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Real Algebraic Sets

Michel Coste
Université de Rennes
I.R.M.A.R.
Campus de Beaulieu
Rennes, France

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Michel Coste

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These lecture notes come from a mini-course given in the winter school of the network "Real Algebraic and Analytic Geometry" organized in Aussois (France) in January 2003. The aim of these notes is to present the material needed for the study of the topology of singular real algebraic sets via algebraically constructible functions. The first chapter reviews basic results of semialgebraic geometry, notably the triangulation theorem and triviality results which are crucial for the notion of link, which plays an important role in these notes. The second chapter presents some results on real algebraic sets, including Sullivan's theorem stating that the Euler characteristic of a link is even, and the existence of a fundamental class. The third chapter is devoted to constructible and algebraically constructible functions; the main tool which make these functions useful is integration against Euler characteristic. We give an idea of how algebraically constructible functions give rise to combinatorial topological invariants which can be used to characterize real algebraic sets in low dimensions.

These notes are still in a provisional form. Remarks are welcome!

Contents

Chapter 1

Semialgebraic Sets

In this chapter we present some basic topological facts concerning semialgebraic sets, which are subsets of $\mathbb{R}^n$ defined by combinations of polynomial equations and inequalities. One of the main properties is the fact that a compact semialgebraic set can be triangulated. We introduce the notion of link, which is an important invariant in the local study of singular semialgebraic sets. We also define a variant of Euler characteristic on the category of semialgebraic sets, which satisfies nice additivity properties (this will be useful for integration in chapter 3).

We do not give here the proofs of the main results. We refer the reader to [BR, BCR, Co1].

Most of the results presented here hold also for definable sets in o-minimal structures (this covers, for instance, sets defined with the exponential function and with any real analytic function defined on a compact set). We refer the reader to [D, Co2].

1.1 Semialgebraic sets, Tarski-Seidenberg

A *semialgebraic subset* of $\mathbb{R}^n$ is the subset of $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semialgebraic subsets of $\mathbb{R}^n$ form the smallest class $\mathcal{SA}_n$ of subsets of $\mathbb{R}^n$ such that:

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n \ ; \ P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n \ ; \ P(x) > 0\} \in \mathcal{SA}_n$.
2. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ are in $\mathcal{SA}_n$.

The fact that a subset of $\mathbb{R}^n$ is semialgebraic does not depend on the choice of affine coordinates. Some stability properties of the class of semialgebraic sets follow immediately from the definition.

1. All algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{SA}_n$. Recall that an algebraic subset is a subset defined by a finite number of polynomial equations

$$P_1(x_1, \ldots, x_n) = \ldots = P_k(x_1, \ldots, x_n) = 0 \ .$$

2. $\mathcal{SA}_n$ is stable under the boolean operations, i.e. finite unions and intersections and taking complement. In other words, $\mathcal{SA}_n$ is a Boolean subalgebra of the powerset $\mathcal{P}(\mathbb{R}^n)$.

3. The cartesian product of semialgebraic sets is semialgebraic. If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_p$, then $A \times B \in \mathcal{SA}_{n+p}$.

Sets are not sufficient, we need also maps. Let $A \subset \mathbb{R}^n$ be a semialgebraic set. A map $f : A \to \mathbb{R}^p$ is said to be semialgebraic if its graph $\Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}^p = \mathbb{R}^{n+p}$ is semialgebraic. For instance, the polynomial maps and the regular maps (i.e. those maps whose coordinates are rational functions such that the denominator does not vanish) are semialgebraic. The function $x \mapsto \sqrt{1-x^2}$ for $|x| \le 1$ is semialgebraic.

The most important stability property of semialgebraic sets is known as "Tarski-Seidenberg theorem". This central result in semialgebraic geometry is not obvious from the definition.

Theorem 1.1 (Tarski-Seidenberg) *Let A be a semialgebraic subset of $\mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, the projection on the first n coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.*

It follows from the Tarski-Seidenberg theorem that images and inverse images of semialgebraic sets by semialgebraic maps are semialgebraic. Also, the composition of semialgebraic maps is semialgebraic. Other consequences are the following. Let $A \subset \mathbb{R}^n$ be a semialgebraic set; then its closure $\mathrm{clos}(A)$ is semialgebraic and the function "distance to A" on $\mathbb{R}^n$ is semialgebraic.

A *Nash manifold* $M \subset \mathbb{R}^n$ is an analytic submanifold which is a semialgebraic subset. A *Nash map* $M \to \mathbb{R}^p$ is a map which is analytic and semialgebraic.

1.2 Cell decomposition and stratification

The semialgebraic subsets of the line are very simple to describe: they are the finite unions of points and open intervals. We cannot hope for such a simple description of semialgebraic subsets of $\mathbb{R}^n$, $n > 1$. However, we have that every semialgebraic set has a finite partition into semialgebraic subsets diffeomorphic to open boxes (i.e. cartesian product of open intervals). We give a name to these pieces:

Definition 1.2 *A* (Nash) cell *in $\mathbb{R}^n$ is a (Nash) submanifold of $\mathbb{R}^n$ which is (Nash) diffeomorphic to an open box $(-1,1)^d$ (d is the* dimension *of the cell).*

Every semialgebraic set can be decomposed into a disjoint union of Nash cells. More precisely:

Theorem 1.3 *Let $A_1, \dots, A_p$ be semialgebraic subsets of $\mathbb{R}^n$. Then there exists a finite semialgebraic partition of $\mathbb{R}^n$ into Nash cells such that each A_j is a union of some of these cells.*

This cell decomposition is a consequence of the so-called "cylindrical algebraic decomposition" (cad), which is the main tool in the study of semialgebraic sets. Actually, the Tarski-Seidenberg theorem can be proved by using cad. A cad of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many semialgebraic subsets (the cells of the cad), satisfying certain properties. We define a cad of $\mathbb{R}^n$ by induction on n.

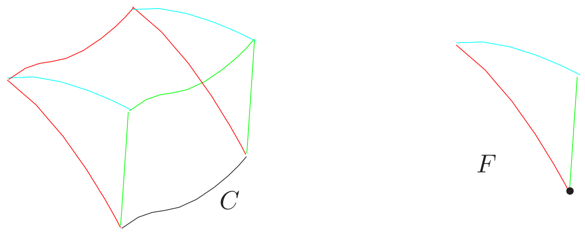


Figure 1.1: Local triviality: a neighborhood of C is homeomorphic to $C \times F$.

- A cad of $\mathbb{R}$ is a subdivision by finitely many points $a_1 < \ldots < a_\ell$. The cells are the singletons $\{a_i\}$ and the open intervals delimited by these points.

- For $n > 1$, a cad of $\mathbb{R}^n$ is given by a cad of $\mathbb{R}^{n-1}$ and, for each cell C of $\mathbb{R}^{n-1}$, Nash functions

$$\zeta_{C,1} < \ldots < \zeta_{C,\ell_C} : C \to \mathbb{R} .$$

  The cells of the cad of $\mathbb{R}^n$ are the graphs of the $\zeta_{C,j}$ and the bands in the cylinders $C \times \mathbb{R}$ delimited by these graphs.

Observe that every cell of a cad is indeed Nash diffeomorphic to an open box. This is easily proved by induction on n.

The main result about cad is that, given any finite family $A_1, \ldots, A_p$ of semialgebraic subsets of $\mathbb{R}^n$, one can construct a cad of $\mathbb{R}^n$ such that every A_j is a union of cells of this cad. This gives Theorem 1.3.

Moreover, the cell decomposition of Theorem 1.3 can be assumed to be a *stratification*: this means that for each cell C, the closure $\mathrm{clos}(C)$ is the union of C and of cells of smaller dimension. This property of incidence between the cells may not be satisfied by a cad (where the cells have to be arranged in cylinders whose directions are given by the coordinate axes), but it can be obtained after a generic linear change of coordinates in $\mathbb{R}^n$. In addition, one can ask the stratification to satisfy a local triviality condition.

Definition 1.4 *Let $\mathcal{S}$ a finite stratification of $\mathbb{R}^n$ into Nash cells; then we say that $\mathcal{S}$ is* locally (semialgebraically) trivial *if for every cell C of $\mathcal{S}$, there exist a neighborhood U of C and a (semialgebraic) homeomorphism $h : U \to C \times F$, where F is the intersection of U with the normal space to C at a generic point of C, and $h(D \cap U) = C \times (D \cap F)$ for every cell D of $\mathcal{S}$.*

Theorem 1.5 *Let $A_1, \ldots, A_p$ be semialgebraic subsets of $\mathbb{R}^n$. Then there exist a finite semialgebraic stratification $\mathcal{S}$ of $\mathbb{R}^n$ into Nash cells such that*

- *$\mathcal{S}$ is locally trivial,*

- *every A_j is a union of cells of $\mathcal{S}$.*

1.3 Connected components, dimension

Every Nash cell is obviously arcwise connected. Hence, from the decomposition of a semialgebraic set into finitely many Nash cells, we obtain:

Proposition 1.6 *A semialgebraic set has finitely many connected components, which are semialgebraic.*

The cell decomposition also leads to the definition of the dimension of a semialgebraic set as the maximum of the dimensions of its cells. This works well.

Proposition 1.7 *Let $A \subset \mathbb{R}^n$ be a semialgebraic set, and let $A = \bigcup_{i=1}^p C_i$ be a decomposition of A into a disjoint union of Nash cells C_i. The number* $\max\{\dim(C_i)\ ;\ i = 1, \ldots, p\}$ *does not depend on the decomposition. The dimension of A is defined to be this number.*

The dimension is even invariant by any semialgebraic bijection (not necessarily continuous):

Proposition 1.8 *Let A be a semialgebraic subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}^k$ a semialgebraic map (not necessarily continuous). Then* $\dim f(A) \leq \dim A$. *If f is one-to-one, then* $\dim f(A) = \dim A$.

Using a stratification, we obtain immediately the following result.

Proposition 1.9 *Let A be a semialgebraic subset of $\mathbb{R}^n$. Then* $\dim(\mathrm{clos}(A) \setminus A) < \dim(A)$.

1.4 Triangulation

First we fix the notation. A k-simplex σ in $\mathbb{R}^n$ (where $0 \leq k \leq n$) is the convex hull of $k+1$ points $a_0, \ldots, a_k$ which are not contained in a $(k-1)$-affine subspace; the points a_i are the vertices of σ. A (proper) face of σ is a simplex whose vertices form a (proper) subset of the set of vertices of σ. The open simplex $\mathring{\sigma}$ associated to a simplex σ is σ minus the union of its proper faces. A map $f : \sigma \to \mathbb{R}^k$ is called linear if $f(\sum_{i=0}^k \lambda_i a_i) = \sum_{i=0}^k \lambda_i f(a_i)$ for every $(k+1)$-tuple of nonnegative real numbers λ_i such that $\sum_{i=0}^k \lambda_i = 1$.

A finite simplicial complex in $\mathbb{R}^n$ is a finite set K of simplices such that

- every face of a simplex of K is in K,
- the intersection of two simplices in K is either empty or a common face of these two simplices.

If K is a finite simplicial complex, we denote by $|K|$ the union of its simplices. It is also the disjoint union of its open simplices. The simplicial complex K is called a simplicial subdivision of the polyhedron $|K|$. Let P be a compact polyhedron in $\mathbb{R}^n$. A continuous map $f : P \to \mathbb{R}^k$ is called piecewise linear (PL) if there is a simplicial subdivision K of P such that f is linear on each simplex of K. A PL map is obviously semialgebraic.

The first result is that every compact semialgebraic set can be triangulated. This result can be obtained by subdividing the cells of a convenient cad.

Theorem 1.10 *Let $A \subset \mathbb{R}^n$ be a compact semialgebraic set, and $B_1, \ldots, B_p$, semialgebraic subsets of A. Then there exists a finite simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $h : |K| \to A$, such that each B_k is the image by h of a union of open simplices of K.*

Moreover, one can assume that $h|_{\mathring{\sigma}}$ is a Nash diffeomorphism for each open simplex $\mathring{\sigma}$ of K.

The semialgebraic homeomorphism $h : |K| \to A$ will be called a semialgebraic triangulation of A (compatible with the B_j).

Continuous semialgebraic functions can also be triangulated, in the following sense.

Theorem 1.11 *Let $A \subset \mathbb{R}^n$ be a compact semialgebraic set, and $B_1, \ldots, B_p$, semialgebraic subsets of A. Let $f : A \to \mathbb{R}$ be a continuous semialgebraic function. Then there exists a semialgebraic triangulation $h : |K| \to A$ compatible with the B_j and such that $f \circ h$ is linear on each simplex of K.*

The method to prove this theorem is to triangulate the graph of f in $\mathbb{R}^n \times \mathbb{R}$ in a way which is "compatible" with the projection on the last factor. The fact that f is a function with values in $\mathbb{R}$ and not a map with values in $\mathbb{R}^k$, $k > 1$, is crucial here. Actually, the blowing up map $[-1, 1]^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x, xy)$ cannot be triangulated (it is not equivalent to a PL map).

An important and difficult result is the uniqueness of semialgebraic triangulation, which means the following.

Theorem 1.12 *Let P and Q be two compact polyhedra. If P and Q are semialgebraically homeomorphic, then they are PL homeomorphic.*

Let us explain in which sense this result means the uniqueness of semialgebraic triangulation. If $h : |K| \to A$ and $h' : |K'| \to A$ are semialgebraic triangulation of the compact semialgebraic set A, then $|K|$ is PL homeomorphic to $|K'|$. This is equivalent to the fact that the complexes K and K' have simplicially isomorphic subdivisions (we hope that this does not need a formal definition).

The triangulation theorem can be applied to the non compact case in the following way. Let A be a noncompact semialgebraic subset of $\mathbb{R}^n$. Up to a semialgebraic homeomorphism, we can assume that A is bounded. Indeed, $\mathbb{R}^n$ is semialgebraically homeomorphic to the open ball of radius 1 by $x \mapsto (1+\|x\|^2)^{-1/2}x$. Then one can take a triangulation of the compact semialgebraic set $\mathrm{clos}(A)$ compatible with A. So we obtain:

Proposition 1.13 *Let A be any semialgebraic set and $B_1, \ldots, B_p$ semialgebraic subsets of A. There exist a finite simplicial complex K, a union U of open simplices of K and a semialgebraic homeomorphism $h : U \to A$ such that each B_j is the image by h of a union of open simplices contained in U.*

1.5 Curve selection

The triangulation theorem allows one to give a short proof of the following.

Theorem 1.14 (Curve selection lemma) *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Let $x \in \mathrm{clos}(S)$, $x \notin S$. Then there exists a continuous semialgebraic mapping $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma((0, 1]) \subset S$.*

Proof. Replacing S with its intersection with a ball with center x and radius 1, we can assume S bounded. Then $\operatorname{clos}(S)$ is a compact semialgebraic set. By the triangulation theorem, there is a finite simplicial complex K and a semialgebraic homeomorphism $h : |K| \to \operatorname{clos}(S)$, such that $x = h(a)$ for a vertex a of K and S is the union of some open simplices of K. In particular, since x is in the closure of S and not in S, there is a simplex σ of K whose a is a vertex, and such that $h(\mathring{\sigma}) \subset S$. Taking a linear parameterization of the segment joining a to the barycenter of σ, we obtain $\delta : [0,1] \to \sigma$ such that $\delta(0) = a$ and $\delta((0,1]) \subset \mathring{\sigma}$. Then $\gamma = h \circ \delta$ satisfies the property of the theorem. $\square$

Actually, the curve in the curve selection lemma can be assumed to be analytic, or rather Nash. We explain the reason for this fact, without giving a complete proof.

The ring of germs of Nash functions at the origin of $\mathbb{R}$ can be identified (via Taylor expansion) to the ring $\mathbb{R}[[t]]_{\mathrm{alg}}$ of real algebraic series in t (algebraic means, as above, satisfying a non trivial polynomial equation $P(t, y) = 0$). Every algebraic series is actually convergent.

On the other hand, one can introduce the ring of germs of continuous semialgebraic functions $[0, \epsilon) \to \mathbb{R}$ on some small interval. This ring can be identified with the ring of real algebraic Puiseux series $\bigcup_p \mathbb{R}[[t^{1/p}]]_{\mathrm{alg}}$. Indeed, the graph of a semialgebraic function $y = f(t)$, restricted to a sufficiently small interval $(0, \epsilon)$, is a branch of a real algebraic curve $P(t, y) = 0$ which can be parameterized by a Puiseux series $y = \sigma(t)$ where σ is a root of the polynomial $P \in \mathbb{R}[t][y]$ in the field of fractions of real Puiseux series in t (for the expansion in Puiseux series of a root of a polynomial in two variables, the reader may consult [W]) . If f extends to 0 by continuity, the series σ has no term with negative power of t, and $f(0)$ is the constant term of σ. For instance, the expansion in Puiseux series of $y = \sqrt{t - t^2}$ is $y = t^{1/2} - \frac{1}{2}t^{3/2} - \frac{1}{8}t^{5/2} \ldots$. In the other direction, every algebraic Puiseux series $\sigma(t)$ is convergent: by definition, there is a positive integer p and an ordinary series $\widetilde{\sigma}$ such that $\widetilde{\sigma}(u) = \sigma(u^p)$ and, since $\widetilde{\sigma}$ satisfies the equation $P(u^p, y) = 0$, it converges.

The change of variable $t = u^p$ that we used above shows the following: if $f : [0, 1) \to \mathbb{R}$ is a continuous semialgebraic function, there is a positive integer p and a Nash function $\widetilde{f} : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $\widetilde{f}(u) = f(u^p)$ for every $u \in [0, \epsilon)$. From this fact we easily deduce an improved curve selection lemma.

Theorem 1.15 (Analytic curve selection) *For A and x as in theorem 1.14, there exists a Nash curve $\gamma : (-1, 1) \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma((0, 1)) \subset A$.*

Note also that the Puiseux series expansion gives the following fact.

Proposition 1.16 *Every semialgebraic curve $\gamma : (0, 1) \to A$, where A is a compact subset of $\mathbb{R}^n$, has a limit $\gamma(0) \in A$.*

1.6 Trivialization

A continuous semi-algebraic mapping $p : A \to \mathbb{R}^k$ is said to be *semialgebraically trivial over a semialgebraic subset* $C \subset \mathbb{R}^k$ if there is a semialgebraic set F

and a homeomorphism $h : p^{-1}(C) \to C \times F$, such that the following diagram commutes

$$\begin{tikzcd} A \supset p^{-1}(C) \arrow[rr, "h"] \arrow[dr, "p"'] & & C \times F \arrow[dl, "\text{projection}"] \\ & \mathbb{R}^k \supset C & \end{tikzcd}$$

The homeomorphism h is called a *semi-algebraic trivialization* of p over C. We say that the trivialization h is *compatible with a semialgebraic subset* $B \subset A$ if there is a semialgebraic subset $G \subset F$ such that $h(B \cap p^{-1}(C)) = C \times G$.

Theorem 1.17 (Hardt's semialgebraic triviality) *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set and $p : A \to \mathbb{R}^k$, a continuous semi-algebraic mapping. There is a finite semialgebraic partition of $\mathbb{R}^k$ into $C_1, \ldots, C_m$ such that p is semialgebraically trivial over each C_i. Moreover, if $B_1, \ldots, B_q$ are finitely many semialgebraic subsets of A, we can ask that each trivialization $h_i : p^{-1}(C_i) \to C_i \times F_i$ is compatible with all B_j.*

In particular, if b and b' are in the same C_i, then $p^{-1}(b)$ and $p^{-1}(b')$ are semialgebraically homeomorphic, since they are both semialgebraically homeomorphic to F_i. Actually we can take for F_i a fiber $p^{-1}(b_i)$, where b_i is a chosen point in C_i, and we ask in this case that $h_i(x) = (x, b_i)$ for all $x \in p^{-1}(b_i)$.

We can easily derive from Hardt's theorem a useful information about the dimensions of the fibers of a continuous semialgebraic mapping. We keep the notation of the theorem. For every $b \in C_i$:

$$\dim p^{-1}(b) = \dim F_i = \dim p^{-1}(C_i) - \dim C_i \leq \dim A - \dim C_i \ .$$

From this observation follows:

Corollary 1.18 *Let $A \subset \mathbb{R}^n$ be a semialgebraic set and $f : A \to \mathbb{R}^k$, a continuous semialgebraic mapping. For $d \in \mathbb{N}$, the set*

$$\{b \in \mathbb{R}^k \ ; \ \dim(p^{-1}(b)) = d\}$$

is a semialgebraic subset of $\mathbb{R}^k$ of dimension not greater than $\dim A - d$.

Let A be a semialgebraic subset of $\mathbb{R}^n$ and a, a nonisolated point of A: for every $\varepsilon > 0$ there is $x \in A$, $x \neq a$, such that $\|x - a\| < \varepsilon$. Let $\overline{B}(a, \varepsilon)$ (resp. $S(a, \varepsilon)$) be the closed ball (resp. the sphere) with center a and radius ε. We denote by $a * (S(a, \varepsilon) \cap A)$ the cone with vertex a and basis $S(a, \varepsilon) \cap A$, i.e. the set of points in $\mathbb{R}^n$ of the form $\lambda a + (1 - \lambda)x$, where $\lambda \in [0, 1]$ and $x \in S(a, \varepsilon) \cap A$.

Theorem 1.19 (Local conic structure) *For $\varepsilon > 0$ sufficiently small, there is a semialgebraic homeomorphism $h : \overline{B}(a, \varepsilon) \cap A \longrightarrow a * (S(a, \varepsilon) \cap A)$ such that $\|h(x) - a\| = \|x - a\|$ and* $h_{|S(a,\varepsilon)\cap A} = \mathrm{Id}$.

Proof. We apply Hardt's theorem to the continuous semialgebraic function $x \mapsto \|x - a\|$ on $\mathbb{R}^n$. For every sufficiently small positive ϵ, there is a semialgebraic trivialization of $\|x - a\|$ over $(0, \epsilon]$:

$$\begin{array}{rcl} \{x \in \mathbb{R}^n \ ; \ 0 < \|x - a\| \leq \epsilon\} & \longrightarrow & (0, \epsilon] \times S(a, \varepsilon) \\ x & \longmapsto & (\|x - a\|, \widetilde{h}(x)) \ , \end{array}$$

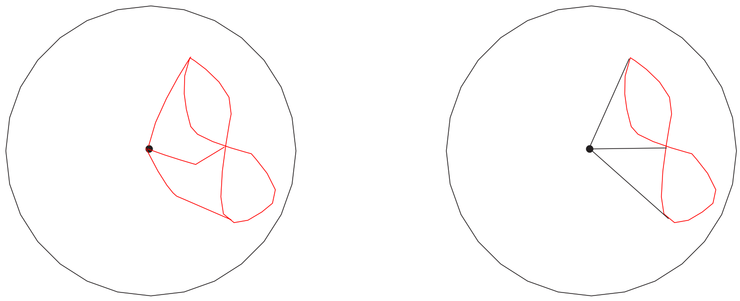


Figure 1.2: Local conic structure: the intersection with a ball is homeomorphic to the cone on the intersection with the sphere.

compatible with A and such that $\widetilde{h}|_{S(a,\varepsilon)} = \mathrm{Id}$. Then we just set $h(x) = a + (\|x-a\|/\epsilon)\,(\widetilde{h}(x) - a)$. $\square$

1.7 Links

Let A be a locally compact semialgebraic subset of $\mathbb{R}^n$ and let a be a point of A. Then we define *the link of a in A* as $\mathrm{lk}(a, A) = A \cap S(a, \epsilon)$ for $\epsilon > 0$ small enough. Of course, the link depends on ϵ, but the semialgebraic topological type of the link does not depend on ϵ, if it is sufficiently small: there is ϵ_1 such that, for every $\epsilon \leq \epsilon_1$, there is a semialgebraic homeomorphism $A \cap S(a, \epsilon) \simeq A \cap S(a, \epsilon_1)$. This is a consequence of the local conic structure theorem.

More generally, let K be a compact semialgebraic subset of A. We define $\mathrm{lk}(K, A)$, the link of K in A, as follows. Choose a proper continuous semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = K$ and $f(x) \geq 0$ for every $x \in \mathbb{R}^n$. We can take for instance for f the distance to K. Now set $\mathrm{lk}(K, A) = f^{-1}(\epsilon) \cap A$ for $\epsilon > 0$ sufficiently small.

Proposition 1.20 *The semialgebraic topological type of the link* $\mathrm{lk}(K, A)$ *does not depend on ϵ nor on f.*

Proof. By Hardt's theorem, there is a semialgebraic trivialization of f over a small interval $(0, \epsilon_1]$, compatible with A. This shows that $f^{-1}(\epsilon) \cap A$ is semialgebraically homeomorphic to $f^{-1}(\epsilon_1) \cap A$ for every ϵ in $(0, \epsilon_1]$.

Let N be a compact neighborhood of K in A. We can take a semialgebraic triangulation of $f|_N$. So we can assume that we have a finite simplicial complex L with $A = |L|$ and a nonnegative function f linear on each simplex of L. Moreover, $K = f^{-1}(0)$ is a union of closed simplices of L. Then, for $\epsilon > 0$ sufficiently small, $f^{-1}(\epsilon) \cap A$ is PL-homeomorphic to the PL-link of K in L. Hence, by uniqueness of semialgebraic triangulation and uniqueness of PL-links, we find that the semialgebraic topological type of $\mathrm{lk}(K, A)$ does not depend on the choice of f. $\square$

The preceding result shows that the semialgebraic topological type of the link is a semialgebraic invariant of the pair (A, K): if h is a semialgebraic homeomorphism from A onto B, then $\mathrm{lk}(K, A)$ and $\mathrm{lk}(h(K), B)$ are semialgebraically homeomorphic. The Euler characteristic of the link is a topological invariant of the pair. Indeed, Hardt's theorem shows that $\mathrm{lk}(K, A)$ is a retract by deformation of $f^{-1}((0, \epsilon)) = f^{-1}([0, \epsilon) \setminus K)$. Hence, we have

$$\chi(\mathrm{lk}(K, A)) = \chi(K) - \chi(A, A \setminus K) .$$

Now we proceed to define the "link at infinity" in the locally compact semialgebraic set A. We choose a nonnegative proper function g on A. If A is closed in $\mathbb{R}^n$ we can take $g(x) = \|x\|$. Otherwise we can take $\|x\|$ plus the inverse of the distance from x to the closed set $\mathrm{clos}(A) \setminus A$. Now we define $\mathrm{lk}(\infty, A)$ to be $g^{-1}(r)$ for r big enough. The semialgebraic topological type of $\mathrm{lk}(\infty, A)$ is well defined and it is a semialgebraic invariant of A.

Every locally compact noncompact semialgebraic set A has a one-point compactification in the semialgebraic category: there is a compact semialgebraic set $\dot{A}$ with a distinguished point ∞_A and a semialgebraic homeomorphism $A \to \dot{A} \setminus \{\infty_A\}$. If A is closed in $\mathbb{R}^n$ and does not contain the origin, we can take for $\dot{A}$ the union of the origin with the image of A by the inversion with respect to the sphere of radius 1. If A is not closed, we can first replace it with the graph of the function g as above. Note that $\mathrm{lk}(\infty, A) = \mathrm{lk}(\infty_A, \dot{A})$.

1.8 Borel-Moore homology and Euler characteristic

We denote by $H_i^{\mathrm{BM}}(A; \mathbb{Z}/2)$ the Borel-Moore homology of a locally compact semialgebraic set (with coefficients in $\mathbb{Z}/2$). We shall not need the definition of this homology. The following properties explain how we can compute it from the ordinary homology.

If A is compact, $H_i^{\mathrm{BM}}(A; \mathbb{Z}/2)$ coincides with the usual homology $H_i(A; \mathbb{Z}/2)$. If A is not compact, we can take an open semialgebraic embedding of A into a compact semialgebraic set B, and this embedding induces an isomorphism of $H_i^{\mathrm{BM}}(A; \mathbb{Z}/2)$ onto the relative homology group $H_i(B, B \setminus A; \mathbb{Z}/2)$. If F is a closed semialgebraic subset of A, then both F and $A \setminus F$ are locally compact and we have a long exact sequence

$$\ldots \to H_{i+1}^{\mathrm{BM}}(A \setminus F; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(F; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(A; \mathbb{Z}/2) \to H_i^{\mathrm{BM}}(A \setminus F; \mathbb{Z}/2) \to \ldots$$

For instance, we have $H_d^{\mathrm{BM}}((-1, 1)^d; \mathbb{Z}/2) \simeq \mathbb{Z}/2$ and $H_i^{\mathrm{BM}}((-1, 1)^d; \mathbb{Z}/2) = 0$ if $i \neq d$, by imbedding $(-1, 1)^d$ as the sphere minus one point.

We can define an Euler characteristic from the Borel-Moore homology. We still denote it by χ, although it would be more usual to denote it as χ_c, since it coincides with the Euler characteristic with compact support. We have $\chi(A) = \sum_i (-1)^i \dim_{\mathbb{Z}/2} H_i^{\mathrm{BM}}(A; \mathbb{Z}/2)$ for any locally compact semialgebraic set A. For instance, $\chi((-1, 1)^d) = (-1)^d$. The Euler characteristic with compact support coincides with the usual Euler characteristic on compact sets. The long exact sequence for Borel-Moore homology implies the following additivity property: if A is a locally compact semialgebraic set and F a closed semialgebraic subset of A, then $\chi(A) = \chi(F) + \chi(A \setminus F)$.

The Euler characteristic with compact support can be computed using a Nash stratification into cells.

Lemma 1.21 *Let A be a locally compact semialgebraic set and let $A = \sqcup_k C_k$ be a finite stratification into Nash cells C_k Nash diffeomorphic to $(-1,1)^{d_k}$. Then $\chi(A) = \sum_k (-1)^{d_k}$.*

Proof. Let d be the dimension of A, and let $A^{<d}$ be the union of the cells of dimension $< d$. By the properties of a stratification, $A^{<d}$ is closed in A. The cells of dimension d are the connected components of the complement $A \setminus A^{<d}$. Using this fact and the additivity property mentioned just above, we obtain

$$\chi(A) = (-1)^d \mathrm{card}(\{k\,;\; d_k = d\}) + \chi(A^{<d})\ .$$

Hence, by induction, the lemma is proved. □

The following theorem allows us to extend the Euler characteristic with compact support and its additivity property to all semialgebraic sets. This will be very convenient in Chapter 3, when we integrate against this Euler characteristic.

Theorem 1.22 *The Euler characteristic with compact support on locally compact semialgebraic sets can be extended uniquely to a semialgebraic invariant (still denoted χ) on all semialgebraic sets satisfying*

$$\begin{array}{rcll} \chi(A \sqcup B) & = & \chi(A) + \chi(B) & \textit{disjoint union} \\ \chi(A \times B) & = & \chi(A) \times \chi(B) & \textit{product.} \end{array}$$

Proof. If such an extension χ exists, it must satisfy the following property. Let $A = \sqcup_k C_k$ be a finite semialgebraic partition of a semialgebraic set into pieces C_k semialgebraically homeomorphic to $(-1,1)^{d_k}$; then $\chi(A) = \sum_k (-1)^{d_k}$. Define $\tilde{\chi}(A) = \sum_k (-1)^{d_k}$, and let us show that this alternating sum does not depend on the semialgebraic partition of A. Since any two finite semialgebraic partitions of A have a common refinement to a finite stratification into Nash cells, it suffices to show that the alternating sum does not change after refinement to a finite stratification into Nash cells. So assume we have a finite stratification of A into Nash cells D_ℓ, such that each C_k is a union of some of these cells. The D_ℓ contained in C_k form a stratification of this locally compact set, so by lemma 1.21 we have $(-1)^{d_k} = \chi(C_k) = \sum_{D_\ell \subset C_k} (-1)^{\dim D_\ell}$. It follows that $\sum_k (-1)^{d_k} = \sum_l (-1)^{\dim D_\ell}$.

We have proved that $\tilde{\chi}(A)$ is well defined, and it is clearly a semialgebraic invariant: if A is semialgebraically homeomorphic to B, then $\tilde{\chi}(A) = \tilde{\chi}(B)$. Lemma 1.21 shows that $\tilde{\chi}(A) = \chi(A)$ if A is locally compact. So we drop the tilde. The additivity for disjoint union is established by taking a finite semialgebraic partition of $A \sqcup B$ into cells such that A and B are union of cells. The product property is established by taking finite semialgebraic partitions of A and B into cells; then $A \times B$ is partitioned by the products of a cell C of A with a cell D of B, which are cells of dimension $\dim C + \dim D$. □

Chapter 2

Real Algebraic Sets

In this chapter we present a few basic facts on real algebraic varieties. First, we have to make precise what we mean by real algebraic varieties. Since we are mainly interested in the topological properties of the set of real points, we forget about complex points (which can be a disadvantage for some questions). This has the consequence that all quasi-projective varieties become affine, and so we can limit ourselves to real algebraic sets.

We recall the resolution of singularities, which we shall use in Chapter 3. We introduce the fundamental class of a real algebraic set. We state Sullivan's theorem on Euler characteristic of links (this theorem will be reproved in Chapter 3).

We do not give proofs of the results. We refer the reader to [AK, BCR].

2.1 Zariski topology, affine real algebraic varieties

The algebraic subsets of $\mathbb{R}^n$ are the closed subsets of the *Zariski topology.* If A is a subset of $\mathbb{R}^n$, we denote by $\overline{A}^Z$ its Zariski closure, i.e. the smallest algebraic subset containing A.

Let X be an algebraic subset of $\mathbb{R}^n$. We denote by $\mathcal{P}(X)$ the ring of polynomial functions on X. If U is a Zariski open subset of X, we denote by $\mathcal{R}(U)$ the ring of regular functions on U; this is the ring of quotients P/Q where P and Q are polynomial functions on X and Q has no zero on U. A regular map $f : U \to V$ between Zariski open subsets of real algebraic sets is a map whose coordinates are regular functions. A biregular isomorphism $f : U \to V$ is a bijection such that both f and f^{-1} are regular.

We can define real algebraic varieties by gluing Zariski open subsets of real algebraic sets along biregular isomorphisms. This is best formalized by using the language of ringed spaces. The rings $\mathcal{R}(U)$, for U Zariski open subset of a real algebraic set X, form a sheaf on X equipped with its Zariski topology, the sheaf of regular functions that we denote by $\mathcal{R}_X$. A ringed space isomorphic to $(X, \mathcal{R}_X)$ is called a *real affine algebraic variety.*

It must be stressed that this definition of real algebraic variety is different from the usual definition of algebraic variety over $\mathbb{R}$ (separated reduced scheme of finite type over $\mathbb{R}$). We forget the complex points when we consider real algebraic varieties. For instance the torus T embedded in three-dimensional affine space with equation $(x^2+y^2+z^2+3)^2-16(x^2+y^2)=0$ is non singular as real algebraic variety, but not as algebraic variety over $\mathbb{R}$: it has a singular point $(0,0,i\sqrt{3})$. There are more morphisms of real algebraic varieties than of varieties over $\mathbb{R}$. For instance the torus T is isomorphic to the product of unit circles $S^1 \times S^1$ as real algebraic varieties, but not as algebraic varieties over $\mathbb{R}$.

The fact that more morphisms are allowed explains why every quasi-projective real algebraic variety (that is, Zariski open in a projective real algebraic set) is affine. So in the following we shall restrict our attention to real algebraic sets.

The first step is to show that the real projective space is affine.

Theorem 2.1 *The real projective space $P_n(\mathbb{R})$ is an affine variety: it is biregularly isomorphic to a real algebraic set in $\mathbb{R}^{(n+1)^2}$.*

Proof. We embed $P_n(\mathbb{R})$ in $\mathbb{R}^{(n+1)^2}$ by the morphism

$$\begin{array}{ccc} P_n(\mathbb{R}) & \hookrightarrow & \mathbb{R}^{(n+1)^2} \\ (x_0:\ldots:x_n) & \longmapsto & \left(\dfrac{x_j x_k}{\sum x_i^2}\right)_{0\le j,k\le n} \end{array} .$$

The image of the embedding is the real algebraic set V of points $(y_{j,k})_{0\le j,k\le n} \in \mathbb{R}^{(n+1)^2}$ such that $\sum y_{j,j}^2 = 1$, $y_{j,k} = y_{k,j}$ and $\sum_k y_{j,k} y_{k,\ell} = y_{j,\ell}$ (these equations mean that the $y_{j,k}$ are the entries of the matrix of an orthogonal projection on a line in $\mathbb{R}^{n+1}$. The inverse image of a point $(y_{j,k}) \in V$ such that $y_{j,j} \neq 0$ is $(y_{0,j}:y_{1,j}:\ldots:y_{n,j})$. □

The second step is to show that Zariski open subsets of real affine algebraic varieties are affine. Indeed, the complement of the algebraic subset defined by polynomial equations $P_1 = \ldots = P_k = 0$ in an algebraic set X is the principal open subset defined by $P_1^2 + \cdots + P_k^2 \neq 0$. This open subset is biregularly isomorphis to the real algebraic set of couples $(x,y) \in X \times \mathbb{R}$ such that $y(P_1^2(x) + \cdots + P_k^2(x)) = 1$.

We give two results concerning the dimension of algebraic sets.

Theorem 2.2 *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Its dimension as a semialgebraic set (cf. 1.7) is equal to the dimension, as an algebraic set, of its Zariski closure $\overline{S}^Z$. In particular, if $V \subset \mathbb{R}^n$ is an irreducible algebraic set, its dimension as a semialgebraic set is equal to the transcendence degree over $\mathbb{R}$ of the field of fractions of $\mathcal{P}(V)$).*

It is sufficient to prove the theorem for a cell $C \subset \mathbb{R}^n$ of a cylindrical algebraic decomposition. The proof is by induction on n.

Proposition 2.3 *Let A be a semialgebraic subset of $\mathbb{R}^n$. Then* $\dim(\overline{A}^Z) = \dim(A)$.

2.2 One-point compactification of real algebraic sets

Proposition 2.4 *Let X be a non compact algebraic set. Then there exists a compact algebraic set $\dot{X}$ with a distinguished point ∞_X such that X is biregularly isomorphic to $\dot{X} \setminus \{\infty_X\}$.*

Proof.
We may assume that $X \subset \mathbb{R}^n$ and the origin $0 \in \mathbb{R}^n$ does not belong to X. The inversion mapping $i : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, $i(x) = x/\|x\|^2$, is a biregular isomorphism. Thus, $i(X)$ is a Zariski closed subset of $\mathbb{R}^n \setminus \{0\}$, and $\dot{X} = i(X) \cup \{0\}$, which is the closure of $i(X)$ in the Euclidean topology, is a compact algebraic subset of $\mathbb{R}^n$. $\square$

For instance taking the algebraic set V defined by the equation $z^2 = xy^2(x^2 + y^2 + 2x)^2$, we translate it so that the origin does not belong to V, take its image by the inversion with respect to the unit sphere and add the origin to find the algebraic one-point compactification $\dot{V}$ defined by the equation $(z+\rho)^2\rho^5 = xy^2(x^2+y^2+2x\rho)^2$, where $\rho = x^2+y^2+z^2$.

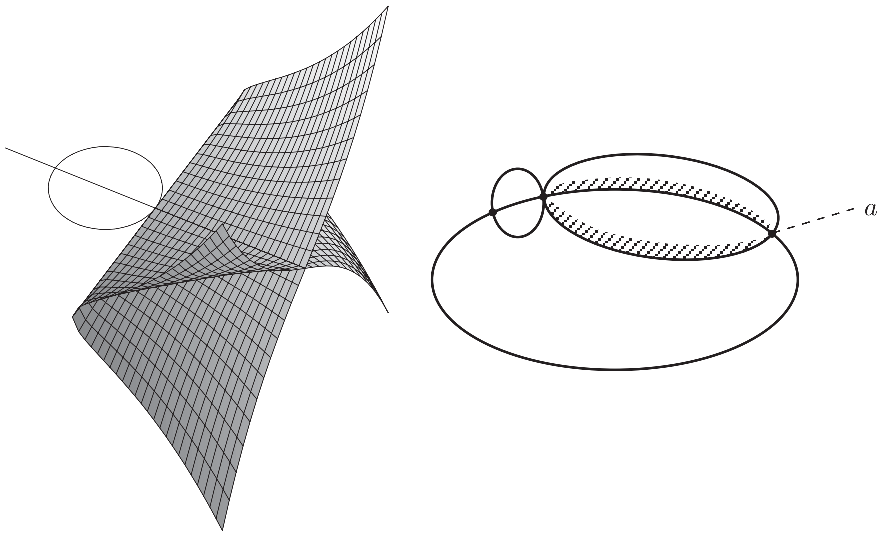


Figure 2.1: The real algebraic set V and its compactification $\dot{V}$ obtained by adding the point a.

2.3 Nonsingular algebraic sets. Resolution

A point x of a real algebraic set $X \subset \mathbb{R}^n$ is *nonsingular* if there are polynomials $P_1, \ldots, P_k$ and a Zariski neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U = \{y \in U \ ; \ P_1(y) = \ldots = P_k(y) = 0\}$ and the gradients of P_i at x are linearly independent. This amounts to say that the ring of germs of regular functions $\mathcal{R}_{X,x}$ is a regular local ring. A real algebraic set X is nonsingular if all its points are nonsingular.

The following important result shows that nonsingular real algebraic sets have no specific topological properties.

Theorem 2.5 (Nash - Tognoli) *Every compact smooth manifold is diffeomorphic to a nonsingular real algebraic set.*

We recall that a singular real algebraic set can be made nonsingular by a sequence of blowups. This is Hironaka's resolution of singularities.

Theorem 2.6 (Hironaka) *Let X be a real algebraic set. Then there exists a nonsingular real algebraic set $\widetilde{X}$, a proper regular map $\pi : \widetilde{X} \to X$ and an algebraic subset Y of X of smaller dimension such that $\pi|_{\widetilde{X}\setminus\pi^{-1}(Y)}$ is a biregular isomorphism onto $X \setminus Y$. Moreover, we can ask that $\pi^{-1}(Y)$ is a divisor with normal crossings of $\widetilde{X}$.*

Nash-Tognoli theorem and Hironaka's resolution of singularities led S. Akbulut and H. King to introduce "resolution towers" in order to characterize topologically singular real algebraic sets. A *topological resolution tower* is a collection of compact smooth manifolds $(V_i)_{i=0,\dots,n}$, each V_i with a collection $\mathcal{A}_i = \bigsqcup_{j<i} \mathcal{A}_{j,i}$ of codimension 1 smooth manifolds with normal crossings, together with collection of maps $(p_{j,i} : V_{j,i} \to V_j)_{0<j<i<n}$, where $V_{j,i} \subset V_i$ is the union of the submanifolds in $\mathcal{A}_{j,i}$, satisfying certain conditions that we shall not state explicitly. The *realization* of a topological resolution tower is the result of the gluing of all V_i's along the maps $p_{j,i}$.

Every real algebraic set is the realization of a topological resolution tower; this is a consequence of the resolution of singularities. We describe in Figure 2.2 a topological resolution tower for the algebraic set $\dot{V}$. It consists of a Klein bottle, two circles and three points. The letters on the drawing indicate how the gluing is done. For instance, the curve labeled $[a, b]$ on the Klein bottle is folded along the segment $[a, b]$ in the realization; the curves labeled a and b collapse to the corresponding points.

There are some difficulties in putting an algebraic structure on the realization of a topological resolution tower. This approach is fully successful in dimension up to 3: Akbulut and King have proved that the compact real algebraic set of dimension at most 3 coincide exactly with the realizations of topological towers of the same dimension. We shall return in chapter 3 to the topological characterization of real algebraic sets of dimension ≤ 3.

2.4 Fundamental class

Proposition 2.7 *Let X be a compact real algebraic set of dimension d, and $\Phi : |K| \to X$, a semi-algebraic triangulation of X. The sum of all d-simplices of K is a cycle with coefficients in $\mathbb{Z}/2$, representing a nonzero element of $H_d(X; \mathbb{Z}/2)$. This element, which is independent of the choice of the triangulation, is called the* fundamental class *of X and is denoted by $[X]$.*

The sum of the d-simplices of K is a $\mathbb{Z}/2$-cycle if and only if every $(d-1)$-simplex σ of K is the face of an even number of d-simplices. This can be proved by taking a generic affine subspace normal to the image $\Phi(\sigma)$. The intersection of this transversal with X is a real algebraic curve, and one is reduced to proving

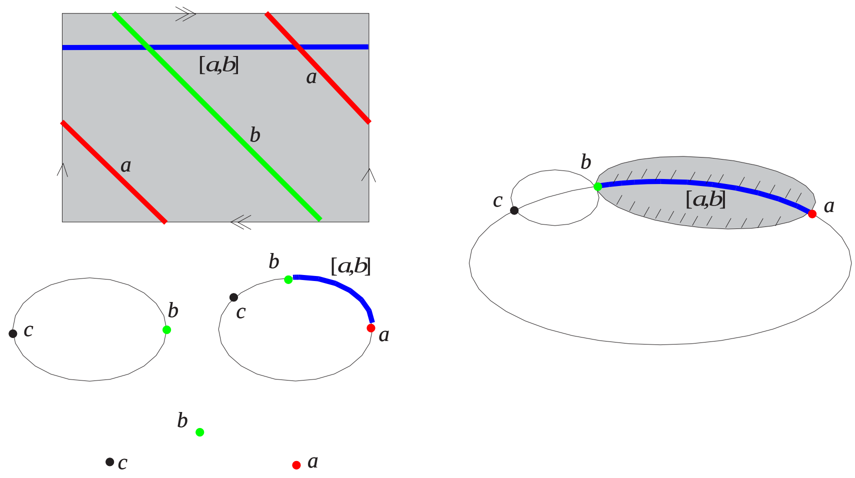


Figure 2.2: A topological resolution tower for $\dot{V}$.

that the link of a point in a real algebraic curve consists of an even number of points.

The fact that the fundamental class does not depend on the triangulation can be proved by noting that, for every point x in X which is nonsingular in dimension d, the image of $[X]$ in $H_d(X, X\setminus x; \mathbb{Z}/2) \simeq \mathbb{Z}/2$ is the nonzero element.

If X is a non compact real algebraic set of dimension d, its fundamental class $[X]$ can be defined in the Borel-Moore homology group $H_d^{\mathrm{BM}}(X;\mathbb{Z}/2)$ as follows: one takes a one-point algebraic compactification $\dot{X}$ (identified with $X \cup \{\infty_X\}$) of X, and $[X]$ is the image of $[\dot{X}] \in H_d(\dot{X};\mathbb{Z}/2)$ by the mapping

$$H_d(\dot{X};\mathbb{Z}/2) \longrightarrow H_d(\dot{X}, \infty_X;\mathbb{Z}/2) = H_d^{\mathrm{BM}}(X;\mathbb{Z}/2)\ .$$

2.5 Euler sets

Theorem 2.8 (Sullivan) *Let X be a real algebraic set. For every $x \in X$, the Euler characteristic $\chi(\mathrm{lk}(x,X))$ of the link of x in X is even*

We shall give in chapter 3 a proof of this theorem (and actually of a more general result).

Definition 2.9 *Let A be a locally compact semialgebraic set. Then A is said to be* Euler *if, for every $x \in A$, the Euler characteristic of the link of x in A is even.*

If A is a non compact Euler set, then its one-point compactification $\dot{A}$ is also Euler. We shall give a proof of this fact in chapter 3.

Every Euler set A of dimension d has a fundamental class $[A]$ in $H_d(A,\mathbb{Z}/2)$ (in $H_d^{\mathrm{BM}}(A,\mathbb{Z}/2)$ if A is non compact). As for algebraic sets, the fundamental class can be obtained by taking the sum of all simplices of dimension d in a

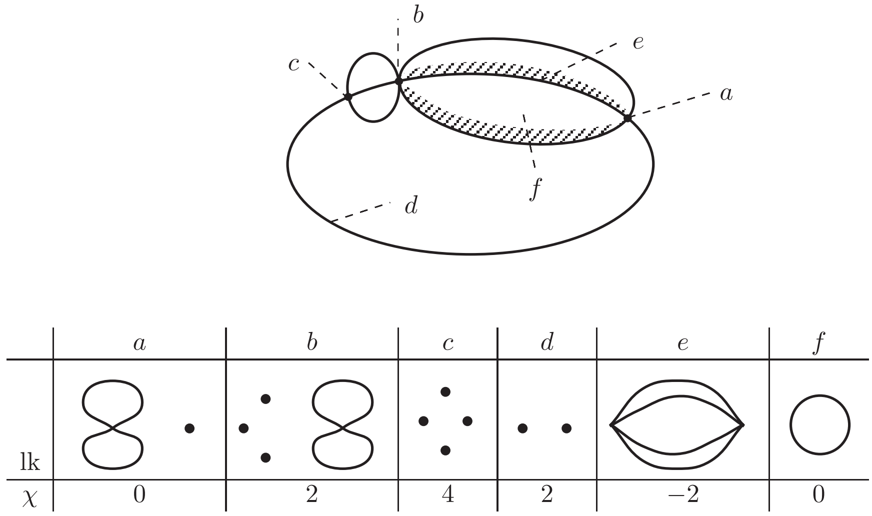


Figure 2.3: The different possible links in $\dot{V}$ and their Euler characteristics .

triangulation of A. The Euler condition implies that every $(d-1)$-simplex is the face of an even number of d-simplices.

In dimension ≤ 2, the Euler condition suffices to characterize topologically the real algebraic sets.

Theorem 2.10 (Akbulut-King, Benedetti-Dedo) *Let A be an Euler set of dimension at most 2. Then A is homeomorphic to a real algebraic set.*

Chapter 3

Constructible Functions

In this chapter we present the theory of constructible and algebraically constructible functions and its application to the topology of singular real algebraic sets. This theory, due to C. McCrory and A. Parusiński, is developed in the papers [MP1, MP2, MP3, PS]. The most important tool is the integration against Euler characteristic, which can be found in a paper by O. Viro [V]. Here we use a variant of Euler characteristic which was introduced in Chapter 1; its nice additivity properties allow us to use integration rather easily.

3.1 The ring of constructible functions on a semialgebraic sets

Let S be a semialgebraic set.

Definition 3.1 *A* constructible function *on S is a function $\varphi : S \to \mathbb{Z}$ which takes finitely many values and such that, for every $n \in \mathbb{Z}$, $\varphi^{-1}(n)$ is a semialgebraic subset of S.*

In other words, a constructible function φ is a function that can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i \mathbf{1}_{X_i} \ , \tag{3.1}$$

where, for each $i \in I$, m_i is an integer and $\mathbf{1}_{X_i}$ is the characteristic function of a semialgebraic subset X_i of S. If φ is a constructible function on S, the $\varphi^{-1}(n)$ form a finite semialgebraic partition of S. Hence, there exists a semialgebraic triangulation of S *compatible with* φ. This means that there is a finite simplicial complex K and a semialgebraic homeomorphism θ from a union U of open simplices of K onto U such that $\varphi \circ \theta$ is constant on each open simplex contained in U.

The sum and the product of two constructible functions on S are again constructible. The constructible functions on S form a commutative ring that we shall denote by $F(S)$. If $\varphi : S \to \mathbb{Z}$ is a constructible function, we define its *support* to be $\{x \in S \ ; \ \varphi(x) \neq 0\}$.

3.2 Integration and direct image

In this chapter we take for χ the Euler characteristic on semialgebraic sets which was defined in section 1.8. This χ is characterized by the following properties:

- it coincides for compact semialgebraic sets with the usual Euler characteristic,
- it satisfies the additivity property $\chi(A \sqcup B) = \chi(A) + \chi(B)$ for disjoint unions,
- it is invariant by semialgebraic homeomorphism.

It follows from these properties that the χ we use coincide with Euler characteristic with compact support (or Euler characteristic for Borel-Moore homology) for locally compact semialgebraic sets, in particular for real algebraic set. Moreover, it also satisfies $\chi(X \times Y) = \chi(X) \times \chi(Y)$.

Definition 3.2 *Let φ be a constructible function on S. The* Euler integral *of φ over a semialgebraic subset X of S is*

$$\int_X \varphi \, d\chi = \sum_{n \in \mathbb{Z}} n \chi(\varphi^{-1}(n) \cap X) .$$

If we have a representation φ as in 3.1, then by additivity of χ we obtain

$$\int_S \varphi \, d\chi = \sum_{i \in I} m_i \chi(X_i) .$$

If φ has relatively compact support we can assume that all X_i are compact, and then $\chi(X_i)$ is the usual Euler characteristic.

Definition 3.3 *Let $f : S \to T$ be a continuous semialgebraic map and φ a constructible function on S. The* pushforward *$f_*\varphi$ of φ along f is the function from T to $\mathbb{Z}$ defined by*

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi \, d\chi . \tag{3.2}$$

Proposition 3.4 *The pushforward of a constructible function is constructible.*

Proof. Assume $\varphi = \sum_{i \in I} m_i \mathbf{1}_{X_i}$. By Hardt's theorem 1.17, there is a finite semialgebraic partition $T = \bigcup_{j \in J} Y_j$ such that, over every Y_j, there is a trivialization of f compatible with all X_i. Then $f_*\varphi$ is constant on each Y_j. □

A continuous semialgebraic map $f : S \to T$ induces a morphism of additive groups $f_* : F(S) \to F(T)$. It also induces a morphism of rings $f^* : F(T) \to F(S)$ defined by $f^*(\varphi) = \varphi \circ f$.

Theorem 3.5 (Fubini's theorem) *Let $f : S \to T$ be a semialgebraic map and φ a constructible function on S. Then*

$$\int_T f_*\varphi \, d\chi = \int_S \varphi \, d\chi . \tag{3.3}$$

Proof. We keep the notation of the proof of the preceding proposition. Choose $y_j \in Y_j$ for each $j \in J$. Then, for every $i \in I$, $f^{-1}(Y_j) \cap X_i$ is semialgebraically homeomorphic to $Y_j \times (f^{-1}(y_j) \cap X_i)$. Hence,

$$\begin{aligned}
\int_T f_*\varphi \, d\chi &= \sum_{j\in J} (\chi(Y_j)\, f_*\varphi(y_j)) \\
&= \sum_{j\in J} \left[\chi(Y_j) \sum_{i\in I} \big(m_i\, \chi(f^{-1}(y_j) \cap X_i)\big) \right] \\
&= \sum_{i\in I} \left(m_i \sum_{j\in J} \chi(f^{-1}(Y_j) \cap X_i) \right) \\
&= \sum_{i\in I} (m_i\, \chi(X_i)) = \int_S \varphi \, d\chi \, .
\end{aligned}$$

□

Corollary 3.6 *Let $f : S \to T$ and $g : T \to U$ be semialgebraic maps. Then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Just apply Fubini to $\int_{g^{-1}(z)} f_*(\varphi)\, d\chi$. □

3.3 The link operator

Definition 3.7 *Let φ be a constructible function on the semialgebraic set S. The* link *of φ is the function $\Lambda\varphi : S \to \mathbb{Z}$ defined by*

$$\Lambda\varphi(x) = \int_{\mathrm{lk}(x,S)} \varphi \, d\chi \, . \tag{3.4}$$

We can assume in this section that S is a locally compact semialgebraic set in order to agree with the definition of the link given in section 3.7. Actually, we can replace S with its closure in the affine space, and extend φ by 0 to the closure.

Assume $\varphi = \sum_{i\in I} m_i \mathbf{1}_{X_i}$. Since $\mathrm{lk}(x, S)$ together with the intersections $X_i \cap \mathrm{lk}(x, S) = \mathrm{lk}(x, X_i)$ are well defined up to a semialgebraic homeomorphism (see section 3.7), the link $\Lambda\varphi$ is well defined.

Proposition 3.8 *The link of a constructible function is a constructible function. The link operator $\varphi \mapsto \Lambda\varphi$ is a homomorphism from the additive group of $F(S)$ to itself.*

Proof. Choose a semialgebraic triangulation of S compatible with the constructible function φ. Then $\Lambda\varphi$ is constant on the image of each open simplex by the triangulation. The second part of the proposition follows from the additivity of the integral. □

It will be useful to have some examples at hand. Let σ be an open simplex in a simplicial complex K, and $\overline{\sigma}$ its closure. We have

$$\Lambda \mathbf{1}_\sigma = (-1)^{d-1} \mathbf{1}_{\overline{\sigma}} + \mathbf{1}_\sigma \quad \text{and} \quad \Lambda \mathbf{1}_{\overline{\sigma}} = \mathbf{1}_{\overline{\sigma}} + (-1)^{d-1} \mathbf{1}_\sigma \, . \tag{3.5}$$

From the equalities 3.5, one can deduce several properties by using triangulations compatible with constructible functions. Define another operator Ω on constructible functions by $\Omega\varphi = 2\varphi - \Lambda\varphi$. Then:

- $\Omega\Lambda = \Lambda\Omega = 0$.
- If the support of the constructible function φ has dimension at most d and d is even (resp. odd), then the support of $\Lambda\varphi$ (resp. $\Omega\varphi$) has dimension at most $d-1$.

Proposition 3.9 *The link operator commutes with proper pushforward. If $f : S \to T$ is a continuous proper semialgebraic map and $\varphi : S \to \mathbb{Z}$ a constructible function, then $\Lambda(f_*\varphi) = f_*(\Lambda\varphi)$.*

Proof. If y is a point of T, then $f^{-1}(y)$ is compact, the link of $f^{-1}(y)$ in S is well defined and $\mathrm{lk}(f^{-1}(y), S) = f^{-1}(\mathrm{lk}(y, T))$. Hence, by Fubini's theorem 3.5,

$$\Lambda(f_*\varphi)(y) = \int_{\mathrm{lk}(f^{-1}(y),S)} \varphi \, d\chi \, .$$

The conclusion follows from the formula 3.6 of the next lemma, with $Y = f^{-1}(y)$. □

Lemma 3.10 *Let Y be a compact semialgebraic subset of a semialgebraic set S, and $\varphi : S \to \mathbb{Z}$ a constructible function. Then*

$$\int_{\mathrm{lk}(Y,S)} \varphi \, d\chi = \int_Y \Lambda\varphi \, d\chi \, . \tag{3.6}$$

Proof. Using a semialgebraic triangulation of S compatible with Y and φ, we can assume that S is a union of open simplices of a finite simplicial complex K, Y a union of closed simplices and φ is constant on open simplices. By additivity it suffices to prove the formula 3.6 for $\varphi = \mathbf{1}_\sigma$, where σ is an open simplex of K.

By subdivision of K, we can assume that for every open simplex σ the intersection $\overline{\sigma} \cap Y$ is a closed face $\overline{\tau}$ (possibly empty) of $\overline{\sigma}$. It follows that $\mathrm{lk}(Y, S) \cap \sigma$ is semialgebraically homeomorphic to an open $(d-1)$-cell if $\sigma \cap Y = \emptyset$ and $\overline{\sigma} \cap Y \neq \emptyset$, and empty otherwise. Hence, we have

$$\int_{\mathrm{lk}(Y,S)} \mathbf{1}_\sigma \, d\chi = \chi(\mathrm{lk}(Y, S) \cap \sigma) = \begin{cases} (-1)^{d-1} & \text{if } \sigma \cap Y = \emptyset \text{ and } \overline{\sigma} \cap Y \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce from this and the formula 3.5 that

$$\int_{\mathrm{lk}(Y,S)} \mathbf{1}_\sigma \, d\chi = \int_Y \Lambda \mathbf{1}_\sigma \, d\chi \, ,$$

which completes the proof of the lemma □

Corollary 3.11 *Let $\varphi : S \to \mathbb{Z}$ be a constructible function on a compact semialgebraic set S. Then $\int_S \Lambda\varphi \, d\chi = 0$.*

Proof. Apply proposition 3.9 to the map from S to a point. □

3.4 Algebraically constructible functions

Definition 3.12 *Let V be a real algebraic set. An* algebraically constructible function *on V is a constructible function $\varphi : V \to \mathbb{Z}$ which can be written as a finite sum*

$$\varphi = \sum_{i \in I} m_i (f_i)_*(\mathbf{1}_{W_i}) \ , \tag{3.7}$$

where f_i are regular mappings from real algebraic sets W_i to V.

Algebraically constructible functions on a real algebraic set V form a ring denoted by $A(V)$.

In other words, the algebraically constructible functions form the smallest class of constructible functions on algebraic sets containing the constant functions and stable by pushforward along regular mappings. Phrased differently, the algebraically constructible functions are the functions $V \ni x \mapsto \chi(f^{-1}(x))$, where $f : W \to V$ is a regular mapping.

Lemma 3.13 *Let φ be an algebraically constructible function on a real algebraic set V. Then there exists a representation 3.7 of φ where*

1. *all f_i are proper regular mappings*
2. *all W_i nonsingular.*

Proof. We begin with condition 1. Consider a regular map $f : W \to V$. Replacing W with the graph of f, one can assume that W is a real algebraic subset of $\mathbb{R}^n \times V$ and that f is the projection on V. Now embed $\mathbb{R}^n \times V$ in $S^n \times V$ using the inverse stereographic projection which is a biregular isomorphism between $\mathbb{R}^n$ and the sphere $S^n \subset \mathbb{R}^{n+1}$ minus its north pole $P = (0, \dots, 0, 1)$. Then set $W' = W \cup (\{P\} \times V)$ and denote by $p : W' \to V$ the projection, which is proper. Then $f_*(\mathbf{1}_W) = p_*(\mathbf{1}_{W'}) - \mathbf{1}_V$.

Now we realize condition 2. We use the resolution of singularities and an induction on dimension. We start with a representation

$$\varphi = \sum_{i \in I} m_i (f_i)_*(\mathbf{1}_{W_i}) \ ,$$

where all f_i are proper regular maps. Let d be the maximum of the dimensions of those among the W_i which are singular (assume that these are $W_1, \dots, W_k$). Then take a resolution of singularities $\pi_i : \widetilde{W_i} \to W_i$ for $i = 1, \dots, k$. There are algebraic subsets $Z_i \subset W_i$ such that π_i is a biregular isomorphism from $\widetilde{W_i} \setminus \pi_i^{-1}(Z_i)$ onto $W_i \setminus Z_i$, and $\pi_i^{-1}(Z_i)$ and Z_i are of dimension $< d$. Hence, we have

$$(f_i)_*(\mathbf{1}_{W_i}) = (f_i \circ \pi_i)_*(\mathbf{1}_{\widetilde{W_i}}) - (f_i \circ \pi_i)_*(\mathbf{1}_{\pi_i^{-1}(Z_i)}) + (f_i)_*(\mathbf{1}_{Z_i}) \ ,$$

and in this way we have decreased the maximum dimension of singular algebraic sets appearing in the representation of φ. Notice that $(f_i \circ \pi_i)$ is proper. □

The behavior of algebraically constructible functions under the link operator is particularly interesting. As we shall see, it encompasses many local topological properties of real algebraic sets.

Theorem 3.14 *Let φ be an algebraically constructible function on a real algebraic set V. Then $\Lambda\varphi$ takes only even values, and $\frac{1}{2}\Lambda\varphi$ is again algebraically constructible.*

Proof. By lemma 3.13, we can assume that

$$\varphi = \sum_{i\in I} m_i (f_i)_*(\mathbf{1}_{W_i}) ,$$

where all W_i are non singular of dimension d_i and all f_i are proper regular maps. Then we have, by proposition 3.9,

$$\Lambda\varphi = \Lambda(\sum_{i\in I} m_i (f_i)_*(\mathbf{1}_{W_i})) = \sum_{i\in I} m_i (f_i)_*(\Lambda\mathbf{1}_{W_i}) .$$

Since W_i is nonsingular of dimension d_i, $\Lambda\mathbf{1}_{W_i}$ is the constant 2 if d_i is odd and 0 if d_i is even. It follows that

$$\frac{1}{2}\Lambda\varphi = \sum_{i\in I,\ d_i\ \text{odd}} m_i (f_i)_*(\mathbf{1}_{W_i}) ,$$

which proves the theorem. □

Remark that a semialgebraic set S is Euler (see section 2.5) if and only if $\Lambda\mathbf{1}_S$ is even. Hence, theorem 3.14 implies Sullivan's theorem 2.8.

We can give the promised proof that a locally closed semialgebraic set S is Euler if and only if its one-point compactification $\dot{S}$, identified with $S \cup \{\infty_S\}$, is Euler. The non trivial part is to prove that $\dot{S}$ is Euler, assuming S Euler. We have $\int_{\dot{S}} \Lambda\mathbf{1}_{\dot{S}}\, d\chi = 0$ by corollary 3.11. On the other hand, $\int_S \Lambda\mathbf{1}_S\, d\chi = 0$ is even since S is Euler. The difference, which is the value of $\Lambda\mathbf{1}_{\dot{S}}$ at ∞_S, is also even.

A constructible function φ on a semialgebraic set S will be called an *Euler function* if $\Lambda\varphi$ takes only even values. The theorem above says that algebraically constructible functions are Euler. Of course, there are Euler functions which are not algebraically constructible.

Example. Let $\varphi = \mathbf{1}_{\{x>0,y>0\}}$ be the characteristic function of the open first quadrant on $\mathbb{R}^2$.

- This function is constructible, but not Euler since the value of its link at the origin is -1.
- The function 2φ is, of course, Euler but it is not algebraically constructible. Indeed, consider its "half-link" $\psi = \frac{1}{2}\Lambda(2\varphi)$. If 2φ were algebraically constructible, so would be ψ by theorem 3.14. But $\psi \times \mathbf{1}_{\{y=0\}}$ is not Euler, since the value of its link at the origin is 1.
- The function 4φ is algebraically constructible. Indeed, $4\varphi = p_*\mathbf{1}_Y$, where

$$Y = \{(x,y,t,u)\ ;\ t^2x = 1, u^2y = 1\} \quad \text{and} \quad p(x,y,t,u) = (x,y) .$$

The fact that the function 4φ above is algebraically constructible is a particular case of the following result.

Proposition 3.15 *Let V be a real algebraic set of dimension d. For every constructible function $\varphi : V \to \mathbb{Z}$, the function $2^d\varphi$ is algebraically constructible.*

Proof. Every semialgebraic subset of V can be represented as a finite disjoint union of subsets of the form

$$S = \{x \in V \ ; \ f(x) = 0, \ g_1(x) > 0, \ldots, g_d(x) > 0\} \ , \tag{3.8}$$

where $f, g_1, \ldots, g_d$ are polynomials on V. It is important that we can take the number of inequalities to be the dimension of V: this is the celebrated Broecker-Scheiderer theorem (6.5.1 in [BCR]). So every constructible function on V is a linear combination with integer coefficients of characteristic functions of semialgebraic sets S as in 3.8. Set

$$W = \{(x, t_1, \ldots, t_d) \in V \times \mathbb{R}^d \ ; \ f(x) = t_1^2 g_1(x) - 1 = \ldots = t_d^2 g_d(x) - 1 = 0\} \ ,$$

and let $p : W \to V$ be the projection. Then $2^d \mathbf{1}_S = p_*(\mathbf{1}_W)$, and the proposition follows. □

Remark. We are using the Euler characteristic with nice additive properties, which differs from usual Euler characteristic. Actually, we would get the same algebraically constructible functions using the usual Euler characteristic χ_{us}. First, by lemma 3.13, every algebraically constructible function can be presented as a linear combination with integer coefficients of usual Euler characteristic of fibers of proper regular maps. In the other direction, if $f : W \to V$ is a regular map, then $x \mapsto \chi_{\mathrm{us}}(f^{-1}(x))$ is algebraically constructible. We can assume $W \subset \mathbb{R}^n \times V$ and f is the projection. For every $x \in V$, there is $r(x) \in \mathbb{R}$ such that $f^{-1}(x)$ retracts by deformation on the intersection of $f^{-1}(x)$ with the closed ball $\overline{B}^n(r(x))$ (in $\mathbb{R}^n \times \{x\}$ identified with $\mathbb{R}^n$), and moreover $f^{-1}(x) \setminus (f^{-1}(x) \cap \overline{B}^n(r(x))$ is semialgebraically homeomorphic to $(f^{-1}(x) \cap S^{n-1}(r(x))) \times (r(x), +\infty)$. Hence we have

$$\begin{aligned}\chi_{\mathrm{us}}(f^{-1}(x)) &= \chi_{\mathrm{us}}(f^{-1}(x) \cap \overline{B}^n(r(x)) = \chi(f^{-1}(x) \cap \overline{B}^n(r(x)) \\ &= \chi(f^{-1}(x)) - \chi(f^{-1}(x) \cap S^{n-1}(r(x))) \ .\end{aligned}$$

We can take $r(x)$ to be semialgebraic and, hence, continuous semialgebraic on a semialgebraic dense subset of V. It follows that we can take $r(x)$ to be a regular function on V minus an algebraic subset W of dimension smaller than V: indeed, any continuous semialgebraic function on a Zariski open subset U of a real algebraic set can be bounded from above by a regular function on U. Then the union of $f^{-1}(x) \cap S^{n-1}(r(x))$ for $x \in V \setminus W$ is Zariski closed in $\mathbb{R}^n \times (V \setminus W)$, which shows that $\chi_{\mathrm{us}}(f^{-1}(x))$ is algebraically constructible on $V \setminus W$. We can then conclude by induction on dimension of algebraic sets.

3.5 Sums of signs of polynomials

Theorem 3.16 *Let V be a real algebraic set. A function $\varphi : V \to \mathbb{Z}$ is algebraically constructible if and only if there are finitely many polynomials $f_1, , \ldots, f_p$ on V such that*

$$\varphi = \sum_{i=1}^{p} \mathrm{sign}(f_i) \ .$$

Proof. The easy part of the equivalence is to prove that the sign of a polynomial $f : V \to \mathbb{R}$ is algebraically constructible. Indeed, set

$$W = \{(t,x) \in \mathbb{R} \times V \ ; \ t^2 = f(x)\} ,$$

and let $p : W \to V$ be the projection defined by $p(t,x) = x$. Then $\operatorname{sign} f = p_*(\mathbf{1}_W) - \mathbf{1}_V$.

For the reverse implication, it suffices to prove that, for every regular map $p : W \to V$, the function $p_*(\mathbf{1}_W)$ is a sum of signs of polynomials on V. Replacing W with the graph of p, we can assume that W is an algebraic subset of $V \times \mathbb{R}^n$ and p is the projection to V. Let h be a positive equation of W in $V \times \mathbb{R}^n$. Then $\mathbf{1}_W = \operatorname{sign} 1 + \operatorname{sign}(-h)$. Proceeding by induction on n, we see that it suffices to prove that the pushforward of the sign of a polynomial along the projection $\mathbb{R} \times V \to V$ is a sum of signs of polynomials on V. This will be done in the next two lemmas. □

Lemma 3.17 *Let V be an irreducible real algebraic set. Let $f : \mathbb{R} \times V \to \mathbb{R}$ be a polynomial function. Denote by $p : \mathbb{R} \times V \to V$ the projection on the second factor. Then there are polynomial functions $g_1, \dots, g_\ell$ on V such that the equality*

$$p_*(\operatorname{sign} f) = \sum_{i=1}^{\ell} \operatorname{sign} g_i$$

holds generically on V (i.e. outside an algebraic subset of smaller dimension).

Proof. The central idea of the proof is that, generically on V, $p_*(\operatorname{sign} f)$ is the signature of a quadratic form with coefficients in the field $\mathbb{R}(V)$ of rational functions on V.

First assume that $f = a_d X^d + \cdots + a_0$ is a polynomial in one variable over $\mathbb{R}$ (with $a_d \neq 0$). Let $t_1 < t_2 < \ldots < t_p$ be the real roots of f and set $t_0 = -\infty$, $t_{p+1} = +\infty$. The integral $\int_{\mathbb{R}} \operatorname{sign} f \, d\chi$ is equal to the number of intervals (t_{i-1}, t_i) where f is negative minus the number of those intervals where f is positive. The sign of f on the interval $(t_p, +\infty)$ is the sign of a_d. If t_i is a root of order m, the sign of f on (t_{i-1}, t_i) is $(-1)^m$ times the sign of the m-th derivative $f^{(m)}(t_i)$. Hence,

$$\int_{\mathbb{R}} \operatorname{sign} f \, d\chi = -\operatorname{sign} a_d - \sum_{j=1}^{d} (-1)^j N_j ,$$

where

$$N_j = \#\{f^2 + \cdots + (f^{(j-1)})^2 = 0, f^{(j)} > 0\} - \#\{f^2 + \cdots + (f^{(j-1)})^2 = 0, f^{(j)} < 0\} .$$

The quantity N_j can be computed as the signature of a symmetric matrix Q_j (of dimension $2d$) whose entries are rational functions of the coefficients of f (see for instance [Co1], Exercise 1.15 p. 18). Then N_j is the sum of the signs of the diagonal entries of any diagonal matrix isometric to Q_j.

Now we consider the case where the coefficients of f are polynomial functions over the irreducible real algebraic set V. The matrices Q_j now have coefficients in the field $\mathbb{R}(V)$. There are $P_j \in \mathrm{GL}(2d, \mathbb{R}(V))$ such that the matrices ${}^tP_j Q_j P_j$

are diagonal with entries $g_{j,1},\ldots,g_{j,2d}$ on the diagonal. Without loss of generality we can assume that all $g_{j,k}$ are polynomials. Let $x \in V$ be a point which is not a zero of the denominator of some entry of Q_j or P_j nor a zero of the determinant of a P_j. Then

$$p_* f(x) = -\operatorname{sign} a_d(x) - \sum_{j=1}^{d}(-1)^j \sum_{k=1}^{2d} \operatorname{sign} g_{j,k}(x) ,$$

which proves the lemma. □

Lemma 3.18 *Let V be a real algebraic set. Let $f : \mathbb{R} \times V \to \mathbb{R}$ be a polynomial function. Denote by $p : \mathbb{R} \times V \to V$ the projection on the second factor. Then there are polynomial functions $g_1,\ldots,g_\ell$ on V such that the equality*

$$p_*(\operatorname{sign} f) = \sum_{i=1}^{\ell} \operatorname{sign} g_i$$

holds everywhere on V.

Proof. We proceed by induction on the dimension of V. Set $d = \dim V$ and assume the lemma proved for all real algebraic set of dimension $< d$. Let $V_1,\ldots,V_m$ be the irreducible components of dimension d of V. By lemma 3.17, there exist polynomials $g_{1,j},\ldots,g_{\ell_j,j}$ on V such that

$$p_*(\operatorname{sign} f) = \sum_{i=1}^{\ell_j} \operatorname{sign} g_{i,j}$$

on $V_j \setminus Z_j$, where Z_j is a proper algebraic subset of V_j. Let h_j be a positive equation for the union of Z_j and all irreducible components of V different from V_j. Then there is an algebraic subset W of V of dimension $< d$ such that

$$\sum_{j=1}^{m}\sum_{i=1}^{\ell_j} \operatorname{sign}(h_j g_{i,j}) = \begin{cases} 0 & \text{on} \quad W \\ p_*(\operatorname{sign} f) & \text{on} \quad V \setminus W \end{cases} .$$

By the inductive assumption, there are polynomials $f_1,\ldots,f_p$ on V such that

$$p_*(\operatorname{sign} f) = \sum_{i=1}^{p} \operatorname{sign} g_i \quad \text{on } W .$$

Let h be a positive equation of W in V. Then

$$p_*(\operatorname{sign} f) = \sum_{j=1}^{m}\sum_{i=1}^{\ell_j} \operatorname{sign}(h_j g_{i,j}) + \sum_{i=1}^{p}(\operatorname{sign} g_i + \operatorname{sign}(-g_i h)) \quad \text{on } V .$$

□

As an easy consequence of Theorem 3.16 we obtain:

Corollary 3.19 *Let V be an irreducible real algebraic set.*

- *Every algebraically constructible function on V is generically constant modulo 2.*
- *Let $f : W \to V$ be a regular map. If $\chi(f^{-1}(x))$ is odd for x in a Zariski dense semialgebraic subset of V, then* $\dim(V \setminus f(W)) < \dim(V)$.

The preceding corollary can be obtained by other ways than the representation as sum of signs of polynomials. The next one depends heavily on this representation. It exhibits a stability property of algebraically constructible functions which is not a consequence of those that we have already encountered.

Corollary 3.20 *Let φ be an algebraically constructible function on a real algebraic set φ. Then $\frac{1}{2}(\varphi^4 - \varphi^2)$ is again algebraically constructible.*

Proof. We start with a representation $\varphi = \sum_{i=1}^{p} \operatorname{sign} f_i$, where the f_i are polynomial functions on V. Set $\sigma_i = \operatorname{sign} f_i$. Each σ_i is algebraically constructible, and $\sigma_i^4 = \sigma^2$. Then

$$\varphi^2 = \sum_i \sigma_i^2 + 2\sum_{i<j} \sigma_i \sigma_j = \sum_i \sigma_i^2 + 2\psi_2 \ ,$$

where ψ_2 is algebraically constructible. Taking the square we obtain

$$\varphi^4 = \sum_i \sigma_i^4 + 2\sum_{i<j} \sigma_i^2 \sigma_j^2 + 4\psi_2 \sum_i \sigma_i^2 + 4\psi_2^2 = \sum_i \sigma_i^2 + 2\psi_4 \ ,$$

where ψ_4 is algebraically constructible. The corollary follows immediately. $\square$

The preceding corollary is not the only result of this kind: one can find in [MP2] the characterization of all polynomials P with rational coefficients such that $P(\varphi)$ is algebraically constructible for every algebraically constructible φ.

3.6 Combinatorial topological properties of real algebraic sets

Let S be a locally compact semialgebraic set. Denote by $\widetilde{\Lambda}(S)$ the smallest subring of $F(S)\left[\frac{1}{2}\right]$ containing $\mathbf{1}_S$ and stable by the half-link operator $\widetilde{\Lambda} = \frac{1}{2}\Lambda$.

Theorem 3.21 *If S is homeomorphic to a real algebraic set, then all functions in $\widetilde{\Lambda}(S)$ have values in $\mathbb{Z}$.*

Proof. Let $h : S \to V$ be a homeomorphism from S to a real algebraic set. Since h preserves the link operator, it induces an isomorphism $h^* : \varphi \mapsto \varphi \circ h$ from $\widetilde{\Lambda}(V)$ to $\widetilde{\Lambda}(S)$. Since $\widetilde{\Lambda}(V)$ is contained in $A(V)$ (a consequence of Theorem 3.14), it consists of functions with values in $\mathbb{Z}$. $\square$

The ring $\widetilde{\Lambda}(S)$ is clearly a semialgebraic invariant of S: a semialgebraic homeomorphism induces an isomorphism of the corresponding ring. Actually, it is a topological invariant: this corresponds to the fact that the Euler characteristic of the link is a topological invariant (although the link itself is only a semialgebraic invariant). For more details, see the appendix of [MP1].

The preceding theorem provides obstructions for a semialgebraic set S to be homeomorphic to an algebraic set. We are going to analyze more precisely these obstructions.

First note that we can use either the half-link operator $\widetilde{\Lambda}$ or the operator $\widetilde{\Omega} = \frac{1}{2}\Omega$ for the generation of $\widetilde{\Lambda}(S)$ (recall that $\widetilde{\Omega}\varphi = \varphi - \widetilde{\Lambda}\varphi$). We describe now generators of the additive group of $\widetilde{\Lambda}(S)$. These generators will be organized by *depth*. They are obtained according to the following rules:

1. $\mathbf{1}_S$ is the only generator of depth 0.
2. If γ is a generator of depth δ and $\dim(S) - \delta$ is even (resp. odd), then $\widetilde{\Lambda}\gamma$ (resp. $\widetilde{\Omega}\gamma$) is a generator of depth $\delta + 1$.
3. If $\gamma_1, \ldots, \gamma_k$ are generators of depths $\delta_1, \ldots, \delta_k$, then the product $\gamma_1 \cdots \gamma_k$ is a generator of depth $\max(\delta_1, \ldots, \delta_k)$.

By construction, the support of a generator of depth δ has codimension at least δ in S. Hence we have to consider generators with depth $\leq \dim(S)$ only. The functions of $\widetilde{\lambda}(S)$ have values in $\mathbb{Z}$ if and only if this is the case for the generators. Of course, it suffices to check this for generators of the form $\widetilde{\Lambda}\varphi$ or $\widetilde{\Omega}\varphi$ obtained by application of rule 2. Actually, it suffices to check this *for a finite number* among these generators. We explain this in the particular case where $\dim(S) = 3$. We use the following observations:

1. $\widetilde{\Omega}\widetilde{\Lambda} = \widetilde{\Lambda}\widetilde{\Omega} = 0$.
2. If φ has values in $\mathbb{Z}$, then, for any positive integer k, φ^k is congruent modulo 2 to φ and congruent modulo 4 to a linear combination with integer coefficients of φ, φ^2 and φ^3.
3. If φ and ψ have values in $\mathbb{Z}$ and are congruent modulo 2^k, where k is a positive integer, and if $\widetilde{\Lambda}\varphi$ has values in $\mathbb{Z}$, then this is also the case for $\widetilde{\Lambda}\psi$ and $\widetilde{\Lambda}\psi$ is congruent to $\widetilde{\Lambda}\varphi$ modulo 2^{k-1} (the same for $\widetilde{\Omega}$).

For depth one, we have to check that $\alpha = \widetilde{\Omega}\mathbf{1}_S$ has values in $\mathbb{Z}$. This means that S is an Euler set. We assume that this holds.

For depth 2, there appears no new obstruction. Indeed, we have $\beta_1 = \widetilde{\Lambda}\alpha = 0$ by observation 1, and then, for every positive integer k, $\beta_k = \widetilde{\Lambda}(\alpha^k)$ has values in $\mathbb{Z}$ by observations 2 and 3.

For depth 3, It suffices to check to check that the four generators

$$\widetilde{\Omega}(\beta_2\beta_3),\ \widetilde{\Omega}(\alpha\beta_2),\ \widetilde{\Omega}(\alpha\beta_3),\ \widetilde{\Omega}(\alpha\beta_2\beta_3)$$

have values in $\mathbb{Z}$. Indeed we have $\widetilde{\Omega}\beta_2 = \widetilde{\Omega}\beta_3 = 0$ by observation 1, and the observations 2 and 3 explain why the β_k for $k > 3$ are superfluous and why no power > 1 of α, β_2 or β_3 is needed.

Remark that the obstructions that are obtained in the way described above are local ones: the value of a function in $\widetilde{\Lambda}(S)$ at a point x of S depends only on the link $\mathrm{lk}(x, S)$. Moreover, an obstruction of depth δ has to be checked only on the codimension δ skeleton of a triangulation (or of a locally trivial stratification) of S. Each local obstruction actually lies in $\mathbb{Z}/2\mathbb{Z}$: let φ be a generator of depth $\delta - 1$ and assume that it has values in $\mathbb{Z}$; then the fact that $\widetilde{\Lambda}\varphi(x)$ (or $\widetilde{\Omega}\varphi(x)$) is an integer is equivalent to

$$\int_{\mathrm{lk}(x,S)} \varphi \, d\chi \equiv 0 \pmod 2 .$$

We can reformulate the obstructions of depth 3 in the following way. Let S be a compact semialgebraic set of dimension 3 and assume that S is Euler. We define

$$\begin{aligned} \theta : S &\longrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \\ x &\longmapsto \left(\int_{\mathrm{lk}(x,S)} \varphi_i \, d\chi \bmod 2 \right)_{i=1,\ldots,4} , \end{aligned}$$

where

$$\varphi_1 = \beta_2\beta_3, \ \varphi_2 = \alpha\beta_2, \ \varphi_3 = \alpha\beta_3, \ \varphi_4 = \alpha\beta_2\beta_3 .$$

We have seen that the vanishing of θ everywhere on S is a necessary condition for S to be homeomorphic to a real algebraic set (this vanishing has to be checked only at the vertices of a triangulation or a locally trivial stratification of S).

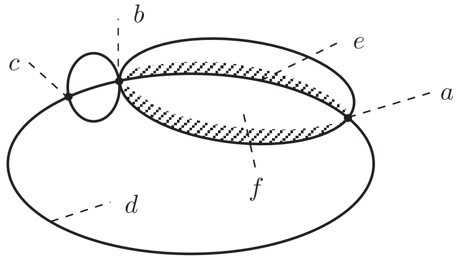
We now give an illustration of the use of the obstructions for the first example given by Akbulut and King of a polyhedron of dimension 3 which is Euler but not homeomorphic to a real algebraic set. This is the suspension of the compact algebraic set $\dot{V}$.

First we state some general facts about the suspension ΣS of a compact semialgebraic set S. Assume that S is contained in $\mathbb{R}^n$ and embed it as $S \times \{0\}$ in $\mathbb{R}^{n+1}$. Take the two suspension point $P_- = (0, -1)$ and $P_+ = (0, 1)$ in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}$. Then the suspension ΣS is the union of the two cones $P_- * S$ and $P_+ * S$. It can be viewed as $S \times [-1, 1]$ with all points of $S \times \{-1\}$ identified to give the suspension point P_- and all points of $S \times \{1\}$ identified to give the other suspension point P_+.

We obtain a stratification of ΣS by taking the two suspension points as 0-strata and the products of strata of a stratification of S with $(-1, 1)$. Every function φ in $\widetilde{\Lambda}(\Sigma S)$ has to be constant on the strata of this stratification. Hence, φ is determined by its restriction to S and its value at the suspension points. If $x \in S$, one remarks that the suspension of $\mathrm{lk}(x, S)$ is $\mathrm{lk}(x, \Sigma S)$. From this remark follow $(\widetilde{\Lambda}\varphi)|_S = \widetilde{\Omega}(\varphi|_S)$ and $(\widetilde{\Omega}\varphi)|_S = \widetilde{\Lambda}(\varphi|_S)$. Hence, the restriction of an element of $\widetilde{\Lambda}(\Sigma S)$ to S is in $\widetilde{\Lambda}(S)$. Remark also that the link of a suspension point in ΣS is S, and that $\widetilde{\Lambda}\varphi(P_\pm) = \frac{1}{2}\int_S \varphi \, d\chi$.

Now consider the suspension $\Sigma\dot{V}$. By the preceding discussion and since $\dot{V}$ is algebraic, all elements of $\widetilde{\Lambda}(\Sigma\dot{V})$ have values in $\mathbb{Z}$ outside of the suspension points. Since the Euler characteristics of $\dot{V}$ is even, $\Sigma\dot{V}$ is Euler ($\alpha = \widetilde{\Omega}\mathbf{1}_{\Sigma\dot{V}}$ has values in $\mathbb{Z}$ also at the suspension point). We check now the obstruction of depth 3 given by $\widetilde{\Omega}(\alpha\beta_2)$ at a suspension point. We compute the restriction of $\alpha\beta_2$ to $\dot{V}$ in Figure 3.1 (the value on strata labeled f is 0 for all functions computed). Now we have

$$\int_{\mathrm{lk}(P_+,\Sigma\dot{V})} \alpha\beta_2 \, d\chi = \int_S \alpha\beta_2 \, d\chi = -1 + 4 = 3 \not\equiv 0 \pmod 2 .$$



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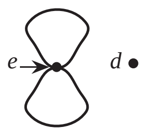
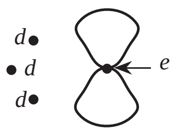
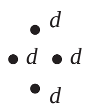
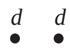
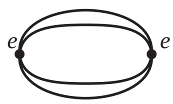
lk					
$\alpha\|_{\dot V} = \widetilde{\Lambda}\mathbf{1}_{\dot V}$	0	1	2	1	-1
$\alpha^2\|_{\dot V}$	0	1	4	1	1
$\beta_2\|_{\dot V} = \widetilde{\Omega}(\alpha^2\|_{\dot V})$	-1	-1	2	0	0
$(\alpha\beta_2)\|_{\dot V}$	0	-1	4	0	0

Figure 3.1: Computation of some elements of $\widetilde{\Lambda}(\dot V)$.

This obstruction shows that $\Sigma\dot V$, although Euler, is not homeomorphic to a real algebraic set.

The vanishing of the four local obstructions modulo 2 of depth 3, together with the condition of being Euler, is equivalent to the necessary and sufficient conditions given by Akbulut and King (Theorem 7.1.1 in [AK]) for a compact 3-dimensional triangulable topological set to be homeomorphic to a real algebraic set. Hence, we have the following result:

Theorem 3.22 (Akbulut-King) *A compact semialgebraic set S of dimension 3 is homeomorphic to a real algebraic set if and only if it is Euler and the four local obstructions*

$$\left(\int_{\mathrm{lk}(x,S)} \varphi_i \, d\chi \bmod 2\right)_{i=1,\ldots,4}$$

defined above vanish everywhere on S.

The analysis of the local obstructions given by the theory of algebraically constructible functions can be pushed further. In dimension 4, they give a total of $2^{43} - 43$ independent local obstructions, if we take into account the stability properties of corollary 3.20! Moreover, it is not known in this case whether the vanishing of these obstructions suffices to characterize topologically real algebraic sets of dimension 4.

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