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Sufficiency of jets with line singularities

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SUFFICIENCY OF JETS WITH LINE SINGULARITIES

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§0. Introduction

In this paper we will study sufficiency of jets with line singularities. Let $z : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be an r-jet identified with a polynomial of degree r with $r > 2$. Let $\Sigma(z)$ denote z's critical set and assume that $\Sigma(z)$ is a 1-dimensional manifold L. After a change in coordinates, we may assume that $L = \mathbf{R} \times \{0\} \subset \mathbf{R} \times \mathbf{R}^n$. We say that z is jet with line singularities. Let $\mathcal{E}^L_{[r]}$ be the set of C^r mappings whose critical set contains L. Let $\mathcal{R}^L_0$ be the set of homeomorphism germs $h : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1}, 0)$ leaving L invariant.

Definition. *We say that a jet $z \in J^r(n+1, 1)$ is sufficient in $\mathcal{E}^L_{[r]}$ if any two f, g in $\mathcal{E}^L_{[r]}$ with $j^r f(0) = j^r g(0) = z$ are $\mathcal{R}^L_0$-equivalent*

In this paper we will give a necessary and sufficient condition for a jet to be sufficient in $\mathcal{E}^L_{[r]}$.

Before we state our main theorem, we will however put this theorem in relation with now classical results about sufficiency of jets and determinacy of mappings. Let z be a jet in $J^r(n, p)$, $\mathcal{E}_{[r]}(n, p)$ the set of C^r germs $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, R an equivalence relation on $\mathcal{E}_{[r]}(n, p)$ and E a subset of $\mathcal{E}_{[r]}(n, p)$. We will say that z is R-sufficient in E if any two $f, g \in E$ with $j^r f(0) = j^r g(0) = z$ are R-equivalent. The study of sufficiency of jets started with classical papers of Kuiper [5], Kuo [6] and [7] and Bochnak and Łojasiewicz [1]. In these papers sufficiency of r-jets in $\mathcal{E}_{[r]}(n, 1) = \mathcal{E}_{[r]}$ and $\mathcal{E}_{[r+1]}$ with respect to topological right-equivalence and sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n, p)$ with respect to $\mathcal{V}$-equivalence (two mappings f, g are $\mathcal{V}$-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic) are studied and necessary and sufficient conditions for sufficiency are given(see [11] for a detailed survey of these results). I all cases the necessary and sufficient conditon is formulated in terms of a Łojasiewicz inequality which has to be satisfied. This Łojasiewicz inequality implies that every realization of the jet is, in some sense, non-singular outside 0. If we consider an unfolding (f_t, t) of the given jet with all levels of the unfolding $\mathcal{E}_{[r]}$ (or $\mathcal{E}_{[r+1]}$) realizations of this jet, the Łojasiewicz inequality will imply that we can define vectorfields, which will have continous flows, trivializing the unfolding (in the case of $\mathcal{V}$-sufficiency the flow will trivialize the varities $f_t^{-1}(0)$). On the other hand if the Łojasiewicz inequality does not hold for the jet z, we can find two realizations of z, one which have a singularities outside 0 and one which is non-singular outside 0 and which are not topologically equivalent. What we here mean by singular and non-singular depends on the toplogical equivalence relation we consider. When we are considering topological right-equivalence among $\mathcal{E}_{[r]}$ (or

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$\mathcal{E}_{[r+1]}$) function-germs, the germs which are non-singular outside 0 are germs which are submersions outside 0, but in the case of $\mathcal{V}$-equivalence we consider germs f such that their varieties $f^{-1}(0)$ are non-singular outside 0. In [2] the author studies sufficiency of jets in $\mathcal{E}_{[r]}(n,p)$ with respect to left-topological equivalence. Again the necessary and sufficient condition is a Łojasiewicz inequality implying that every $\mathcal{E}_{[r]}$ realization of the given jet is one-to-one and also an immersion outside 0. In a forth-coming paper, [4], we will prove some sufficiency theorems with respect to topological-left-right equivalence for a restricted class of jets from the plane to the plane . Here sufficiency is characterized by Łojasiewicz inequalities giving that realizations of the jets have no worse than fold singularities outside 0 and no critical double points. Looking for necessary and sufficient condition characterizing sufficiency of jets with line-singularities, we are therefore seeking Łojasiewicz inequalities which imply that all $\mathcal{E}^L_{[r]}$-realizations of z have in some sense well-behavied singularities outside 0 relevant for $\mathcal{R}^L_0$-equivalence. Looking at the cases of sufficiency of jets with repect to topological right-equivalence, $\mathcal{V}$-equivalenc, left-equivalence and left-right-equivalence we have discussed above, we find that the non-singular behaviour or well-behaved singularities outside 0 we require of our realizations of germs are the same as those required of complex analytic-finite determined germs with respect to analytic right-equivalence, contact-equivalence, left-equivalence or left-right-equivalence. Same non-singular or nice singular behavior are also required in the case of smooth infinite-determinacy of map-germs (see [11] Theorem 2.1 and 6.1 and [3] for further details). The cases of finite determinacy of complex analytic functions or smooth-infinitely determinacy of smooth functions with line singularities are studied in [8] and [9]. The equivalences in these cases are either complex analytic or smooth right-equivalence leaving the singular set L invariant. The finite or infinite determined germs are here among those which are non-singular outside L and the singularities along L outside 0 are Morse-singularities in the direction transverse to L . In the case of sufficiency, we will therefore need Łojasiewicz inequalities which will give that every $\mathcal{E}^L_{[r]}$-realization of the jet is non-singular outside 0 and have only Morse-singularties along L outside 0 in the direction transverse to L.

We will now formulate the Łojasiewicz inequalities relevant for sufficiency of jets with line singularities. Let us identify a jet $z \in J^r(n+1,1)$ with a polynomial $P : \mathbf{R}^{n+1} \to \mathbf{R}$. Assume $\Sigma(P) = L$. Let $f \in \mathcal{E}^L_{[r]}$ with $j^r f(0) = z$. Let us denote the coordinates in $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ by $(x,y) = (x, y_1, ..., y_n)$. Since L the becomes the x-axis, and the partial derivatives of f vanish along L all the partial derivatives of the form $\dfrac{\partial^k f}{\partial x^k}$ and $\dfrac{\partial^{(k-1)}}{\partial x^{(k-1)}} \dfrac{\partial f}{\partial y_i}$ must vanish along L when $0 < k \leq r$. Especially, we get $\dfrac{\partial^k P}{\partial x^k}(0) = \dfrac{\partial^k f}{\partial x^k}(0) = 0$ and $\dfrac{\partial^{(k-1)}}{\partial x^{(k-1)}} \dfrac{\partial P}{\partial y_i}(0) = \dfrac{\partial^{(k-1)}}{\partial x^{(k-1)}} \dfrac{\partial f}{\partial y_i}(0) = 0$, and from this it is easy to see that P must have the form $P(x,y) = \sum_{1 \leq i,j \leq n} y_i y_j P_{ij}(x,y)$ where P_{ij} is a polynomial of degree $r-2$ and $P_{ij} = P_{ji}$. Let $\mathrm{Sym}(n)$ denote the symmetric $n \times n$ matrices, and let $\Lambda(n) \subset \mathrm{Sym}(n)$ denote the subset of singular matrices. Let $D^2_y P(x) \in \mathrm{Sym}(n)$ denote $n \times n$ matrix $(P_{ij}(x,0))$. (Note that allthough the $P_{ij}(x,y)$'s are not uniquely determined the matrix $D^2_y P(x)$ is determined by P beeing the Hessian matrix of P in the y-direction.) The following theorem gives necessary and sufficient conditions for a jet to be sufficient in $\mathcal{E}^L_{[r]}$.

Theorem. *The following two conditions are equivalent*

(1) z is sufficient $\mathcal{E}^L_{[r]}$.

(2) There exists a constant $C > 0$ and a neighborhood U of 0 such that

(i) $|\frac{\partial P}{\partial x}(x,y)|\|(x,y)\| + (\sum_{i=1}^{n} |\frac{\partial P}{\partial y_i}(x,y)|)\|y\| \geq C\|y\|^2\|(x,y)\|^{r-2}$ *for* $(x,y) \in U$, *and*

(ii) $dist(D^2_y P(x), \Lambda(n)) \geq C\|x\|^{r-2}$ *for* $x \in U \cap L$.

The rest of the article is organized as folows: In section 1 we will prove the necessity of the inequalities of (2) and in section 2 we will prove that the inequalities are sufficient conditions. In section 3 we will give some examples of sufficient jets with line singularities.

§1. Proof of (1)⇒(2).

Assume (2) fails. We will construct different representations of z which cannot be $\mathcal{R}^L_0$-equivalent. First assume that (i) fails. Then there exists a sequence $(x_m, y_m) \to 0$ with $y_m \neq 0$ such that

$$|\frac{\partial P}{\partial x}(x_m,y_m)|\|(x_m,y_m)\| + (\sum_{i=1}^{n} |\frac{\partial P}{\partial y_i}(x_m,y_m)|)\|y_m\| = o(\|y_m\|^2\|(x,y)\|^{r-2}).$$

We may assume that $\|(x_{m+1}, y_{m+1})\| \leq \frac{1}{2}\|(x_m,y_m)\|$. We may also assume that $|y_{m1}| \geq |y_{mi}|$ for all m and i. For each m consider the linear function h_{m1} defined by

$$h_{m1}(x,y) = \frac{1}{2y_{m1}}\frac{\partial P}{\partial y_1}(x_m,y_m) - \sum_{i=2}^{n} \frac{y_{mi}}{2y^2_{m1}}\frac{\partial P}{\partial y_i}(x_m,y_m) + \frac{1}{y^2_{m1}}\frac{\partial P}{\partial x}(x_m,y_m)(x - x_m),$$

and for $i = 2, ..n$ the constant functions

$$h_{mi}(x,y) = \frac{\partial P}{\partial y_i}(x_m,y_m)\frac{1}{y_{m1}}.$$

Let $r_m = \frac{\|(x_m,y_m)\|}{4}$ and D_m the disc $D((x_m,y_m), r_m)$. It is easy to see that for all i and m we have $|h_{mi}(x,y)| = o(\|(x,y)\|^{r-2})$ when $(x,y) \in D_m$, $|\frac{\partial h_{m1}}{\partial x}|(x,y) = o(\|(x,y)\|^{r-3})$ and all other partial derivatives of h_{mi} vanish for each i and m. A standard construction gives us that for each m there exist smooth functions p_m such that $0 \leq p_m \leq 1$, p_m vanishes outside D_m and $p_m \equiv 1$ on the smaller disq $D((x_m,y_m), \frac{1}{2}r_m)$ and such that there exists constants C_k independent of m such that $\frac{\partial^{|\alpha|} p_m}{\partial (x,y)^\alpha} \leq \frac{C_{|\alpha|}}{r_m^{|\alpha|}}$ for each multiindex α. Let us redefine each h_{mi} by putting $h_{mi} := p_m h_{mi}$. Now the sum $\sum_{m=1}^{\infty} h_{mi}$ defines a smooth map h_i on $\mathbf{R}^{n+1} - \{0\}$ such that $\frac{\partial^{|\alpha|} h_i}{\partial (x,y)^\alpha} = o(\|(x,y)\|^{r-2-|\alpha|})$ for each multiindex α. It easy to see that these inequalities allow us to extend this function to a C^{r-2} function at the origin with all derivatives vanishing at 0, and that the function $h(x,y) = \sum_{i=1}^{n} y_1 y_i h_i$ becomes a C^r function with all derivatives vanishing at 0. A straight forward calculation now shows that $f(x,y) = P(x,y) - h(x,y)$ have critical points along the sequence (x_m, y_m) and also along L. It is an easy exercise to see that we can perturb h further such that we actually can assume that f has Morse singularities along (x_m, y_m) and such that f still has critical points also along the x-axis (this can actually be done by adding a suitable smooth function which is flat at 0).

Let $c_m = f(x_m, y_m)$. Since $\{c_m\}$ is a sequence converging to 0, we may either assume that all the c_m's are distinct and $\neq 0$ or each of them are equal 0, so we can consider their union as a 0-dimensional manifold. We wish to construct a representative of P, g, such that g resticted to $\mathbf{R}^{n+1} - L$ has the sequence (c_m) as regular values. To this end, consider the map $F(x,y,a) = P(x,y) + \sum_{i=1}^{n} a_i y_i^{r+1}$. Here $(x,y) \in \mathbf{R}^{n+1} - L$ and $a = (a_1, ..., a_n) \in \mathbf{R}^n$. For $(x,y) \in \mathbf{R}^{n+1} - L$ we must have $y_i \neq 0$ for some i. Hence $\dfrac{\partial F}{\partial a_i}(x,y) = y_i^{r+1} \neq 0$. This shows that F is a submersion and therefore is transverse to the manifold $\cup c_m$. By an application of Sard's Theorem there exsits a reidual set in $\mathbf{R}^n$ such that each map F_a also is transverse to this mainifold on $\mathbf{R}^{n+1} - L$ when a is in this set. So, put $g = F_a$ for such a. Now g is another representative of P so if f and g are $\mathcal{R}_0^L$-equivalent, the set $g^{-1}(c_m) - L$ and $f^{-1}(c_m) - L$ much be homeomorphic and the germ of f at (x_m, y_m) must be C^0-right equivalent with the germ of g at some point in $g^{-1}(c_m) - L$. This is however impossible since the first germ is a Morse singularity and the other germ is non-singular.

Assume that (ii) fails. Then there exists a sequence (x_m) such that $dist(D_y^2 P(x_m), \Lambda(n)) = o(\|x_m\|^{r-2})$. We may assume that each x_m is in the same component of $L - \{0\}$. Since P is a polynomial, we must either have $D_y^2 P(x) \in \Lambda(n)$ for all x in a neighborhood of 0, or that $D_y^2 P(x) \notin \Lambda(n)$ when $x \neq 0$. In the first case we will show that we can find a polynomial representative f of P such 0 is isolated in $(D_y^2 f)^{-1}\Lambda(n)$. Again, since P is a polynomial the rank of $D_y^2 P(x)$ must be constant for $x \neq 0$ say, $k < n$. Let $I = \{i_1, .., i_k\}$ be a subsets of $\{1, ..n\}$ of cardinality k let A be an $n \times n$ symmetric matrix and $A(I)$ be the $k \times k$ submatrix of A we get by removing the lines and columns corresponding to the the index set $\{1, .., n\} - I$. It is an exercise in linear algebra to see that if A is symmetric of rank k, there exists I such that $A(I)$ is non-singular. Using this and the fact that P is algebraic we may assume that the upperleft $k \times k$ submatrix of $D_y^2 P(x)$ is non-singular for $x \neq 0$. Let $D(x)$ denote the corresponding $k \times k$ minor. Let

$$Q(x,y) = x^{r-1}(\sum_{i=k+1}^{n} y_i^2).$$

A straightforward calculation of determinants shows that

$$\det D_y^2(P+Q)(x) = x^{(n-k)(r-1)} D(x).$$

From above it follows that we can find a polynomial representative f of P such that 0 is isolated in $(D_y^2 f)^{-1}\Lambda(n)$. From continuity it is clear that the index of $D_y^2 f(x)$

is constant on each component of $L - \{0\}$. It is an easy exercise in linear algebra to show that if $A \in \Lambda(n)$, then A is infinetely close to two non-singular matrices with different indices. Since we have assumed that (ii) fails, we can therefore find a sequence $(x_m, 0) \in L - \{0\}$ such that $x_m \to 0$, and a sequence $A_m \in \mathrm{Sym}(n)$ such that $\|A_m\| = o(\|x_n\|^{r-2})$ and such that $D_y^2 P(x_m) + A_m$ is non-singular symmetric matrix chosen such that the indices of these matrices are different for m and $m+1$ (so the index is not a constant function of m for m large). Using an argument similar to one we used above, we can extend the map $(x_m, 0) \to A_m$ to a smooth map $A : \mathbf{R}^{n+1} - \{0\} \to Sym(n)$ such that $\dfrac{\partial^{|\alpha|} A}{\partial(x,y)^\alpha} = o(\|(x,y)\|^{r-2-|\alpha|})$, and we can extend it further to a C^{r-2} map on $\mathbf{R}^{n+1}$ with all derivatives vanishing at 0. Write $A(x,y) = (A_{ij}(x,y))$ and define $h(x,y) = \sum_{i,j} \frac{1}{2} y_i y_j A_{ij}(x,y)$. It is easy to see that h becomes a C^r function with all derivatives vanishing at 0. Put $g = P + h$. Then $D_y^2 g(x_m) = D_y^2 P(x_m) + A_m$. Assume f and g are $\mathcal{R}_0^L$-equivalent, then for each m there exists a point z_m in L such that the germ of f at z_m is right-equivalent with the germ of g at x_m and the equivalence will leave L invariant. Since the x_m's belong to the same component of $L - \{0\}$ and the equivalences of the germs at x_m and z_m come from the same equivalence in $\mathcal{R}_0^L$, the z_m's must also all belong to a common component of $L - \{0\}$. So for each m we have a germ of a homeomorphism H_m of form $H_m(x,y) = (h_m(x,y), k_m(x,y))$ with $k_m(x,0) = 0$, $H_m(x_m, 0) = (z_m, 0)$ and $f(h_m(x,y), k_m(x,y)) = g(x,y)$. Let us distinguish the germs of the coordinate function in L at x_m and z_m by denoting them by x and z respectively. For each x and z let g_x and f_z denote the map germs $y \to g(x,y)$ and $y \to f(z,y)$ respectively. Hence we get deformations $x \to g_x$ and $z \to f_z$ of g_{x_m} and f_{z_m} respectively. Both these deformations consist of germs which are singular at 0, and since $g_{x_m} = g_m$ and $f_{z_m} = f_m$ both are Morse function the deformations are trivial, and can be trivialized by one-parameter families of smooth diffeomorphisms of germs $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and these diffeomorphisms depend smoothly of the parameter. Redefining H_m by composing with these families in a suitable manner, we may suppose that the germs of $g(x,y)$ and $f(z,y)$ at x_m and z_m are independent of x and z respectively, so $f(z,y) = f_m(y)$ and $g(x,y) = g_m(y)$, and we still have $f \circ H_m = g$. We will now show that this is impossible. To this end we will need a lemma.

Lemma. *Consider the two non-degenerate quadratic function Q and R on $L \times \mathbf{R}^n$ defined by $Q(x, y_1, ..., y_n) = -y_1^2 - \cdots - y_r^2 + y_{r+1}^2 + \cdots + y_n^2$ and $R(x, y_1, ..., y_n) = -y_1^2 - \cdots - y_l^2 + y_{l+1}^2 + \cdots + y_n^2$ where $0 \le r < l \le n$. Then Q and R are not $\mathcal{R}_0^L$-equivalent.*

Proof. The case $n = 1$ is obvious. Assume $n > 1$. If the germs are $\mathcal{R}_0^L$-equivalent, the set-germs $Q^{-1}(a)$ and $R^{-1}(a)$ must be homeomorphic for any value a. If $r = 0$, then $Q^{-1}(a) = \varnothing$ and $R^{-1}(a) \neq \varnothing$ for $a < 0$. So these sets are not homeomorphic. The case $l = n$ is similar. If $0 < r < l < n$, it is easy to see that for $a < 0$, $Q^{-1}(a)$ and $R^{-1}(a)$ is homotopically equivalent with $\mathbf{S}^{r-1}$ and $\mathbf{S}^{l-1}$ respectively. Since these spheres have different homology, $Q^{-1}(a)$ and $R^{-1}(a)$ cannot be homeomorphic. This proves the lemma.

Let us complete the proof of (1)$\Rightarrow$(2). Since the indices of $g_m(y)$ and $g_{m+1}(y)$ are different and the indices of all $f_m(y)$'s are the same
(because the z_m's belong to the same component of $L - \{0\}$), we may assume that

the indices of $g(x,y) = g_m(y)$ and $f(z,y) = f_m(y)$ are different.We may therefore apply Morse-Lemma and suppose that f_m and g_m have the form of Q and R in the Lemma above (since they have different indices). It follows directly from the conclusion of this lemma that there exists no map H_m such that $f \circ H_m = g$.

§2. Proof of (2)⇒ (1).

Assume (2). Let $h : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be a germ of a C^r mapping with $L \subset \Sigma(h)$ and $j^r h(0) = 0$. It is obviously sufficient to prove that P and $P + h$ are $\mathcal{R}_0^L$-equivalent. Let $F(x,y,t) = (P(x,y) + th(x,y), t)$. Put $f(x,y,t) = f_t(x,y) = P(x,y) + th(x,y)$. Using Morse-Lemma, we find that h can be written in the form $h(x,y) = \sum_{1 \le i,j \le n} y_i y_j h_{ij}(x,y)$ where h_{ij} are C^{r-2} functions with $r-2$ jet equal 0 at 0. From this it is clear that

$$|\frac{\partial h}{\partial x}(x,y)| \|(x,y)\| = o(\|y\|^2 \|(x,y)\|^{r-2})$$

and that

$$(\sum_{i=1}^{n} |\frac{\partial h}{\partial y_i}(x,y)|) \|y\| = o(\|y\|^2 \|(x,y)\|^{r-2}).$$

From this and the inequality in (i) it follows that

$$|\frac{\partial f_t}{\partial x}(x,y)| \|(x,y)\| + (\sum_{i=1}^{n} |\frac{\partial f_t}{\partial y_i}(x,y)|) \|y\| \ge (C/2) \|y\|^2 \|(x,y)\|^{r-2}$$

for $t \in [0,1]$ and (x,y) in a perhaps smaller neighbourhood contained in U.

Consider the vector field $X(x,y,t)$ on $\mathbf{R}^{n+1} \times \mathbf{R}$ defined by

$$X(x,y,t) = \begin{cases} (0,0,1) - \dfrac{(0,0,1) \cdot \nabla f}{\|\nabla f\|^2} \nabla f & \text{when} \quad y \neq 0 \\ (0,0,1) \quad \text{when} \quad y = 0 \end{cases}$$

Let us consider $\mathbf{R}^{n+1} \times \mathbf{R}$ as a stratified space with $\{0\} \times \mathbf{R}$, $(L - \{0\}) \times \mathbf{R}$ and $(\mathbf{R}^{n+1} - L) \times \mathbf{R}$ as strata. We wish to see that $X(x,y,t)$ is a rugose stratified vector field in the sense of Verdier (see [10]). We have

$$\begin{aligned} &|\frac{\partial f_t}{\partial x}(x,y)| \|(x,y)\| + (\sum_{i=1}^{n} |\frac{\partial f_t}{\partial y_i}(x,y)|) \|(x,y)\| \\ &\ge |\frac{\partial f_t}{\partial x}(x,y)| \|(x,y)\| + (\sum_{i=1}^{n} |\frac{\partial f_t}{\partial y_i}(x,y)|) \|y\| \\ &\ge (C/2) \|y\|^2 \|(x,y)\|^{r-2}. \end{aligned}$$

It follows that

$$\|\nabla f(x,y,t)\| \ge (C/2(n+1)) \|y\|^2 \|(x,y)\|^{r-3}.$$

Since also $|h(x,y)| = o(\|y\|^2 \|(x,y)\|^{r-2})$, we get that

$$\|\frac{(0,0,1)\cdot\nabla f}{\|\nabla f\|^2}\nabla f\| = \frac{|h(x,y)|}{\|\nabla f(x,y,t)\|} = o(\|(x,y)\|).$$

This proves that X restricted to the strata $\{0\}\times\mathbf{R}$ and $(\mathbf{R}^{n+1}-L)\times\mathbf{R}$ satisfies Verdier'rugosity condition. That X restricted to the strata $\{0\}\times\mathbf{R}$ and $(L-\{0\})\times\mathbf{R}$ satisfies Verdier'rugosity condition is obvious. Let us now consider the strata $(L-\{0\})\times\mathbf{R}$ and $(\mathbf{R}^{n+1}-L)\times\mathbf{R}$. Given $(x_0,0,t_0)\in(L-\{0\})\times\mathbf{R}$, we need to prove that there exists a neighbourhood V around $(x_0,0,t_0)$ and a constant $C>0$, such that for every pair $(x,0,t)$ and (x',y',t') in this neighbourhood we have

$$\|X(x,0,t) - X(x',y',t')\| \leq C\|(x,0,t)-(x',y',t')\|.$$

We will need the following Lemma

Lemma. *Let* $\mathrm{gl}(n)$ *be the space of* $n\times n$ *matrices equipped with the usual Eucledean norm (by identifying* $\mathrm{gl}(n)$ *with* $\mathbf{R}^{n^2}$*). Let* $A\in Sym(n)$*. Then* $dist(A,\Lambda(n)) = |\lambda|$ *where* λ *is an eigenvalue of* A *with minimal absolute value.*

Proof. This is left to the reader.

For each (x,y,t) let $D_y^2(f_t)(x,y)$ be the linear operator on $\mathbf{R}^n$ with matrix representation $\left(\frac{\partial^2 f_t}{\partial y_i \partial y_j}(x,y)\right)$. We find that $D_y^2(f_0)(x,0) = D_y^2 P(x)$. From (ii) in the Theorem and the lemma above it follows that $\|D_y^2 P(x)v\| \geq C\|x\|^{r-2}\|v\|$ for any vector $v\in\mathbf{R}^n$. Since h is of the form $\sum_{1\leq i,j\leq n} y_i y_j h_{ij}(x,y)$, where the $r-2$ jet of each h_{ij} at 0 is 0, it is easy to see that we must have $\|(D_y^2 P(x) - D_y^2(f_t)(x,0))v\| \leq (C/2)\|x\|^{r-2}\|v\|$ for all $t\in[0,1]$ in a neighbourhood of 0 and any vector $v\in\mathbf{R}^n$. From continuity it follows that given $(x_0,0,t_0)$ we can find a neighborhood V around $(x_0,0,t_0)$ such that $\|D_y^2(f_t)(x,0) - D_y^2(f_t)(x,y)\| < (C/4)\|x\|^{r-2}$ for all (x,y,t) in this neighbourhood where this time, in abuse of notation, $\|\ldots\|$ denotes the operator norm. Let $\nabla_y f_t(x,y) = (\frac{\partial f_t}{\partial y_1}(x,y),\ldots,\frac{\partial f_t}{\partial y_n}(x,y))$. We have $\nabla_y f_t(x,y) = \int_0^1 D_y^2(f_t)(x,ty)y dt$. From above, we get that for $(x,y,t)\in V$, we have

$$\|\nabla f(x,y,t)\| \geq \|\nabla_y f_t(x,y)\| = \|\int_0^1 D_y^2(f_t)(x,ty)y dt\| \geq \|D_y^2 P(x)y\| -$$
$$\int_0^1 \|(D_y^2 P(x) - D_y^2(f_t)(x,0))y\| dt - \int_0^1 \|(D_y^2(f_t)(x,ty) - D_y^2(f_t)(x,0))y\| dt \geq$$
$$C\|x\|^{r-2}\|y\| - (C/2)\|x\|^{r-2}\|y\| - (C/4)\|x\|^{r-2}\|y\| = (C/4)\|x\|^{r-2}\|y\|.$$

Since $|h(x,y)| = o(\|y\|^2\|(x,y)\|^{r-2})$, we get that

$$\|\frac{(0,0,1)\cdot\nabla f}{\|\nabla f\|^2}\nabla f\| = \frac{|h(x,y)|}{\|\nabla f(x,y,t)\|} = o(\|y\|).$$

Verdier's rugosity condition for the strata $(L-\{0\})\times\mathbf{R}$ and $(\mathbf{R}^{n+1}-L)\times\mathbf{R}$ follows directly from this. So since X is rugose, we can integrate this vectorfield and obtain a continous flow. Since the vectorfield is tangent to every level-surface of f with t-component of form $1+o(\|(x,y)\|)$ and other components equal $o(\|(x,y)\|)$ the flow will obviously trivialize the family f_t and it will also fix the L-axis, proving that $f_0 = P$ and $f_1 = P+h$ are $\mathcal{R}_0^L$-equivalent.

§3. Examples.

We will now give examples of sufficient jets with line singularities. These examples are all given in [9], where it is shown that regarded as smooth functions they are infinitely determined among functions with line singularities. We will show that they are sufficient in $\mathcal{E}^L_{[r]}$ regarded as r-jets.

1) $P(x,y) = xy^2$. We have $|\frac{\partial P}{\partial x}|\|(x,y)\| = y^2\|(x,y)\|$, so (i) of (2) holds with $r = 3$. Furthermore $D^2_y P(x) = (x)$ hence $dist(D^2_y P(x), \Lambda(1)) = |x|$ and (ii) of (2) also holds with $r = 3$. So P is sufficient.

2) $P(x,y) = x^2y^2 + y^r$, $r \geq 3$. Let us first consider the case $r = 3$. Assume $|x| \geq |y|$. Then $|x| \geq \frac{1}{\sqrt{2}}\|(x,y)\|$, and we get $|\frac{\partial P}{\partial x}|\|(x,y)\| = 2|x|y^2\|(x,y]\| \geq \frac{2}{\sqrt{2}}y^2\|(x,y)\|^2$, and (i) holds since P is a 4-jet. Assume $|y| \geq |x|$. Then $|y| \geq \frac{1}{\sqrt{2}}\|(x,y)\|$. We have $|\frac{\partial P}{\partial y}| = |2x^2y + 3y^2| \geq \frac{3}{2}y^2$, and we get $|\frac{\partial P}{\partial y}||y| \geq \frac{3}{2\sqrt{2}}y^2\|(x,y)\|$, so (i) holds also in this case. Let $r > 3$. Then $\frac{\partial P}{\partial y} = 2x^2y + ry^{r-1}$. Assume that $2x^2 \leq \frac{r}{2}|y|^{r-2}$. Then $|x| \leq |y|$, so $|y| \geq \frac{1}{\sqrt{2}}\|(x,y)\|$. Futhermore $|\frac{\partial P}{\partial y}| \geq \frac{r}{2}|y|^{r-1}$, and therefore $|\frac{\partial P}{\partial y}||y| \geq \frac{r}{2}|y|^r \geq Cy^2\|(x,y)\|^{r-2}$ for a suitable constant C. So (i) holds since P is an r-jet. Assume that $2x^2 \geq \frac{r}{2}|y|^{r-2}$. We have $|\frac{\partial P}{\partial x}| = 2|x|y^2$. If $|x| \geq |y|$, we will then get $|\frac{\partial P}{\partial x}|\|(x,y)\| \geq \frac{2}{\sqrt{2}}y^2\|(x,y)\|^2$, and (i) holds since P is an $r \geq 4$-jet. If $|y| \geq |x|$, $|x| \geq \frac{\sqrt{r}}{2}|y|^{\frac{r-2}{2}} \geq C\|(x,y)\|^{\frac{r-2}{2}}$. So $|\frac{\partial P}{\partial x}|\|(x,y)\| \geq 2Cy^2\|(x,y)\|^{\frac{r}{2}}$. Now since $r \geq 4$, $\frac{r}{2} \leq r-2$ and (i) holds. For all r we have $D^2_y P(x) = (x^2)$ and $dist(D^2_y P(x), \Lambda(1)) = x^2$, and (ii) also holds. It follows that P is sufficient for all $r \geq 3$.

3) $P(x,y) = (y_1^2 + y_2^2)(x^2 + y_1^2 + y_2^2)$. We have

$$|\frac{\partial P}{\partial x}(x,y)|\|(x,y)\| + (\sum_{i=1}^{n}|\frac{\partial P}{\partial y_i}(x,y)|)\|y\| =$$
$$2|x|\|y\|^2\|(x,y)\| + 2(|y_1| + |y_2|)(x^2 + 2\|y\|^2)\|y\| \geq 2\|y\|^2\|(x,y)\|^2,$$

so (i) holds since P is a 4-jet. Furtermore we have $D^2_y P(x) = \begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix}$, and it is easy to see that $dist(D^2_y P(x), \Lambda(1)) = x^2$, so (ii) holds and P is sufficient.

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