see [53]). A property of such curves $\bar{C}$ is that their singular points are on the line at infinity. We give several important such curves.

Proposition 19. *Assume that C is an irreducible smooth curve of degree d in $\mathbb{P}^2$. Then $\pi_1(\mathbb{C}^2_L - C) \cong \mathbb{Z}$ for any line at infinity L.*

Proof. The assertion is well-known if L is generic. For non-generic line at infinity, we consider the polynomial $f(x, y)$ which defines C in $\mathbb{C}^2_L$. Then it is easy to see that $f : \mathbb{C}^2_L \to \mathbb{C}$ has no atypical value at infinity. Thus the assertion follows from Proposition 18. □

Remark 20. *The assertion is not true without the smoothness of C. For example, for a cubic curve $C : Y^2Z - X^3 = 0$, $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/3\mathbb{Z}$ but $\pi_1(\mathbb{C}^2_L - C)$ is isomorphic to the braid group B_3 where $L = \{Z = 0\}$. Thus non-generic affine complements contain more geometrical informations.*

Example 21. *Assume that $f(x, y)$ be a strongly convenient polynomial which has a non-degenerate outside Newton boundary in the sense of Kouchnirenko* [20] *and* [39]*. Then f :*

$\mathbb{C}^2 \to \mathbb{C}$ *has no atypical value at infinity (Theorem 3.10,*[39]*). Thus if* $C := f^{-1}(0)$ *is smooth in* $\mathbb{C}^2$, $\pi_1(\mathbb{C}^2 - C)$ *and also* $\pi_1(\mathbb{P}^2 - C)$ *are abelian.*

For example, we can take $f(x, y) = x^5y^6 + x + y$.

Definition 22. *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve of degree* d *and let* $\xi_0 \in C$ *be a singular point such that* (C, ξ_0) *is locally irreducible. We say* (C, ξ_0) *has* the maximal tangency at ξ_0 *if there is a line* L *passing through* ξ_0 *such that the local intersection multiplicity* $I(C, L, \xi_0) = d$.

Proposition 23. *Assume that* C *is an irreducible curve with a single singularity* ξ_0 *such that* (C, ξ_0) *is irreducible and has a maximal tangent line* L. *Then* $\pi_1(\mathbb{C}^2_L - C)$ *and* $\pi_1(\mathbb{P}^2 - C)$ *are abelian.*

Proof. Taking L as a line at infinity and $\xi_0 = (1 : 0 : 0)$, let $f(x, y)$ be the defining polynomial of C in $\mathbb{C}^2_L$. We know that f has no atypical value at infinity ([**?**, 1]. Thus $\pi_1(\mathbb{C}^2_L - C) = \mathbb{Z}$. □

3.2. **Class formula and flex formula.** As we have seen in the previous subsection, to have a non-abelian fundamental group $\pi_1(\mathbb{P}^2 - C)$, it is necessary that C has some singularities. Also if it has a flex of order greater than or equal to deg $C - 3$, it is abelian. Thus let us recall the following formula about the number of flex points. Let $d = \text{degree}\,(C)$, $\check{d}$ be the degree of the dual curve $\check{C}$, $\Sigma(C)$ be the singular points of C and let α be the number of the flex points.

$$\check{d} = d(d-1) - \sum_{P \in \Sigma(C)} (\mu(C, P) + m(C, P) - 1) \tag{1}$$

$$\alpha = 3d(d-2) - \sum_{P \in \Sigma(C)} \gamma(P, C) \tag{2}$$

where $m(C, p)$ is the multiplicity of C at P and $\gamma(P, C)$ is the flex defect of the singularity P [28, 40]. (In [40], we have denoted $\gamma(P, C)$ as $\delta(P, C)$.) For the formula $\gamma(P, C)$, see [40].

By the above consideration, the existence of a flex is very strong for curves of low degree. In fact, the lowest degree where there exists an irreducible curve C with non-abelian $\pi_1(\mathbb{P}^2 - C)$ is 4, as an irreducible cubic has a flex. Note also that irreducible quartic has at least one flex except three cuspidal quartics. However for $\pi_1(\mathbb{C}^2_L - C)$, $d = 3$ is the lowest. In fact, we have

Proposition 24. *(1) Let* C *be an irreducible curve of degree 3. (a) Then* $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/3\mathbb{Z}$.

(b) On the other hand, for a cuspidal cubic and L *is a (in fact unique) flex tangent line,* $\pi_1(\mathbb{C}^2_L - C) = \pi_1(\mathbb{C}^2 - \{x^3 + y^2 = 0\}) \cong B_3$.

(2) Let C *be an irreducible curve of degree 4. (a) Then* $\pi_1(\mathbb{P}^2 - C)$ *is abelian except* C *has 3* A_2.

(b) $\pi_1(\mathbb{C}^2_L - C)$ *is abelian except*

- $\Sigma(C) = \{3A_2\}$ *and for any* L.
- $\Sigma(C) = \{E_6\}$ *with a flex of order 2 and* L *is the flex tangent line.*
- $\Sigma(C) = \{2A_2 + A_1\}$ *with a flex of order 2 and* L *is the flex tangent line.*

Note that except above three configurations, irreducible quartics have at least 3 flexes. Let us give some more information for later purpose.

(1) Let $C = f(x, y) = 0$ be a 3 cuspidal quartic where

$$f(x,y) := -\frac{1}{2}\,y^4 - \frac{1}{2} - 3\,x^2\,y^2 + y^2 + \frac{3}{2}\,x^4 - 4\,x^3 + 3\,x^2$$

For generic L, for example $\{z = 0\}$,

$$\pi_1(\mathbb{C}^2_L - C) = \langle \xi, \zeta | \xi\,\zeta\,\xi = \zeta\,\xi\,\zeta,\ \xi^2 = \zeta^2 \rangle$$

For $L = \{y = 0\}$ (this is the tangent cone of a cusp),

$$\pi_1(\mathbb{C}^2_L - C) = \langle \xi, \zeta | \xi\,\zeta\,\xi = \zeta\,\xi\,\zeta \rangle = B_3 \qquad \text{(braid group)}$$

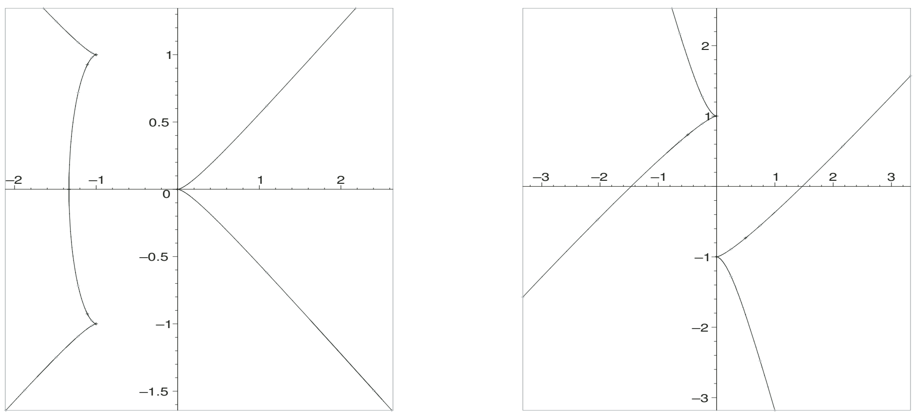

FIGURE 2. 3 cuspidal Quartic(left:$L = \{z = 0\}$, right: $L = \{y = 0\}$)

(2) Consider the following degenerated quartic with E_6 and one flex of order 2 at $P = (0, 1, 0)$.

$$C: \quad f(x, y) = y^3 + x^4 = 0$$

Take the flex line $z = 0$. Then

$$\pi_1(\mathbb{C}^2_L - C) \cong G(3, 4)$$

(3) We consider quartic with $\Sigma(C) = 2A_2 + A_1$ with a degenerated flex of order 2 at infinity:

$$C: \quad \frac{16}{9}\,z^3 - 2\,z^2 - 6\,z + \frac{15}{2} - 6\,y^2\,z - 9\,y^2 - \frac{1}{2}\,y^4 = 0$$

$$\pi_1(\mathbb{C}^2_L - C) = \langle \xi_1, \xi_2, \xi_3 | \xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2,\ \xi_3\xi_2\xi_3 = \xi_2\xi_3\xi_2,\ \xi_3\xi_2\xi_1\xi_3 = \xi_2\xi_1\xi_3\xi_2 \rangle$$

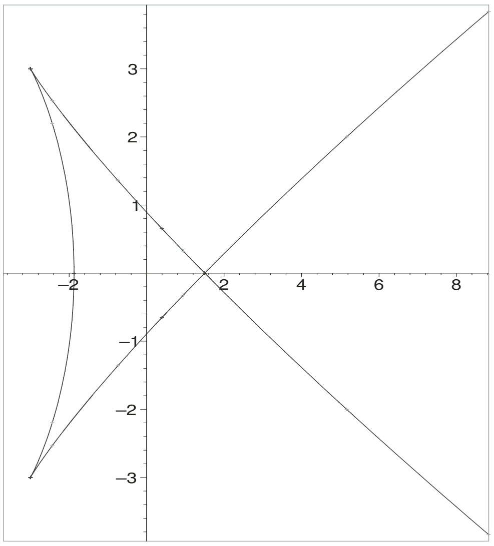

FIGURE 3. $C \cap \mathbb{C}^2_L$

Remark 25. *The other part of the proof is due to a direct computation. However the following fact is useful: a generic quartic with* $\Sigma(C) = \{E_6\}$ *or* $\{2A_2 + A_1\}$*, there are 2 flex points respectively. The above quartics in (2) and (3) are degenerated in the sense that two flexes collapse into a flex of order 2. For other configuration, there are at least 3 flex points, and using this flex points, we can easily compute the fundamental group to be abelian.*

3.3. **Non-abelian case: Curves of torus type.** Assume that p, q are positive integers greater than 1 and consider the curve

$$C_{p,q}: \quad f_p(x,y,z)^q + f_q(x,y,z)^p = 0$$

where $f_j(x,y,z)$ is a polynomial of degree j. $C_{p,q}$ is called a curve of (p,q)-torus type. If two curves $f_p = 0$ and $f_q = 0$ intersects at pq distinct points transversely, $C_{p,q}$ has pq cusp singularities $y^p + x^q = 0$. The fundamental group $\pi_1(\mathbb{P}^2 - C_{p,q})$ is generically isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$ ([35]).

We can also consider a more general curve: let d be a positive integers with two partition $d = \sum_{i=1}^r n_i a_i = \sum_{j=1}^s m_j b_j$ and consider

$$C_{\mathbf{a},\mathbf{b}}: f_1(x,y,z)^{a_1} \cdots f_r(x,y,z)^{a_r} + g_1(x,y,z)^{b_1} \cdots g_s(x,y,z)^{b_s} = 0$$

where degree $f_i(x,y,z) = n_i$, degree $g_j(x,y,z) = m_j$, $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{b} = (b_1, \dots, b_s)$ and we assume the following genericity. The curves $f_i(x,y,z) = 0$ and $g_j(x,y,z) = 0$ intersect transversely and the singularity $\Sigma(C)$ is on the intersections $\bigcup_{i,j} \{(x,y,z); f_i(x,y,z) = g_j(x,y,z) = 0\}$ and any two curves $f_i = f_k = 0$ or $g_j = g_\ell = 0$ intersect outside of C. Then at the intersection $P \in \{f_i = g_j = 0\}$, (C,P) has a singularity (topologically isomorphic to) $x^{a_i} + y^{b_j} = 0$.

Put $a := \gcd(a_1, \dots, a_r)$ and $b := \gcd(b_1, \dots, b_s)$. Assume that $\gcd(a, b) = 1$ to make C irreducible. We can deform C keeping the genericity assumption to a curve

$$C' : h_1(x,z)^{a_1} \cdots h_r(x,z)^{a_r} + k_1(y,z)^{b_1} \cdots k_s(y,z)^{b_s} = 0$$

Thus by [36], $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_a * \mathbb{Z}_b$.

4. Alexander polynomial

4.1. **Infinite cyclic covering and its invariant.** Let X be a topological space which has a homotopy type of a finite CW-complex and assume that we have a surjective homomorphism: $\phi : \pi_1(X) \to \mathbb{Z}$. Let t be a generator of $\mathbb{Z}$ and put $\Lambda = \mathbb{C}[t, t^{-1}]$. Note that Λ is a principal ideal domain. Consider an infinite cyclic covering $p : \widetilde{X} \to X$ such that $p_{\#}(\pi_1(\widetilde{X})) = \operatorname{Ker}\phi$. Then $H_1(\widetilde{X}, \mathbb{C})$ has a structure of Λ-module where t acts as the canonical covering transformation. Thus we have an identification:

$$H_1(\widetilde{X}, \mathbb{C}) \cong \Lambda/\lambda_1 \oplus \cdots \oplus \Lambda/\lambda_n$$

as Λ-modules. We normalize the denominators so that λ_i is a polynomial in t with $\lambda_i(0) \neq 0$ for each $i = 1, \dots, n$. The Alexander polynomial $\Delta(t)$ is defined by the product $\prod_{i=1}^n \lambda_i(t)$.

The classical one is the case $X = S^3 - K$ where K is a knot. As $H_1(S^3 - K) = \mathbb{Z}$, we have a canonical surjective homomorphism $\phi : \pi_1(S^3 - K) \to H_1(S^3 - K, \mathbb{Z})$ induced by the Hurewicz homomorphism. The corresponding Alexander polynomial is called *the Alexander polynomial of the knot* K. If K is a link of a plane curve singularity $p \in C$, it is equal to the characteristic polynomial of the Milnor fibration at P ([26]).

In our situation, we consider a plane curve $C = C_1 \cup \cdots \cup C_r$ defined by a homogeneous polynomial $F(X, Y, Z)$ of degree d and take a line at infinity $L = L_\infty$. Then let ϕ be the composition

$$\phi : \pi_1(\mathbb{P}^2 - C \cup L_\infty) \xrightarrow{\xi} H_1(\mathbb{P}^2 - C \cup L_\infty, \mathbb{Z}) \cong \mathbb{Z}^r \xrightarrow{s} \mathbb{Z}$$

where ξ is the Hurewicz homomorphism and s is defined by $s(a_1, \dots, a_r) = \sum_{i=1}^r a_i$. We call s the summation homomorphism.

Let $\tilde{X} \to \mathbb{P}^2 - C \cup L$ be the infinite cyclic covering corresponding to $\operatorname{Ker}\phi$.

Definition 26. *The corresponding Alexander polynomial is called the* Alexander polynomial with respect to L *we denote it by* $\Delta_C(t; L)$*. If* L *is generic, we call* $\Delta_C(t; L)$ generic Alexander polynomial *of* C *and we denote simply* $\Delta_C(t)$*. It does not depend on the choice of the generic line at infinity* L_∞*. When* C *has a maked point* P *and* $L = T_P C$ *(the tangent line at* $P \in C$*), we denote* $\Delta_C(t; L)$ *as* $\Delta_C(t; P)$ *and we call* the tangential Alexander polynomial at P. *Similarly we define the doubly tangential Alexander polynomial* $\Delta_C(t; P, Q)$ *as* $\Delta_{C \cup T_P C}(t; Q)$*.*

Let $M = F^{-1}(1) \subset \mathbb{C}^3$ be the Milnor fiber of F. The monodromy map $h : M \to M$ is defined by the coordinate-wise multiplication of $\exp(2\pi i/d)$. Randell showed in [47] the following important theorem.

Theorem 27. *The generic Alexander polynomial* $\Delta_C(t)$ *is equal to the characteristic polynomial of the monodromy* $h_* : H_1(M) \to H_1(M)$*. Thus the degree of* $\Delta_C(t)$ *is equal to the first Betti number* $b_1(M)$*.*

Lemma 28. *Assume that C has r irreducible components. Then the multiplicity of the factor $(t-1)$ in $\Delta_C(t)$ is $r-1$.*

Proof. As $h^d = \mathrm{id}_M$, the monodromy map $h_* : H_1(M) \to H_1(M)$ has a period d. This implies that h_* can be diagonalized. Assume that ρ is the multiplicity of $(t-1)$ in $\Delta_C(t)$. Consider the Wang sequence:

$$H_1(M) \xrightarrow{h_*-\mathrm{id}} H_1(M) \to H_1(E) \to H_0(M) \to 0$$

where $E := S^5 - V \cap S^5$ and $V = F^{-1}(0)$. Then we get $b_1(E) = \rho + 1$. On the other hand, by Alexander duality, we have $H_1(E) \cong H^3(S^5, V \cap S^5)$ and $b_1(E) = r$. Thus we conclude that $\rho = r - 1$. □

Definition 29. *We say that Alexander polynomial of a curve C is trivial with respect to L if $\Delta_C(t; L) = (t-1)^{r-1}$ where r is the number of the irreducible components of C.*

The following Lemma describes the relation between the generic Alexander polynomial and local singularities.

Lemma 30. (Libgober [21]) *Let $P_1, \dots, P_k$ be the singular points of C (including those at infinity) and let $\Delta_i(t)$ be the characteristic polynomial of the Milnor fibration of the germ (C, P_i). Then the generic Alexander polynomial $\Delta_C(t; L_\infty)$ divides the product $\prod_{i=1}^k \Delta_i(t)$*

Lemma 31. (Libgober [21]) *Let d be the degree of C. Then the Alexander polynomial $\Delta_C(t; L_\infty)$ divides the Alexander polynomial at infinity $\Delta_\infty(t)$.*

If L_∞ is generic, $\Delta_\infty(t)$ is given by $(t^d-1)^{d-2}(t-1)$. In particular, the roots of the genric Alexander polynomial are d-th roots of unity.

4.2. **Fox calculus.** Suppose that $\phi : \pi_1(X) \to \mathbb{Z}$ is a given surjective homomorphism. Assume that $\pi_1(X)$ has a finite presentation as

$$\pi_1(X) \cong \langle x_1, \dots, x_n \,|\, R_1, \dots, R_m \rangle$$

where R_i is a word of $x_1, \dots, x_n$. Thus we have a surjective homomorphism $\psi : F(n) \to \pi_1(X)$ where $F(n)$ is a free group of rank n, generated by $x_1, \dots, x_n$. Consider the group ring of $F(n)$ with $\mathbb{C}$-coefficients $\mathbb{C}[F(n)]$. The Fox differential

$$\frac{\partial}{\partial x_j} : \mathbb{C}[F(n)] \to \mathbb{C}[F(n)]$$

is $\mathbb{C}$-linear map which is characterized by the property

$$\frac{\partial}{\partial x_j} x_i = \delta_{i,j}, \quad \frac{\partial}{\partial x_j}(uv) = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j},\ u, v \in \mathbb{C}[F(n)]$$

The composition $\phi \circ \psi : F(n) \to \mathbb{Z}$ gives a ring homomorphism $\gamma : \mathbb{C}[F(n)] \to \mathbb{C}[t, t^{-1}]$. *The Alexander matrix* A is $m \times n$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$ and its (i,j)-component is given by $\gamma(\frac{\partial R_i}{\partial x_j})$. Then it is known that the Alexander polynomial $\Delta(t)$ is given by the greatest common divisor of $(n-1)$-minors of A ([9]). The following formula will be useful.

$$\frac{\partial}{\partial x_j}\omega^k = (1 + \omega + \cdots + \omega^{k-1})\frac{\partial}{\partial x_j}\omega, \quad \frac{\partial}{\partial x_j}\omega^{-k} = -\omega^{-k}\frac{\partial}{\partial x_j}\omega^k$$

Example 32. *We gives several examples.*
1. Consider the trivial case $\pi_1(X) = \mathbb{Z}$ *and* ϕ *is the canonical isomorphism. Then* $\pi_1(X) \cong \langle x_1 \rangle$ *(no relation) and* $\Delta(t) = 1$*. More generally assume that* $\pi_1(X) = \mathbb{Z}^r$ *with* $\phi(n_1, \dots, n_r) = n_1 + \cdots + n_r$*. Then*

$$\pi_1(X) = \langle x_1, \dots, x_r \,|\, R_{i,j} = x_i x_j {x_i}^{-1} {x_j}^{-1},\, 1 \le i < j \le r \rangle$$

As we have

$$\gamma\left(\frac{\partial}{\partial x_\ell} R_{i,j}\right) = \begin{cases} 1-t & \ell = i \\ t-1 & \ell = j \\ 0 & \ell \ne i,j \end{cases},$$

we have $\Delta(t) = (t-1)^{r-1}$.
2. Let $C = \{y^2 z - x^3 = 0\}$ *and* $X = \mathbb{C}^2_L - C$, $L = \{z = 0\}$*. Then* L *is the tangent line at the flex point* $P = (0,1,0)$*. Then* $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/3\mathbb{Z}$ *but*

$$\pi_1(X) = \langle x_1, x_2 \,|\, x_1 x_2 x_1 = x_2 x_1 x_2 \rangle.$$

is known as the braid group $B(3)$ *of three strings and the Alexander polynomials are given by* $\Delta_C(t) = 1$ *and* $\Delta_C(t; P) = t^2 - t + 1$.
3. Let us consider the curve $C := \{y^2 z^3 - x^5 = 0\} \subset \mathbb{C}^2$ *and put* $X = \mathbb{C}^2 - C$, $L = \{z = 0\}$, $P = (0,1,0)$*. Then* $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/5\mathbb{Z}$ *and*

$$\pi_1(X) \cong G(2,5) = \langle x_0, x_1 \,|\, x_0 (x_1 x_0)^2 x_1^{-1} (x_1 x_0)^{-2} \rangle$$

In this case, we get

$$\Delta_C(t) = 1, \quad \Delta_C(t; P) = t^4 - t^3 + t^2 - t + 1 = \frac{(t^{10} - 1)(t-1)}{(t^2-1)(t^5-1)}.$$

4.3. **Computation of Alexander polynomials from position data of singularities.** Let C be a given plane curve of degree d defined by $f(x,y) = 0$ and let $\Sigma(C)$ be the singular locus of C and let $P \in \Sigma(C)$ be a singular point. Consider an embedded resolution of C, $\pi : \tilde{U} \to U$ where U is an open neighbourhood of P in $\mathbb{P}^2$ and let $E_1, \dots, E_s$ be the exceptional divisors. Let us choose (u,v) be a local coordinate system centered at P and let k_i and m_i be the order of zero of the canonical two form $\pi^*(du \wedge dv)$ and $\pi^* f$ respectively along the divisor E_i. We consider an ideal of $\mathcal{O}_P$ generated by the function germ ϕ such that the pull-back $\pi^*\phi$ vanishes of order at least $-k_i + [k m_i / d]$ along E_i and we denote this ideal by $\mathcal{J}_{P,k,d}$. Namely

$$\mathcal{J}_{P,k,d} = \{\phi \in \mathcal{O}_P, (\pi^*\phi) \ge \sum_i (-k_i + [k m_i/d]) E_i\}$$

Let us consider the canonical homomorphisms induced by the restrictions:

$$\sigma_{k,P} : \mathcal{O}_P \to \mathcal{O}_P / \mathcal{J}_{P,k,d}, \quad \sigma_k : H^0(\mathbb{P}^2, \mathcal{O}(k-3)) \to \bigoplus_{P \in \Sigma(C)} \mathcal{O}_P / \mathcal{J}_{P,k,d}$$

where the right side of σ_k is the sum over singular points of C. We define two invariants:

$$\rho(P,k) = \dim_{\mathbb{C}} \mathcal{O}_P / \mathcal{J}_{P,k,d}, \quad \rho(k) = \sum_{P \in \Sigma(C)} \rho(P,k)$$

Let ℓ_k be the dimension of the cokernel σ_k. Then the formula of Libgober [22] and Loeser-Vaquié [25], combined with a result of Esnault and Artal [15, 2], can be stated as follows.

Lemma 33. *The generic Alexander polynomial $\Delta_C(t)$ is written as the product*

$$\tilde{\Delta}_C(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k}, \quad k = 1, \dots, d-1$$

where

$$\Delta_k = (t - \exp(\frac{2k\pi i}{d}))(t - \exp(\frac{-2k\pi i}{d})).$$

5. Zariski pairs and marked Zariski pairs

Definition 34. *A pair of reduced plane curves of a same degree $\{C, C'\}$ is called* a Zariski pair *if there is*
(1) a homeomorphism $\tilde{\alpha} : N(C) \to N(C')$ of the respective tubular neighborhood $N(C)$, $N(C')$ which induces by restriction a homeomorphism $\alpha : (C, \Sigma(C)) \to (C', \Sigma(C'))$ so that the two germs (C, P), $(C', \alpha(P))$ are topologically equivalent as germs of plane curves for each singularity $P \in \Sigma(C)$ but
(2) the pair of spaces $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

Corollary 35. ([38]) *Let $\{C, C'\}$ be a Zariski pair and assume that $\pi_1(\mathbb{P}^2 - C')$ satisfies (H.I.C)-condition. Then for any $n \geq m \geq 1$, $\{\mathcal{C}_{m,n}(C), \mathcal{C}_{m,n}(C')\}$ is a Zariski pair.*

See also Shimada [48].

Example 36. *First example of Zariski pair is given by Zariski* [55]. *It is a pair of sextics with $6\,A_2$ such that one is of torus type and the other is not.*

Definition 37. *Now we consider a fixed mark point $P \in C$. There is a line L which passes throught P and the local intersection number $I(L, C; P)$ is sytrictly larger than the multiplicity of (C, P). At a smooth point, it is the usual tangent line. At singular points, it is one of the tangent line to a branch at P. We denote the tangent line at P by $T_P C$. (To make it unique, P must be either a smooth point or a singular point where C is locally irreducible.) We call (C, P)* a curve with a marked point P. *Two curves with marked points (C, P) and (C', P') are called* a marked Zariki pair *if $\{C \cup T_P C, C' \cup T_{P'} C'\}$ is a Zariki pair.*

Similarly we can consider a curve with k-marked points $(C, P_1, \dots, P_k)$. Two curves with k-marked points $(C, P_1, \dots, P_k)$ and $(C', P'_1, \dots, P'_k)$ are called a k-marked Zariki pair *if $\{C \cup_{i=1}^k T_{P_i} C,\ C' \cup \cup_{i=1}^k T_{P'_i} C'\}$ is a Zariki pair.*

5.1. **Tangential Alexander spectrum.** *As we have seen Proposition 24, for irreducible cubic curves, the fundamental group is abelian. For quartic, the only exception is a three cuspidal quartic. In this case, using Fox calculus, we can easily conclude that the generic Alexander polynomial is trivial. We have observed there that non-generic Alexander polynomials are more fruitful.*

Definition 38. *For the sharper usage of Alexander polynomial, we consider the tangential Alexander polynomial* $\Delta_C(t;P) := \Delta_C(t;T_PC)$ *and consider the set*

$$t\text{-}AS(C) := \{\Delta_C(t;P); P \in C\}$$

and we call t-$AS(C)$ *the* tangential Alexander spectrum of C.

We can also define the k-ple tangential Alexander spectrum of C, *denoted as* t-$AS^{(k)}(C)$, *by*

$$t\text{-}AS^{(k)}(C) := \{\Delta_{C\cup T_PC}(t;P_1,\dots,P_k); P_k, \in C\}$$

Example 39. *0. For smooth curves, the tangential Alexander spectrums are trivial by Proposition 19.*

1. Let C *be an irreducible conic. Then* t-$AS(C) = \{1\}$.

2. Let C *be an irreducible cubic. For smooth or nodal cubics, we have* t-$AS(C) = \{1\}$. *For a cuspidal cubic* C,

$$t\text{-}AS(C) = \{1, t^2 - t + 1\}$$

The non-trivial spectrum is given by taking P *at the unique flex point of* C.

3. Let C *be an irreducible conic. Then the secondary Alexander polynomial* $\Delta_C(t;P,Q)$ *is given by* $(t-1)(t^2+1)$. *Note that this is equal to the local Alexander polynomial of* A_3.

However we have

Proposition 40. *Let* C *be a smooth curve of degree* $d \geq 3$. *Then for any non-flex point* Q *and an arbitrary point* $P \neq Q$, *we have* $\Delta_C(t;P,Q) = (t-1)$.

Proof. *We know that* $\pi_1(\mathbb{C}^2 - C) \cong \mathbb{Z}$ *where* $\mathbb{C}^2 = \mathbb{P}^2 - T_PC$. *Fix affine coordinates* (x,y) *and let* $p : \mathbb{C}^2 \to \mathbb{C}$ *be the projection to* x *coordinate. There is a polydisk* $B := B_R \times B_S$ *so that* $B \supset C \cap p^{-1}(B_R)$ *and* $\pi_1(B - C) \cong \mathbb{Z}$. *Take a generic point* Q *near infinity so that* $T_QC \cap B = \emptyset$. *Then we can see that* $T_QC \cap C \cap p^{-1}(B_R) = \emptyset$ *and* $T_QC \cap C$ *has at least one transversal point as we have assumed* $d \geq 3$. *Using pencil section* $C \cap \{x = \eta\}_{\eta\in\mathbb{C}}$, *we see immediately that* $\pi_1(\mathbb{C}^2 - C \cup T_QC) = \mathbb{Z}^2$. □

5.2. **Configuration space.** *Let* Σ *be a finite set of topological equivalent class of curve singularity and let* $\mathcal{M}(\Sigma,n)$ *be the configuration space of plane curves of degree* n *with singularity configuration* Σ. *The generic Alexander polynomial is an invariant of the connected components of the configuration space. However the tangential Alexander spectrum is finer invariant. To understand it better, we introduce the* dual stratification $\mathcal{S}(\mathcal{M}(\Sigma,n))$ of $\mathcal{M}(\Sigma,n)$ *as follows. Consider the Gauss map image* $\check{\mathcal{M}}(\Sigma,n)$ *of* $\mathcal{M}(\Sigma,n)$. *On this image, there is a canonical stratification by the configuration of the singularities* $\Sigma(\check{C})$, $\check{C} \in \check{\mathcal{M}}(\Sigma,n)$. *The dual stratification* $\mathcal{S}(\mathcal{M}(\Sigma,n))$ *is defined by the inverse images of this stratification by the Gauss map. Two points on the same stratum* P,Q *can be joined by a path* P_t *with* $P_0 = P$ *and* $P_1 = Q$. *This induces* μ*-constant family* $C \cup T_{P_t}C$. *Thus*

Proposition 41. *The tangential Alexander spectrum is an invariant of each stratum of the dual stratification.*

Example 42. *We consider $\mathcal{M}(2A_2+A_1,4)$ and $\mathcal{M}(E_6,4)$. By class formula, the dual curve $\check{C}$ of a generic member C of $\mathcal{M}(2A_2+A_1,4)$ or $\mathcal{M}(E_6,4)$ is a quartic with $2A_2+A_1$ in either case. (C has generically 2 flex points.) There are strata corresponding to* degenerated *members which has one flex of order 2. Thus let us consider*

$$\begin{aligned} M_1 &:= \{C \in \mathcal{M}(2A_2+A_1,4); \Sigma(\check{C}) = \{2A_2+A_1\}\}, \\ M_2 &:= \{C \in \mathcal{M}(2A_2+A_1,4); \Sigma(\check{C}) = \{E_6\}\} \\ N_1 &:= \{C \in \mathcal{M}(E_6,4); \Sigma(\check{C}) = \{2A_2+A_1\}\}, \\ N_2 &:= \{C \in \mathcal{M}(E_6,4); \Sigma(\check{C}) = \{E_6\}\} \end{aligned}$$

We can easily see that $\{M_1,\, M_2\}$, $\{N_1,\, N_2\}$ are respective dual stratifications. We observe that $M_2 \subset \bar{M}_1$ and $N_2 \subset \bar{N}_1$. In fact, under the canonical topology of the space of quartics, $\bar{M}_1 = M_1 \cup M_2 \cup N_1 \cup N_2$. The respective tangential Alexander spectrum are

$$t\text{-}AS(M_1) = \{1\}, \quad t\text{-}AS(M_2) = \{1,\, t^2-t+1\}$$
$$t\text{-}AS(N_1) = \{1\}, \quad t\text{-}AS(N_2) = \{1,\, (t^2-t+1)(t^4-t^2+1)\}$$

The non-trivial Alexander polynomials can be obtained by Fox calculus applied to the presentation given in Proposition 24.

5.3. **Degeneration and Alexander polynomial.** *We consider a degeneration $C_t \to C_0$. By Corollary 12 and Fox calculus, we have*

Theorem 43. *Assume that we have a degeneration family of reduced curves $\{C_s \mid s \in U\}$ at $s=0$. Let $\Delta_s(t)$ be the Alexander polynomial of C_s. Then $\Delta_s(t) | \Delta_0(t)$ for $s \neq 0$.*

Remark 44. *Theorem 43 can be generalized to a degeneration of marked curves $(C_s, P_s) \to (C_0, P_0)$ when $C_s \cup T_{P_s}C_s \to C_0 \cup T_{P_0}C_0$ is a degeneration.*

6. Triviality of the Alexander Polynomials.

We have seen that the generic Alexander polynomial is trivial if C is irreducible and $\pi_1(\mathbb{C}^2_L - C)$ is abelian. However this is not a necessary condition, as we will see in the following. Let $F(X,Y,Z)$ be the defining homogeneous polynomial of C and let $M = F^{-1}(1) \subset \mathbb{C}^3$ the Milnor fiber of F.

Theorem 45. *Assume that C is an irreducible curve. The generic Alexander polynomial $\Delta_C(t)$ of C is trivial if and only if the first homology group of the Milnor fiber $H_1(M)$ is at most a finite group.*

Proof. By Theorem 27, the first Betti number of M is equal to the degree of $\Delta_C(t)$. $\square$

Corollary 46. *Assume that $\pi_1(\mathbb{P}^2 - C)$ is a finite group. Then the generic Alexander polynomial is trivial.*

Proof. This is immediate from Theorem 9 as $D(\pi_1(\mathbb{P}^2 - C)) = \pi_1(M)$ and it is a finite group under the assumption. $\square$

6.1. **Examples.** 1. **(Zariski's three cuspidal quartic,** [55]) Let Z_4 be a quartic curve with three A_2-singularities. The corresponding moduli space is irreducible. Then the fundamental groups are given by [55, 37] as

$$\pi_1(\mathbb{C}^2 - Z_4) \cong \langle \rho, \xi \,|\, \rho\,\xi\,\rho = \xi\,\rho\,\xi,\, \rho^2 = \xi^2 \rangle$$
$$\pi_1(\mathbb{P}^2 - Z_4) \cong \langle \rho, \xi \,|\, \rho\,\xi\,\rho = \xi\,\rho\,\xi,\, \rho^2\xi^2 = e \rangle$$

Then by an easy calculation, $\Delta_C(t) = 1$. This also follows from Theorem 45 as $\pi_1(\mathbb{P}^2 - Z_4)$ is a finite group of order 12 by Zariski [55]. By Theorem 15, the generic covering transform $\mathcal{C}_{n,n}(Z_4)$ has also a trivial Alexander polynomial for any n.
2. We have seen that the generic Alexander polynomials are trivial for any irreducible cubics or quartics. We can also prove that this is also true for irreducible quintic with simple singularities, using Lemma 33 but the detail of calculation is left to the reader.
3. **Curves of torus type.** We consider a generic curve of torus type $C_{p,q}$. By the caculation of the fundamental group, we have

Theorem 47. *The generic Alexander polynomial is the same as the characteristic polynomial of the Pham-Brieskorn singularity $B_{p,q}$ which is given by*

$$\Delta_C(t) = \frac{(t^{p_1 q_1 s} - 1)^s\,(t-1)}{(t^p - 1)\,(t^q - 1)} \tag{3}$$

where $s = \gcd(p,q)$, $p = p_1 s$, $q = q_1 s$.

6.2. **Sextics of torus type.** The most interesting torus curve is a sextic of torus type. Let us consider a sextic of torus type

$$C:\; f_2(x,y)^3 + f_3(x,y)^2 = 0,\ \text{degree}\, f_j = j,\ j = 2, 3,$$

as an example. Assume that C is reduced and irreducible. A sextic of torus type is called *tame* if the singularities are on the intersection of the conic $f_2(x,y) = 0$ and the cubic $f_3(x,y) = 0$. A generic sextic of torus type is tame but the converse is not true. Then the possibility of generic Alexander polynomials for sextics of torus type is determined as follows.

Theorem 48. ([43, 41]) *Assume that C is an irreducible sextic of torus type. The generic Alexander polynomial of C is one of the following.*

$$(t^2 - t + 1),\ (t^2 - t + 1)^2,\ (t^2 - t + 1)^3$$

Moreover for tame sextics of torus type, the generic Alexander polynomial is given by $t^2 - t + 1$ and the fundamental group of the complement in $\mathbb{P}^2$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ except the case when the configuration is $[C_{3,9}, 3A_2]$. In the exceptional case, the generic Alexander polynomial is given by $(t^2 - t + 1)^2$.

7. Weakness of generic Alexander polynomial.

We have already observed a weakness of generic Alexander polynomials: it does not tell us any geometrical information if $\pi_1(\mathbb{P}^2 - C)$ is abelian. Another weakness is for reducible curves, which we are going to explain. Let C_1 and C_2 be curves which intersect transversely.

We take a line at infinity L_∞ for $C_1 \cup C_2$ so that L_∞ does not contain any points of $C_1 \cap C_2$. Theorem 13 says that

$$\pi_1(\mathbb{C}^2 - C_1 \cup C_2) \cong \pi_1(\mathbb{C}^2 - C_1) \times \pi_1(\mathbb{C}^2 - C_2)$$

which tell us that the fundamental group of the union of two curves keeps informations about each curves C_1, C_2. On the other hand, the generic Alexander polynomial of $C_1 \cup C_2$ keeps no information about each curves C_1, C_2. In fact, we have

Theorem 49. *Assume that C_1 and C_2 intersect transversely and let $C = C_1 \cup C_2$. Then the generic Alexander polynomial $\Delta_C(t)$ of C is given by given by $(t-1)^{r-1}$ where r is the number of irreducible components of C.*

Proof. Assume that $\pi_1(\mathbb{C}^2 - C_j)$, $j = 1, 2$ is presented as

$$\pi_1(\mathbb{C}^2 - C_1) = \langle g_1, \dots, g_{s_1} \,|\, R_1, \dots, R_{p_1} \rangle,\ \pi_1(\mathbb{C}^2 - C_2) = \langle h_1, \dots, h_{s_2} \,|\, S_1, \dots, S_{p_2} \rangle$$

Then by Theorem 13, we have

$$\pi_1(\mathbb{C}^2 - C) = \langle g_1, \dots, g_{s_1}, h_1, \dots, h_{s_2} \,|\, R_1, \dots, R_{p_1}, S_1, \dots, S_{p_2}, T_{i,j},\ 1 \le i \le s_1,\ 1 \le j \le s_2 \rangle$$

where $T_{i,j}$ is the commutativity relation $g_i\, h_j g_i^{-1}\, h_j^{-1}$. Let $\gamma : \mathbb{C}[g_1, \dots, g_{s_1}, h_1, \dots, h_{s_2}] \to \mathbb{C}[t, t^{-1}]$ be the ring homomorphism defined before (§3.1). Put $g_{s_1+j} = h_j$ for brevity. Then the submatrix of the Alexander matrix corresponding to

$$\left(\gamma(\frac{\partial T_{i,j}}{\partial g_k})\right),\ \{i = 1, \dots, s_1,\ j = s_2\} \text{ or } \{i = s_1,\ j = 1, \dots, s_2 - 1\}, \text{ and } 1 \le k \le s_1 + s_2 - 1$$

is given by $(1-t) \times A$ where

$$A = \begin{pmatrix} E_{s_1} & 0 \\ K & -E_{s_2-1} \end{pmatrix}$$

and E_ℓ is the $\ell \times \ell$-identity matrix and K is a $(s_2-1) \times s_1$ matrix with only the last column is non-zero. Thus the determinant of this matrix gives $\pm(t-1)^{s_1+s_2-1}$ and the generic Alexander polynomial must be a factor of $(t-1)^{s_1+s_2-1}$. As the monodromy of the Milnor fibration of the defining homogeneous polynomial $F(X, Y, Z)$ of C is periodic, this implies that $h_* : H_1(M) \to H_1(M)$ is the identity map. Thus $\Delta_C(t) = (t-1)^{b_1}$ where b_1 is the first Betti number of M. On the other hand, $b_1 = r - 1$ by Lemma 28. $\square$

7.1. **θ-Alexander polynomials and tangential Alexander polynomials.** To cover the weakness of Alexander polynomials for irreducible curves, we propose the tangential Alexander polynomials.

To cover the weakness of generic Alexander polynomials for reducible curves, we propose now is the following. Consider a plane curve with r irreducible components $C_1, \dots, C_r$ with degree $d_1, \dots, d_r$ respectively. We assume that the line at infinity is generic for C. For the generic Alexander polynomial, we have used the summation homomorphism s. This is not enough for reducible curves. We consider a fixed line at infinity $L = L_\infty$ and every possible surjective homomorphism $\theta : \pi_1(\mathbb{C}^2_L - C) \to \mathbb{Z}$ and the corresponding infinite cyclic covering $\pi_\theta : X_\theta \to \mathbb{C}^2_L - C$. The corresponding Alexander polynomial will be denoted by

$\Delta_{C,\theta}(t;L)$ and we call it *the θ-Alexander polynomial of C with respect to L.* Note that a surjective homomorphism θ factors through the Hurewicz homomorphism, and a surjective homomorphism $\theta' : H_1(\mathbb{C}^2 - C) \cong \mathbb{Z}^r \to \mathbb{Z}$. On the other hand, θ' corresponds to a multi-integer $\mathbf{m} = (m_1, \dots, m_r)$ with $\gcd(m_1, \dots, m_r) = 1$. So we denote θ as $\theta_{\mathbf{m}}$ hereafter. We denote the set of all Alexander polynomials by θ-AS$(C;L)$

$$\theta\text{-AS}(C;L) := \{\Delta_\theta(t) \,|\, \theta : \pi_1(\mathbb{C}^2 - C) \to \mathbb{Z} \text{ is surjective}\}$$

and we call θ-AS$(C;L)$ *the θ-Alexander spectrum with respect to L* of C. (In [42], it is called the Alexander set. To compare with the tangential Alexander spectrum, we changed the terminology.) We say θ-AS$(C;L)$ is *trivial* if θ-AS$(C;L) = \{(t-1)^{r-1}\}$. It is easy to see that θ-AS$(C;L)$ is a topological invariant of the complement $\mathbb{P}^2 - C$ for a generic line L.

Theorem 50. [42] *The θ-Alexander spectrum θ-AS$(C;L)$ is not trivial if there exists a component C_{i_0} for which the Alexander polynomial $\Delta_{C_{i_0}}(t;L)$ is not trivial.*

First we define *the radical* $\sqrt{q(t)}$ of a polynomial $q(t)$ to be the generator of the radical $\sqrt{(q(t))}$ of the ideal $(q(t))$ in $\mathbb{C}[t]$.

Lemma 51. *([42]) Assume that C is a reduced curve of degree d with a non-trivial Alexander polynomial $\Delta_C(t;L)$. Assume that C' is irreducible, $\pi_1(\mathbb{C}^2_L - C') \cong \mathbb{Z}$ and the canonical homomorphism $\pi_1(\mathbb{C}^2_L - C \cup C') \to \pi_1(\mathbb{C}^2_L - C) \times \pi_1(\mathbb{C}^2_L - C')$ is isomorphic. Put $D = C \cup C'$. Consider the homomorphism*

$$\theta : H_1(\mathbb{C}^2_L - D) \to \mathbb{Z},\ [g_j] \mapsto t,\ [h] \mapsto t^d.$$

Then $\Delta_{D,\theta}(t;L)$ is divisible by $\sqrt{\Delta_C(t;L)}$.

Proof. First we may assume that the presentation of the respective fundamental groups are given as

$$\pi_1(\mathbb{C}^2_L - C) = \langle g_1, \dots, g_k \,|\, R_1, \dots, R_\ell \rangle$$
$$\pi_1(\mathbb{C}^2_L - D) = \langle g_1, \dots, g_k, h \,|\, R_1, \dots, R_\ell, T_j,\ 1 \le j \le k \rangle$$

where T_j is the commuting relation: $hg_jh^{-1}g_j^{-1}$. Here h is presented by a lasso for C'. Then the image of the differential of the relation T_j by the ring homomorphism

$$\gamma_\theta : \mathbb{C}(F(k+1)) \to \mathbb{C}(\pi_1(\mathbb{C}^2_L - D)) \to \mathbb{C}[t, t^{-1}]$$

gives the raw vector v_j whose j-th component is $(t^d - 1)$, $(k+1)$-th component is $1 - t$. Thus the θ-Alexander matrix of D is given by

$$A' := \begin{pmatrix} A & O \\ (t^d - 1)E_k & (1-t)\vec{w} \end{pmatrix}, \quad O = {}^t(0, \dots, 0),\ \vec{w} = {}^t(1, \dots, 1)$$

where A is the Alexander matrix for C with respect to the summation homomorphism. Take $k \times k$ minor B of A'. If B contains at least a $(k-1) \times (k-1)$ minor of A, $\det B$ is a linear combination of the $(k-1)$-minors of A and therefore divisible by $\Delta_C(t;L)$. Assume that B

does not contain such a minor. Then any $k\times k$ minor of B is divisible by t^d-1. As $\sqrt{\Delta_C(t;L)}$ divides t^d-1 by Proposition 27, we conclude that $\sqrt{\Delta_C(t;L)}$ divides $\Delta_{D,\theta}(t;L)$. □

Assume that $k=2$ in the situation of Lemma 51. So $\pi_1(\mathbb{C}_L^2-C)$ is generated by two elements. Then each coefficients of A is divisible by $\Delta_C(t;L)$ and the Alexander polynomial $\Delta_{D,\theta}(t;L)$ is given by the greatest common divisor of 2×2 minors. Thus we have a sharper statement:

Corollary 52. *Assume that $k=2$ and $\theta$is as above. Then $\Delta_{D,\theta}(t;L)$ is given by $(t-1)\times$ $\gcd(\Delta_C(t;L),t^d-1)$*

Corollary 53. ([42]) *Assume that C is as in Lemma 51 with r irreducible components and let C' be a curve with $\pi_1(\mathbb{C}_L^2-C')=\mathbb{Z}^s$ with s is the number of irreducible components of C'. Suppose that the canonical homomorphism $\pi_1(\mathbb{C}_L^2-C\cup C')\to\pi_1(\mathbb{C}_L^2-C)\times\pi_1(\mathbb{C}_L^2-C')$ is isomorphic. Then $\Delta_{C\cup C',\theta}(t;L)$ is divisible by $\sqrt{\Delta_C(t;L)}$ for $\theta=\theta_{\mathbf{m}}$ where $\mathbf{m}=(\mathbf{u},\mathbf{v})\in\mathbb{Z}^r\times\mathbb{Z}^s$, $\mathbf{u}=(1,\dots,1)$ and $\mathbf{v}=(d,\dots,d)$.*

Proof. Suppose that we have the following presentation.

$$\pi_1(\mathbb{C}_L^2-C)=\langle g_1,\dots,g_k\,|\,R_1,\dots,R_\ell\rangle$$

Then the presentation of $\pi_1(\mathbb{C}_L^2-C\cup C')$ is given by

$$\pi_1(\mathbb{C}_L^2-C\cup C')=\langle g_1,\dots,g_k,h_1,\dots,h_s\,|\,R_1,\dots,R_\ell,T_{j,\ell},\ 1\le j\le k,\ 1\le\ell\le s\rangle$$

where $T_{j,\ell}$ is the commuting relation: $h_\ell g_j h_\ell^{-1}g_j^{-1}$. Consider the homomorphism

$$\theta_{\mathbf{m}}:H_1(\mathbb{C}_L^2-C\cup C')\cong\mathbb{Z}^r\times\mathbb{Z}^s\to\mathbb{Z},\ (\mathbf{a},\mathbf{b})\mapsto\sum_{i=1}^r a_i+d\sum_{j=1}^s b_j$$

Assume that $C'=C_1'\cup\cdots\cup C_s'$ be the irreducible decomposition. Put $D_j=C_1'\cup\cdots\cup C_j'$. We have a family of surjective homomorphisms $\theta_j:H_1(\mathbb{C}_L^2-C\cup D_j)\to\mathbb{Z}$ which give the commutative diagram:

$$\begin{array}{ccc}
H_1(\mathbb{C}_L^2-C\cup C') & \to & H_1(\mathbb{C}_L^2-C\cup D_j)\\
\big\downarrow\theta_{\mathbf{m}} & & \big\downarrow\theta_j\\
\mathbb{Z} & \xrightarrow{=} & \mathbb{Z}
\end{array}$$

Then the assertion follows from Lemma 51, by showing that $C\cup D_j$ has a non-trivial θ_j-Alexander polynomial which is divisible by $\sqrt{\Delta_C(t;L)}$, by the inductive argument on $j=1,\dots,s$. □

Now we are ready to prove the Main theorem.

Proof of Theorem 50. Assume that C has irreducible components $C_1,\dots,C_r$ and assume that an irreducible component C_{i_0} has a non-trivial Alexander polynomial $\Delta_{C_{i_0}}(t;L)$. For simplicity, we assume $i_0=1$. Put $d_1=\operatorname{degree}C_1$. Consider a canonical surjective homomorphism $\phi:\pi_1(\mathbb{C}_L^2-C)\to\pi_1(\mathbb{C}_L^2-C_1)\times\mathbb{Z}^{r-1}$. Then we consider the surjective homomorphism $\theta:H_1(\mathbb{C}_L^2-C)\to\mathbb{Z}$ which is defined by $\theta(a_1,\dots,a_r)=a_1+d_1(a_2+\cdots+a_r)$ and let $\bar\phi:\pi_1(\mathbb{C}_L^2-C)\to\mathbb{Z}$ be the associated surjection. Then $\Delta_{C,\theta}(t;L)$ is divisible by

$\Delta_{\bar\phi}(t)$ and the latter is divisible, by Corollary 53, $\sqrt{\Delta_{C_1}(t;L)}$. Thus $\Delta_{C,\theta}(t;L)$ is divisible by $\sqrt{\Delta_{C_1}(t;L)}$. ☐

7.2. **Relations among the tangential and θ-Alexander polynomials).** Let C be a curve of degree d and let $P \in C$ and let $L = T_P C$. We consider the tangential Alexander polynomial $\Delta_C(t;P)$. Let

$$\langle g_1, \dots, g_k \,|\, R_1, \dots, R_\ell \rangle$$

be a presentation of $\pi_1(\mathbb{C}_L^2 - C)$ by generators and relations. Take a generic line L_∞ for $C \cup L$ and let $\mathbb{C}^2 = \mathbb{P}^2 - L_\infty$. Then by Product theorem 13, we have

$$\pi_1(\mathbb{C}^2 - C \cup L) = \pi_1(\mathbb{C}_L^2 - C \cup L\infty) = \pi_1(\mathbb{C}_L^2 - C) \times \pi_1(\mathbb{C}_L^2 - L_\infty) = \pi_1(\mathbb{C}_L^2 - C) \times \mathbb{Z}$$

and it has a presentation:

$$\begin{aligned}\pi_1(\mathbb{C}^2 - C \cup L) = &\ \langle g_1, \dots, g_k, h, h_\infty \,|\, R_1, \dots, R_\ell, T_1, \dots, T_k, S \rangle \\ = &\ \ \langle g_1, \dots, g_k,\, h_\infty \,|\, R_1, \dots, R_\ell, T_1, \dots, T_k \rangle\end{aligned}$$

where $T_j = h_\infty\, g_j\, h_\infty^{-1}\, g_j^{-1}$ and $S = h_\infty\, h\, \omega$, ω is a big circle containing d points $L_\eta \cap C$ and L_η is a generic line in the chosen pencil of the lines. Now the tangential Alexander polynomial is associated to the surjective homomorphism

$$s : \pi_1(\mathbb{C}_L^2 - C) \to \mathbb{Z} = \langle t \rangle,\ g_i \mapsto t$$

and $\Delta_{C\cup L}(t)$ is associated with the canonical surjective homomorphism

$$s' : \pi_1(\mathbb{C}^2 - C \cup L) \to \mathbb{Z} = \langle t \rangle,\ g_i \mapsto t,\ h \mapsto t$$

Now taking $g_1, \dots, g_k,\, h_\infty$ as generators, s' corresponds to the homomorphism:

$$\theta :\ \pi_1(\mathbb{C}^2 - C \cup L) \to \mathbb{Z},\ g_i \mapsto t,\ h_\infty \mapsto t^{-d-1}$$

The last property is the result of the observation: $s'(\omega) = t^d$. Thus we have shown

Theorem 54. *Let* $\sqrt{\Delta_C(t;P)}_{d+1} := \gcd(\sqrt{\Delta_C(t;P)(t-1)}, t^{d+1}-1)$. *Then we have* $\Delta_{C\cup L}(t) = \Delta_{C\cup L_\infty,\theta}(t;L)$ *and*

$$\sqrt{\Delta_{C\cup L}(t)} \,|\, \Delta_C(t;L)(t-1), \quad \Delta_C(t;P)_{d+1} \,|\, \sqrt{\Delta_{C\cup L}(t)}$$

In other word, $\sqrt{\Delta_{C\cup L}(t)} = \sqrt{\Delta_C(t;P)}_{d+1}$.

Proof. Consider the Alexander matrix of presentation:

$$\pi_1(\mathbb{C}^2 - C \cup L) = \ \langle g_1, \dots, g_k,\, h_\infty \,|\, R_1, \dots, R_\ell, T_1, \dots, T_k \rangle$$

As we have sen in the proof of Lemma 51, the Alexander matrix of this presentation is given by

$$A' := \begin{pmatrix} A & O \\ (t^{d+1}-1)E_k & (1-t)\vec{w} \end{pmatrix}, \quad O = {}^t(0, \dots, 0),\ \vec{w} = {}^t(1, \dots, 1)$$

where A is the Alexander matrix of C with respect to L. The greatest common divisor of the $k \times k$ minors in which $k-1$ rows are from A is given by $\Delta_C(t;L) \times (t-1)$. Any other minors are divisible by $t^{d+1} - 1$. ☐

Example 55. *In [33], we have shown that there exists irreducible quintics B_5 with configuration $4A_2$, $4A_2+A_1$, A_5+2A_2, $A_5+2A_2+A_1$, E_6+2A_2, E_6+A_5, $2A_5$, A_8+A_2, $A_8+A_2+A_1$ or A_{11} has two different flex points of order 1 P, P' such that (B_5,P) and (B_5,P') are marked Zariski pairs. Recall that this implies that the sextics $\{B_5\cup T_PB_5,\, B_5\cup T_{P'}B_5\}$ are Zariski pairs. In fact their generic ALexander polynomials are given as*

$$(t^2-t+1),\quad 1$$

Strictly speaking, this implies that $\{B_5\cup T_PB_5\cup L_\infty,\, B_5\cup T_{P'}B_5\cup L_\infty\}$ is a Zariki pair (this implies $\{\{B_5\cup T_PB_5,\, B_5\cup T_{P'}B_5\}$ is also a Zariski pair.) We can also see this directly. We have seen that $(t^2-t+1)(t-1)\,|\,\Delta_{B_5}(t;P)_6$ but $\sqrt{\Delta_{B_5}(t;P')_6}=1$ by Theorem 54.

The following quintic $B_5:\ f(x,y)=0$ has A_{11} singularity at the origin and 9 flex points. Among them, the flex at $P=(0,1)$ is very important (a flex of torus type). One can easily see that $\pi_1(\mathbb{P}^2-B_5)=\mathbb{Z}/5\mathbb{Z}$ using the pencil $y=\eta$, $\eta\in\mathbb{C}$. However if we kill the relations coming from P adding the tangent line $y-1=0$, all other flex points does not gives relations to make the fundamental group abelian. In fact, $B_5\cup T_PB_5$ is a sextic of torus type [33].

$$f(x,y)=-\frac{33}{64}\,y^5+\left(\frac{7}{8}\,x+\frac{129}{64}\right)y^4+\left(-5/4\,x^2-\frac{15}{8}\,x-5/2\right)y^3+\left(\frac{15}{8}\,x^3+\frac{13}{4}\,x^2+x+1\right)y^2$$
$$+\left(-3/4\,x^4-2\,x^3-2\,x^2\right)y+x^5+x^4$$

Example 56. *We have shown in [33] that there are Zariski pairs of reducible sextics, consisting an irreducible quartic B_4 and two flex tangents, one is of torus type and the other is not of torus type. The corresponding confifurations are*

$$[3A_5+3A_1],\qquad [2A_5+2A_2+3A_1]$$

The corresponding configurations of the quartic B_4 are $[A_5]$, $[2A_2]$ respectively. These example correspond to doubly marked Zariski pairs.

For example, consider the following quartic B_4. It has two A_2 and 8 flexes. Three of then is at $P=(1,0)$, $Q=(-1,0)$, $R=(0,-1)$ and we have shown that $B_4\cup T_PB_4\cup T_QB_4$ is a sextic of torus type and $B_4\cup T_PB_4\cup T_RB_4$ is a sextic of non-torus type. Thus (B_4,P,Q), (B_4,P,R) gives a doubly marked Zariski pair. (The flex tangent line at P,Q are given by $x\mp1=0$.)

$$f(x,y)=\frac{254143}{4096}\,x^4-\frac{235}{16}\,x^3+\frac{43}{32}\,yx^3-\frac{141645}{2048}\,x^2-\frac{5029}{2048}\,y^2x^2$$
$$-\frac{10105}{1024}\,x^2y+1/32\,y^3x+\frac{235}{16}\,x-\frac{43}{32}\,xy+\frac{275}{4096}\,y^4+\frac{29147}{4096}-\frac{235}{1024}\,y^3+\frac{5029}{2048}\,y^2+\frac{10105}{1024}\,y$$

Example 57. *Let C be a smooth cubic curve. Then it has 9 flex points and there are 12 colinear 3 flex points $\{P,Q,R\}$ so that $C\cup T_PC\cup T_QC\cup T_RC$ is a sextic of torus type with $[3A_5+3A_1]$. Other choice of three flex points gives a sextic of non-torus type ([33]). This gives an example of triple marked Zariki pair.*

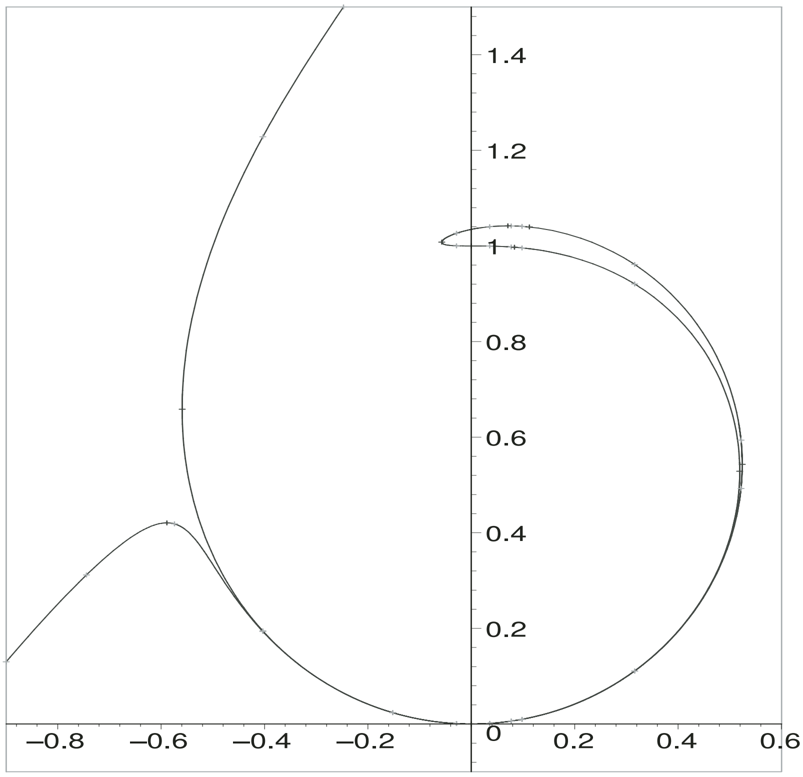

Figure 4. Quintic with A_{11}

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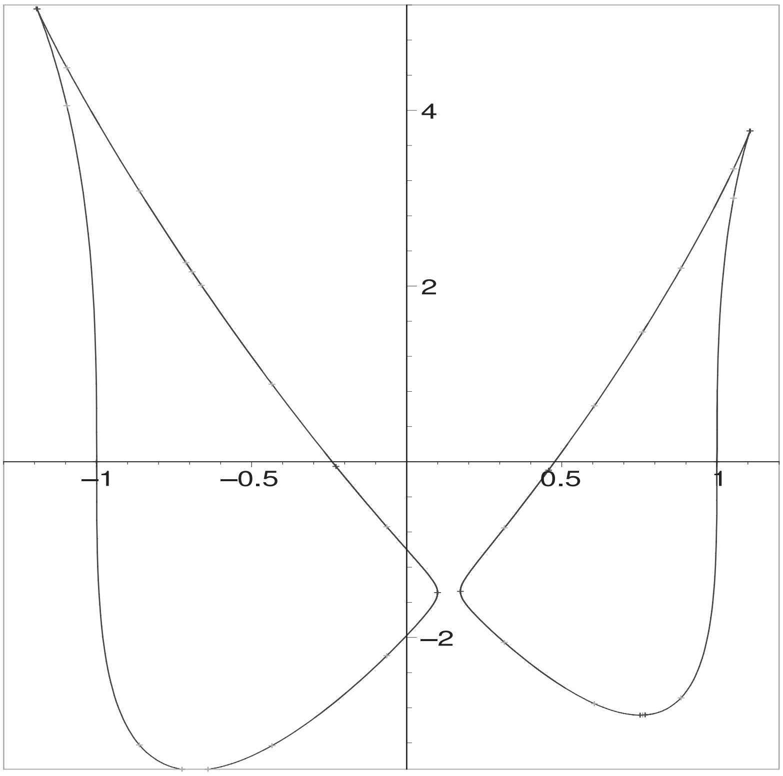

FIGURE 5. Quartic with $2A_2$

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