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Introduction to Basic Toric Geometry

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Introduction

These notes are intended to give a sketchy introduction to some fundamental notions of toric geometry, i.e., geometry of toric varieties, with applications to singularity theory in mind. Toric varieties and their singularities provide a particularly interesting and rich class of examples: On the one hand, though it is a restricted class, these varieties illustrate many concepts of great importance for the general study of algebraic varieties and their singularities. Quoting from the introduction to FULTON's notes, "*toric varieties have provided a remarkably fertile testing ground for general theories*". On the other hand, they admit a surprisingly simple description in terms of objects coming from elementary convex and combinatorial geometry, namely, "rational" convex polyhedral cones (and compatible collections thereof) in a real vector space, with its real dimension equal to the dimension of the variety.

The attribute "toric" refers to the *algebraic torus* of algebraic group theory. In the complex setting that we are dealing with exclusively, the complex algebraic n-torus is an n-fold product $(\mathbb{C}^*)^n$, endowed both with its group structure and its structure as an affine algebraic variety. (The reader should note that this is not the usual torus of topologists, though the complex algebraic n-torus contains the real compact n-torus $(S^1)^n$ as an equivariant deformation retract.) A toric variety is a normal algebraic variety containing a torus as an open dense subset such that the group structure extends to a natural torus action on the variety. It turns out that many familiar algebraic varieties actually are toric. Basic singularities like the two- and the three-dimensional quadric cones $N(xy - z^2) \subset \mathbb{C}^3$ and $N(xy - zw) \subset \mathbb{C}^4$ are toric, too. Toric methods allow to construct and study many more singular varieties with interesting properties, and they also provide an accessible way of "resolving" these singularities.

In our minicourse, we focus on basic parts of the theory that are indispensable if one wants to apply toric methods as a tool for singularity theory. The picture thus obtained is by no means complete: Many interesting applications to singularity theory, let alone to other parts of mathematics, have to be left out.

The reader should be familiar with elementary concepts of algebraic geometry: Affine complex algebraic varieties and their morphisms are in one-to-one (arrow-reversing) correspondence to finitely generated reduced $\mathbb{C}$-algebras and their homomorphisms, since the elements of the algebra yield the regular functions on the variety, and the points of the variety correspond to the maximal ideals of the algebra. Subvarieties correspond to ideals; they are the closed sets of the Zariski topology. Affine varieties can be glued along isomorphic (Zariski) open subsets. If the resulting space satisfies a natural separation property, it is called an algebraic variety. All varieties to be considered here are of finite type, i.e., they can be covered by finitely many open affine subspaces.

We will exclusively deal with *normal* varieties, i.e., the "coordinate algebras" corresponding to their affine open subsets are normal domains. (We recall that an integral domain is called normal if it is integrally closed in its field of fractions.)

Besides of these fundamental notions of algebraic geometry, the reader should be familiar with the basic language of group actions (orbits, invariant subsets, isotropy subgroups, fixed points).

1 Algebraic tori

An "algebraic n-torus" $\mathbb{T} := \mathbb{T}_n$ is by definition the n-fold direct product $(\mathbb{C}^*)^n$ of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. It is both an algebraic variety and a group, even an "algebraic group", i.e. the group operations

$$\mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}, (t, t') \mapsto tt'$$

and

$$\mathbb{T} \longrightarrow \mathbb{T}, t \mapsto t^{-1}$$

are morphisms of complex algebraic varieties. A homomorphism between algebraic tori is both a group homomorphism and a morphism of algebraic varieties. A homomorphism $\chi\colon \mathbb{T}_n \longrightarrow \mathbb{C}^*$ is called a **character** of the torus $\mathbb{T}$, the set of all characters forms a free abelian group $\mathbb{X}(\mathbb{T}) := \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ of rank n, in fact

$$M := \mathbb{Z}^n \longrightarrow \mathbb{X}(\mathbb{T}_n)\,,\ \mu = (\mu_1, \ldots, \mu_n) \mapsto \chi^\mu\colon t = (t_1, \ldots, t_n) \mapsto t^\mu := t_1^{\mu_1} \cdot \ldots \cdot t_n^{\mu_n}$$

is an isomorphism. So a character χ is nothing but a Laurent polynomial in the coordinates $t_1, \ldots, t_n$, and the torus $\mathbb{T}$, being an affine algebraic variety, has as its coordinate ring the Laurent algebra

$$\mathcal{O}(\mathbb{T}) = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = \bigoplus_{\chi \in \mathbb{X}(\mathbb{T})} \mathbb{C}\chi.$$

A homomorphism $\alpha\colon \mathbb{C}^* \longrightarrow \mathbb{T}_n$ is called a **one parameter subgroup**, and

$$N := \mathbb{Z}^n \longrightarrow \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}_n), \nu = (\nu_1, \dots, \nu_n) \mapsto \alpha_\nu\colon s \mapsto (s^{\nu_1}, \dots, s^{\nu_n})$$

is again an isomorphism. We shall use the letter M in order to denote $\mathbb{Z}^n$ if we identify $\mu \in M = \mathbb{Z}^n$ with the character χ^μ, and then even write χ instead of μ; if we deal with one parameter subgroups, we write $N = \mathbb{Z}^n$ and usually replace ν with α. We could have defined $M := \mathbb{X}(\mathbb{T})$ and $N := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$ – the only difference between the left and the right hand side is the fact that the group operation is written as addition and not as multiplication. That is important, since in the following we shall consider both

$$M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n \quad \text{and } N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$$

and use their "vector space geometry".

There is a natural pairing

$$\langle .,. \rangle : M \times N \longrightarrow \mathbb{Z},$$

where $\langle \chi, \alpha \rangle$ is defined by the requirement

$$(\chi \circ \alpha)(s) = s^{\langle \chi, \alpha \rangle}, \ \forall\ s \in \mathbb{C}^*$$

for the homomorphism $\chi \circ \alpha : \mathbb{C}^* \longrightarrow \mathbb{C}^*$. We use the same notation for the extended pairing

$$\langle .,. \rangle : M_\mathbb{R} \times N_\mathbb{R} \longrightarrow \mathbb{R},$$

which agrees with the standard inner product on $\mathbb{R}^n \cong M_\mathbb{R} \cong N_\mathbb{R}$.

The following *complete reducibility property* of algebraic tori is basic in the sequel: Consider an algebraic action

$$\mathbb{T} \times V \longrightarrow V\ ,\ (t, v) \mapsto t \cdot v$$

of the torus $\mathbb{T}$ on a finite dimensional (complex) vector space V by linear automorphisms, i.e. the map $V \longrightarrow V, v \mapsto t \cdot v$, is a linear automorphism of the vector space V for every $t \in \mathbb{T}$. For example, given a character $\chi \in \mathbb{X}(\mathbb{T})$, the definition $t \cdot v := \chi(t)v$ provides such an action. Then any linear action of a torus $\mathbb{T}$ is a direct sum of such actions via characters: Given an action of $\mathbb{T}$ on V, we associate to every character $\chi \in \mathbb{X}(\mathbb{T})$ its "eigenspace"

$$V_\chi := \{v \in V; t \cdot v = \chi(t)v,\ \forall\ t \in \mathbb{T}\}.$$

Then

$$V = \bigoplus_{\chi \in \mathbb{X}(\mathbb{T})} V_\chi,$$

where $V_\chi \neq \{0\}$ for at most finitely many $\chi \in \mathbb{X}(\mathbb{T})$, i.e. V is a finite direct sum of nontrivial eigenspaces.

2 Definition and examples of toric varieties

Definition 2.1. *A ($\mathbb{T}$-)***toric variety** *consists of a normal algebraic variety X together with an effective algebraic action*

$$\mathbb{T} \times X \longrightarrow X\,,\ (t,x) \mapsto t \cdot x$$

*and a distinguished point $x_o \in X$ with trivial isotropy group, such that the orbit $T \cdot x_o \subset X$ is open and dense in X. A morphism $\varphi\colon X \longrightarrow Y$ of $\mathbb{T}$-toric varieties (or for short a "***toric morphism***") is an equivariant morphism (i.e. respecting the $\mathbb{T}$-actions), mapping the distinguished point of X to the distinguished point of Y.*

Example 2.2.
1. Take $X := \mathbb{T}$ with $\mathbb{T}$ acting on itself by group multiplication and $x_o := e := (1, \ldots, 1) \in \mathbb{T}$.
2. Take $X := \mathbb{C}^n \supset \mathbb{T}_n$ with $t \cdot x := (t_1x_1, \ldots, t_nx_n)$ and $x_o := (1, \ldots, 1) \in \mathbb{C}^n$.
3. Take $X := \mathbb{P}_n$ with $t \cdot [x] := [x_0, t_1x_1, \ldots, t_nx_n]$ and distinguished point $[1, \ldots, 1]$.
4. Every nonempty open $\mathbb{T}$-invariant subset $U \subset X$ of a toric variety is itself a toric variety.

Remark 2.3. Using the open embedding $\mathbb{T} \longrightarrow X, t \mapsto t \cdot x_0$, we regard $\mathbb{T}$ as open subset of X and write $\mathbb{T} \subset X$, with $e \in \mathbb{T}$ as distinguished point. Then a toric morphism $\varphi : X \longrightarrow Y$ restricts to the identity from $\mathbb{T} \subset X$ to $\mathbb{T} \subset Y$; in particular, there is at most one toric morphism from X to Y.

The investigation of toric varieties is based on

Theorem 2.4 (Sumihiro). *Every point $a \in X$ admits an affine open $\mathbb{T}$-invariant neighbourhood $U \subset X$.*

Thus, in order to investigate toric varieties, it suffices to consider finitely many affine toric varieties $X_i \supset \mathbb{T}$ and to study how they can be patched together to a toric variety X.

3 Affine toric varieties and cones

Remember that an affine variety is completely determined by its ring $\mathcal{O}(X)$ of regular functions, and since $\mathbb{T} \subset X$ is dense in X, the restriction of functions provides an injective algebra homomorphism $\mathcal{O}(X) \hookrightarrow \mathcal{O}(\mathbb{T})$. So we have to hunt for subalgebras $A \subset \mathcal{O}(\mathbb{T})$ of the form

$$A = \mathcal{O}(X)|_{\mathbb{T}} \subset \mathcal{O}(\mathbb{T})$$

with an affine toric variety $X \supset \mathbb{T}$. Now the action of $\mathbb{T}$ on X induces an action

$$\mathbb{T} \times \mathcal{O}(X) \longrightarrow \mathcal{O}(X)\,,\ (t,f) \mapsto f_t,\ f_t(x) := f(t \cdot x).$$

Of course $\mathcal{O}(X)$ is not a finite dimensional vector space (except for the trivial case $X = \{x_o\}$), but the $\mathbb{T}$-action is "locally finite", i.e., for every regular function $f \in \mathcal{O}(X)$, the functions $f_t, t \in \mathbb{T}$, generate a finite dimensional ($\mathbb{T}$-invariant) subspace on which $\mathbb{T}$ acts linearly. In order to see that consider the pull back of functions

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(\mathbb{T} \times X) = \mathcal{O}(\mathbb{T}) \otimes \mathcal{O}(X)$$

corresponding to the action $\mathbb{T} \times X \longrightarrow X$. The image of $f \in \mathcal{O}(X)$ then is a finite sum

$$\sum_i g_i \otimes h_i; g_i \in \mathcal{O}(\mathbb{T}), h_i \in O(X),$$

in particular $f_t = \sum_i g_i(t)h_i$, and thus all "translates" $f_t, t \in T$ of f belong to the vector space generated by the finitely many $h_i \in \mathcal{O}(X)$. As a consequence,

$$\mathcal{O}(X) = \bigoplus_{\chi \in \mathbb{X}(\mathbb{T})} \mathcal{O}(X)_\chi,$$

the decomposition into eigenspaces. For $X = \mathbb{T}$ that decomposition looks as follows

$$\mathcal{O}(\mathbb{T}) = \bigoplus_{\chi \in \mathbb{X}(\mathbb{T})} \mathbb{C}\chi.$$

So for the $\mathbb{T}$-invariant subalgebra $\mathcal{O}(X) \subset \mathcal{O}(\mathbb{T})$ we have the following alternative: Either $\mathcal{O}(X)_\chi = \{0\}$ or $\mathcal{O}(X)_\chi = \mathbb{C}\chi$, so

$$\mathcal{O}(X) = \bigoplus_{\chi \in S} \mathbb{C}\chi$$

with

$$S = S_{(*)} := \{\chi \in \mathbb{X}(\mathbb{T}) \cong M; \mathcal{O}(X)_\chi \neq 0\}.$$

Since $\mathcal{O}(X)$ is an algebra, the set $S_{(*)} \subset \mathbb{X}(\mathbb{T})$ is a subsemigroup of the (multiplicative) group $\mathbb{X}(\mathbb{T})$ of all characters of the torus $\mathbb{T}$, i.e., $1 \in S_{(*)}$ and $\chi, \chi' \in S_{(*)} \implies \chi\chi' \in S_{(*)}$. Furthermore, since $\mathcal{O}(X)$, by assumption, is integrally closed in $\mathcal{O}(\mathbb{T}) \subset Q(\mathcal{O}(X))$, we see that $S_{(*)} \subset \mathbb{X}(\mathbb{T})$is "saturated", i.e. if $\chi^k \in S_{(*)}$ for some $k \in \mathbb{N}_{>1}$, then also $\chi \in S_{(*)}$: The polynomial $Y^k - \chi^k \in \mathcal{O}(X)[Y]$ is an integral equation for χ. Finally, $\mathcal{O}(X)$ being a finitely generated $\mathbb{C}$-algebra, $S_{(*)} \subset \mathbb{X}(\mathbb{T})$ is finitely generated as a semigroup.

In order to understand the structure of the semigroup $S \subset \mathbb{X}(\mathbb{T})$ better, the "additive point of view", i.e., considering $S \subset M \subset M_\mathbb{R} \cong \mathbb{R}^n$, turns out to be more convenient. Then it satisfies

$$0 \in S; \chi, \chi' \in S \implies \chi + \chi' \in S$$

as well as

$$k\chi \in S \quad \text{with some } k \in \mathbb{N}_{>0} \implies \chi \in S$$

and can be written

$$S = \mathbb{N}\chi_1 + \ldots + \mathbb{N}\chi_r$$

with finitely many generators $\chi_1, \ldots, \chi_r \in S$, since $\mathcal{O}(X)$ is a finitely generated $\mathbb{C}$-algebra. Such a set can be interpreted "geometrically": The set

$$\tau := \mathbb{R}_{\geq 0} \cdot S \subset M_{\mathbb{R}}$$

of all finite linear combinations of vectors in S with non-negative real coefficients is a "polyhedral cone"

$$\tau = \mathbb{R}_{\geq 0}\chi_1 + \ldots + \mathbb{R}_{\geq 0}\chi_r \tag{1}$$

satisfying

$$S = \tau \cap M.$$

Furthermore, τ is not contained in any hyperplane, since otherwise the quotient field $\mathrm{Q}(\mathcal{O}(X))$ would have transcendence degree $\leq n-1$.

Conversely, given a cone of the form (1) with $\chi_1, \ldots, \chi_r \in M$ and not contained in any hyperplane, the semigroup $S := \tau \cap M$ is finitely generated (although not necessarily by $\chi_1, \ldots, \chi_r$), and the maximal spectrum

$$X^\tau := \mathrm{Sp}(\mathbb{C}[\tau \cap M])$$

of the semigroup algebra

$$\mathbb{C}[\tau \cap M] := \bigoplus_{\chi \in \tau \cap M} \mathbb{C}\chi \quad \subset \mathcal{O}(\mathbb{T}),$$

defines a toric variety, with the $\mathbb{T}$-action corresponding to the $\mathbb{T}$-action on $\mathbb{C}[\tau \cap M] \subset \mathcal{O}(\mathbb{T})$ and the distinguished point x_0 being the image of $1 \in \mathbb{T} = \mathrm{Sp}(\mathcal{O}(\mathbb{T}))$ with respect to the morphism $\mathbb{T} \longrightarrow X$ induced by the inclusion $\mathbb{C}[\tau \cap M] \subset \mathcal{O}(\mathbb{T})$.

Definition 3.1. *Let $L \cong \mathbb{Z}^n$ be a "lattice" and $V := L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$. A subset $\tau \subset V$ is called a* **polyhedral cone** *if there are finitely many vectors $v_1, \ldots, v_r \in V$ such that*

$$\tau = \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_r$$

or, equivalently, if it is the finite intersection of closed linear half spaces. It is called an L-cone (or lattice cone or L-rational cone) if we can choose $v_1, \ldots, v_r \in L$. A cone is called **strongly convex** *if it does not contain a line. We define $\dim \tau := \dim \mathrm{span}(\tau)$, where $\mathrm{span}(\tau) := \tau + (-\tau)$ is the linear subspace spanned by τ.*

So we have seen that affine toric varieties $X \supset \mathbb{T}$ are in one-to-one-correspondence to polyhedral M-cones $\tau \subset M_{\mathbb{R}}$ of dimension n. Obviously an inclusion of such cones $\tau \subset \tau'$ induces a morphism $X^{\tau'} \longrightarrow X^{\tau}$ in the opposite direction.

For various purposes, it is important to have a "covariant" description. This can be achieved by passing from $\tau \subset M_{\mathbb{R}}$ to its "dual" cone

$$\check{\tau} := \{v \in N_{\mathbb{R}}; \langle \tau, v \rangle \geq 0\} \subset N_{\mathbb{R}}$$

where $\langle \tau, v \rangle \geq 0$ means that $\langle w, v \rangle \geq 0$ holds for all $w \in \tau$. – The analogous definition applies to cones in $N_{\mathbb{R}}$, providing cones in $M_{\mathbb{R}}$. The reader should note that dualization of cones does not necessarily preserve the dimension.

Proposition 3.2. *The dualization of cones*

$$\tau \mapsto \check{\tau}$$

defines an order reversing bijection between n-dimensional polyhedral M-cones in $M_{\mathbb{R}}$ and strongly convex polyhedral N-cones in $N_{\mathbb{R}}$. Furthermore

$$\check{\check{\tau}} = \tau.$$

Now define for a strongly convex polyhedral N-cone $\sigma \subset N_{\mathbb{R}}$ the toric variety

$$X_\sigma := X^{\check{\sigma}} = \mathrm{Sp}(\mathbb{C}[\check{\sigma} \cap M]).$$

The elements $\chi \in \check{\sigma} \cap M$ are characterized by the fact that $\chi \in \mathcal{O}(\mathbb{T})$ extends to a regular function on X_σ. There is a dual interpretation for the elements $\alpha \in \sigma \cap N$: We have $\alpha \in \sigma \cap N$ iff $\alpha : \mathbb{C}^* \longrightarrow \mathbb{T} \subset X_\sigma$ extends to a map $\mathbb{C} \longrightarrow X_\sigma$. This is seen as follows: Choose semigroup generators $\chi_i \in \check{\sigma} \cap M_{\mathbb{R}}, i = 1, \ldots, r$; they define a closed embedding $X \hookrightarrow \mathbb{C}^r$ (which is even "equivariant", i.e. respects the $\mathbb{T}$-operations, if we let act $t \in \mathbb{T}$ on the i-th component of a vector in $\mathbb{C}^r$ by multiplication with $\chi_i(t)$). Then a one parameter subgroup α extends to $\mathbb{C}$ iff that is true for all

$$\chi_i \circ \alpha : \mathbb{C}^* \longrightarrow \mathbb{C}^* \subset \mathbb{C},\ s \mapsto s^{\langle \chi_i, \alpha \rangle}$$

iff $\langle \chi_i, \alpha \rangle \geq 0$ for $i = 1, \ldots, r$ iff $\langle \check{\sigma}, \alpha \rangle \geq 0$ iff $\alpha \in \check{\check{\sigma}} = \sigma$.

In order to understand affine open $\mathbb{T}$-invariant subspaces of X_σ we need:

Definition 3.3. *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex polyhedral cone. A* **face** *τ of σ is an intersection $\tau = \ker(\varphi) \cap \sigma$, where $\varphi \in N^*_{\mathbb{R}} \cong M_{\mathbb{R}}$ a linear form such that $\varphi|_\sigma \geq 0$. In that case we write $\tau \preceq \sigma$. – A face of codimension one is called a* **facet**.

We remark that a face τ of a strongly convex polyhedral N-cone $\sigma \subset N_{\mathbb{R}}$ is again a strongly convex polyhedral N-cone, and that one always can choose the linear form $\varphi = \chi \in M \subset M_{\mathbb{R}}$. Furthermore, any inclusion $\tau \subset \sigma$ of (strongly convex polyhedral)

N-cones induces an equivariant morphism $X_\tau \longrightarrow X_\sigma$ restricting to the identity from $\mathbb{T} \subset X_\tau$ to $\mathbb{T} \subset X_\sigma$. If $\tau = \ker(\chi) \preceq \sigma$ is even a face of σ, then that morphism is an open embedding, an isomorphism onto the principal open subset $(X_\sigma)_\chi := \{x \in X_\sigma; \chi(x) \neq 0\}$. (Remember here that χ extends to a regular function on X_σ because of $\langle \chi, \sigma \rangle \geq 0$.) Because of that we shall from now on regard X_τ as an open subset of X_σ.

Theorem 3.4. *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex polyhedral N-cone. Then there is an bijection between the set of faces of σ and the $\mathbb{T}$-orbits in X_σ,*

$$\sigma \succeq \tau \mapsto O_\tau := \mathbb{T} \cdot (\lim_{s \to 0} \alpha(s)),$$

where $\alpha \in \overset{\circ}{\tau}$ lies in the "relative interior" $\overset{\circ}{\tau}$, i.e., in the interior of τ as topological subspace of span(τ). *The orbits $O_\tau \subset X_\sigma$ are locally closed subsets of dimension*

$$\dim O_\tau = n - \dim \tau \, .$$

Moreover the correspondence is "order reversing", i.e., $\gamma \preceq \tau$ iff $O_\tau \subset \overline{O}_\gamma$.

In particular $O_o = \mathbb{T}$ for the zero cone $o := \{0\}$, and if $\dim \sigma = n$, *then $O_\sigma = \{x_\sigma\}$ consists of the unique fixed point of X_σ.*

Example 3.5. Assume that $\sigma = \mathbb{R}_{\geq 0}\alpha_1 + \ldots + \mathbb{R}_{\geq 0}\alpha_n$ with a basis $\alpha_1, \ldots, \alpha_n$ of the free abelian group (lattice) N. Denote $\chi_1, \ldots, \chi_n$ the corresponding dual basis of M, i.e., $\langle \chi_j, \alpha_i \rangle = \delta_{ij}$. Then $X_\sigma \cong \mathbb{C}^n$ with the action

$$t \cdot z = (\chi_1(t) z_1, \ldots, \chi_n(t) z_n)$$

and the distinguished point $(1, \ldots, 1)$. The faces of σ are the cones $\sigma_A := \sum_{i \in A} \mathbb{R}_{\geq 0}\alpha_i$ with subsets $A \subset \{1, \ldots, n\}$. The corresponding orbit is

$$O_{\sigma_A} = \{(z_1, \ldots, z_n); z_i = 0 \iff i \notin A\}.$$

Proof of Theorem 3.4. We use induction on $\dim \sigma$. For $\dim \sigma = 0$, we have $\sigma = o := \{0\}$ with $X_o = \mathbb{T}$. In general, we apply the following

Proposition 3.6. *Let $\sigma \subset N_\mathbb{R}$ be an r-dimensional strongly convex polyhedral N-cone. Then there are decompositions*

$$\mathbb{T} \cong \mathbb{T}_\sigma \times \mathbb{T}'_\sigma \tag{2}$$

with tori $\mathbb{T}_\sigma \cong (\mathbb{C}^*)^r, \mathbb{T}'_\sigma \cong (\mathbb{C}^*)^{n-r}$ and

$$X_\sigma \cong Z_\sigma \times \mathbb{T}'_\sigma \tag{3}$$

with a $\mathbb{T}_\sigma$-toric variety Z_σ and $\mathbb{T}'_\sigma$ acting on itself by translation.

Proof (of the proposition). The intersection $N_\sigma := \mathrm{span}(\sigma) \cap N \subset N$ is a saturated subgroup of rank $\dim \sigma$ (σ being N-rational), i.e. $k\alpha \in N_\sigma \Longrightarrow \alpha \in N_\sigma$ holds for $\alpha \in N$ and $k \in \mathbb{N}_{>0}$. Then N/N_σ has no torsion and thus is free; in particular there is a complementary subgroup $N'_\sigma \subset N$ such that $N = N_\sigma \oplus N'_\sigma$. There is a corresponding decomposition $\mathbb{T} = \mathbb{T}_\sigma \times \mathbb{T}'_\sigma$: Take a basis $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1, \ldots, \alpha_r$ generate N_σ, and $\alpha_{r+1}, \ldots, \alpha_n$ the complementary submodule N'_σ. Since $\alpha : \mathbb{T} \longrightarrow \mathbb{T},\ t \mapsto (\alpha_1(t), \ldots, \alpha_n(t))$ is an isomorphism, we can choose $\mathbb{T}_\sigma = \alpha^{-1}((\mathbb{C}^*)^r \times (1, \ldots, 1))$ and $\mathbb{T}'_\sigma = \alpha^{-1}((1, \ldots, 1) \times (\mathbb{C}^*)^{n-r})$. Then $N_\sigma \subset N$ resp. $N'_\sigma \subset N$ is the lattice of one-parameter groups of $\mathbb{T}_\sigma \subset \mathbb{T}$ resp. $\mathbb{T}'_\sigma \subset \mathbb{T}$. Now choose Z_σ as the $\mathbb{T}_\sigma$-toric variety associated to the cone $\sigma \subset (N_\sigma)_\mathbb{R}$. □

Let us continue with the proof of Theorem 3.4: As a consequence of the above result, there is a one-to-one correspondence between ($\mathbb{T}$-) orbits in X_σ and $\mathbb{T}_\sigma$-orbits $O \subset Z_\sigma$ via $O \leftrightarrow O \times \mathbb{T}'_\sigma$. It thus suffices to consider the case $\dim \sigma = n$. The theorem then is true by induction hypothesis for the $X_\tau \subset X_\sigma$ with a proper face $\tau \prec \sigma$. It remains to show that

$$F := X_\sigma \setminus \bigcup_{\tau \prec \sigma} X_\tau$$

consists of one point, the unique fixed point of the $\mathbb{T}$-action and limit of all one-parameter subgroups $\alpha \in \mathring{\sigma}$. Take non-trivial generators $\chi_1, \ldots, \chi_r$ of the semigroup $\check{\sigma} \cap M$ and consider the associated closed embedding $X_\sigma \hookrightarrow \mathbb{C}^r$. Since the principal open sets $(X_\sigma)_{\chi_i}$ are of the form X_τ, the set F is contained in the set of all zeros of the χ_i, and since the χ_i generate $\mathcal{O}(X_\sigma)$, the set F contains at most one point. Choose $\alpha \in \mathring{\sigma}$. Then we have $\langle \chi_i, \alpha \rangle > 0$ for $i = 1, \ldots, r$, whence it follows that $0 = \lim_{s \mapsto 0} \alpha(s) \in X_\sigma$. Obviously there are no fixed points in any $X_\tau \subset X_\sigma$ for a proper face $\tau \precneqq \sigma$. □

Orbit closures. Let us consider an orbit closure $Y := \overline{O}_\tau \hookrightarrow X_\sigma \hookrightarrow \mathbb{C}^r$ with some equivariant embedding $X_\sigma \hookrightarrow \mathbb{C}^r$. From 3.4 applied to $X_\tau \cong Z_\tau \times {\mathbb{T}_\tau}'$, we see that $\mathbb{T}_\tau$ is the isotropy subgroup of any point in $O_\tau = \{z_\tau\} \times {\mathbb{T}_\tau}'$, where z_τ is the unique fixed point of the $\mathbb{T}_\tau$-action on Z_τ. So there is a natural action of the factor torus $\mathbb{T}/\mathbb{T}_\tau \cong {\mathbb{T}_\tau}'$ on the $\mathbb{T}$-orbit O_τ and on its closure Y. We have also a distinguished point $y_o \in Y$: Take any one parameter subgroup $\alpha \in \mathring{\tau}$ and set $y_o := \lim_{s \to 0} \alpha(s)$. In fact, $y_0 \in O_\tau \subset \mathbb{C}^r$ is the unique point in O_τ that only has components 1 or 0. Denote

$$\tau^\perp := \{v \in M_\mathbb{R}; \langle v, \tau \rangle = 0\} .$$

Then we have a vector space decomposition

$$\mathcal{O}(X_\sigma) \cong \mathcal{O}(X_\sigma)^{T_\tau} \oplus I(\overline{O}_\tau)$$

with the subalgebra

$$\mathcal{O}(X_\sigma)^{T_\tau} = \bigoplus_{\chi \in M \cap \check{\sigma} \cap \tau^\perp} \mathbb{C}\chi$$

of all regular function invariant under the action of the subtorus $\mathbb{T}_\tau \hookrightarrow \mathbb{T}$ and the ideal

$$I(\overline{O}_\tau) = \bigoplus_{\chi \in (\check{\sigma} \cap M) \setminus \tau^\perp} \mathbb{C}\chi.$$

Hence

$$\mathcal{O}(\overline{O}_\tau) \cong \mathcal{O}(X_\sigma)^{T_\tau} = \bigoplus_{\chi \in M \cap \check{\sigma} \cap \tau^\perp} \mathbb{C}\chi.$$

Now the characters in $M \cap \check{\sigma} \cap \tau^\perp$ are exactly those in $M \cap \check{\sigma}$ which come from a linear form on N/N_τ, resp. a character of $\mathbb{T}/\mathbb{T}_\tau$ belonging to $\check{\gamma}$ for $\gamma := \sigma/\tau := \pi(\sigma)$ with the quotient map $\pi : N_\mathbb{R} \longrightarrow (N/N_\tau)_\mathbb{R}$. So, finally

$$X_\sigma \hookleftarrow \overline{O}_\tau = X_{\sigma/\tau}. \tag{4}$$

4 The fixed point

The common subject of the courses of these three weeks at the ICTP is the geometry of singularities. In the context of affine toric varieties X_σ with an n-dimensional cone $\sigma \subset N_\mathbb{R}$, the fixed point $x_\sigma \in X_\sigma$ is a good candidate for a singularity.

First of all let us characterize the case when it is a regular point:

Proposition 4.1. *Let $\sigma \subset N_\mathbb{R}$ be an n-dimensional strongly convex N-cone. Then the following statements are equivalent:*

1. *The affine variety X_σ is smooth.*
2. *The unique fixed point of X_σ is a regular point.*
3. *The cone $\check{\sigma}$ is of the form $\check{\sigma} = \mathbb{R}_{\geq 0}\chi_1 + \ldots + \mathbb{R}_{\geq 0}\chi_n$ with a basis $\chi_1, \ldots, \chi_n$ of the (free) lattice M.*
4. *The cone σ is of the form $\sigma = \mathbb{R}_{\geq 0}\alpha_1 + \ldots + \mathbb{R}_{\geq 0}\alpha_n$ with a basis $\alpha_1, \ldots, \alpha_n$ of the (free) lattice N.*
5. *$X_\sigma \cong \mathbb{C}^n$ with an action*

$$t \cdot z = (\chi_1(t)z_1, \ldots, \chi_n(t)z_n)$$

where $\chi_1, \ldots, \chi_n$ constitute a basis of the lattice M.

Proof. "1) $\Longrightarrow$ 2)": Obvious.
"2) $\Longrightarrow$ 3)": The maximal ideal $\mathfrak{m} \subset \mathcal{O}(X_\sigma)$ of all regular functions vanishing at the unique fixed point $x_0 \in X_\sigma$ satisfies

$$\mathfrak{m} = \bigoplus_{\chi \in \mathring{S}} \mathbb{C}\chi$$

with $\overset{\circ}{S} := (\check{\sigma} \cap M) \setminus \{0\}$, while

$$\mathfrak{m}^2 = \bigoplus_{\chi \in \overset{\circ}{S} + \overset{\circ}{S}} \mathbb{C}\chi,$$

so the Zariski tangent space of X_σ at x_0 takes the form

$$T_{x_0} X_\sigma = (\mathfrak{m}/\mathfrak{m}^2)^* \quad , \quad \text{where} \quad \mathfrak{m}/\mathfrak{m}^2 \cong \bigoplus_{\chi \in \overset{\circ}{S} \setminus (\overset{\circ}{S} + \overset{\circ}{S})} \mathbb{C}\chi$$

with the set $\overset{\circ}{S} \setminus (\overset{\circ}{S} + \overset{\circ}{S})$ of all "indecomposable" elements in S. Since x_σ is a regular point, that vector space has dimension $n = \dim X_\sigma$; so $|\overset{\circ}{S} \setminus (\overset{\circ}{S} + \overset{\circ}{S})| = n$. But S is generated as a semigroup by $\overset{\circ}{S} \setminus (\overset{\circ}{S} + \overset{\circ}{S})$ (prove that!), and $M = S - S$ (For every sufficiently "long" vector $\chi \in M \cap \overset{\circ}{\sigma}$ we have $\chi + e_i \in S$ with the standard basis vectors $e_i \in M = \mathbb{Z}^n$). Hence the elements in $\overset{\circ}{S} \setminus (\overset{\circ}{S} + \overset{\circ}{S})$ generate M as a group and thus form a basis, since there are only n of them.
"3) $\Longrightarrow$ 4)": By dualization.
"4) $\Longrightarrow$ 5)": Cf. Ex. 3.5.
"5) $\Longrightarrow$ 1)": Obvious. □

Nearest to the above situation are "simplicial cones":

Definition 4.2. *An r-dimensional strongly convex L-cone $\sigma \subset L_{\mathbb{R}}$ is called* **simplicial** *if*

$$\sigma = \varrho_1 + \ldots + \varrho_r$$

with r rays (one dimensional cones) $\varrho_1, \ldots, \varrho_r \preceq \sigma$. It is called **regular** *if, in addition, $\varrho_i = \mathbb{R}_{\geq 0}\alpha_i$ with $\alpha_1, \ldots, \alpha_r \in L$ occuring as part of a basis $(\alpha_1, \ldots, \alpha_n)$ of the lattice L.*

We now proceed to describe the structure of the affine toric variety X_σ corresponding to a full-dimensional simplicial cone σ. It turns out that X_σ is a quotient variety $\mathbb{C}^n/G$ for a finite subgroup $G \subset \mathbb{T}$ that we describe first: We write $\sigma = \varrho_1 + \ldots + \varrho_n \subset N_{\mathbb{R}}$ with the extremal rays $\varrho_i = \mathbb{R}_{\geq 0}\alpha_i$ with one parameter subgroups $\alpha_i \in N$. Now consider the group homomorphism

$$q\colon \mathbb{T} \longrightarrow \mathbb{T}\,,\ t \mapsto \alpha_1(t_1)\alpha_2(t_2) \cdot \ldots \cdot \alpha_n(t_n).$$

It induces an injective homomorphism

$$dq\colon N \longrightarrow N, e_i \mapsto \alpha_i$$

since the α_i are linearly independent, but dq need not be surjective! In fact its cokernel $N/dq(N)$ is a finite group isomorphic to $\ker(q) \subset \mathbb{T}$: In order to see that, we consider

$$N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$$

and the "exponential map"

$$\exp\colon N_{\mathbb{C}} \longrightarrow \mathbb{T}, (z_1, \ldots, z_n) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n})$$

which induces an isomorphism

$$N_{\mathbb{C}}/N \cong \mathbb{T}$$

and satisfies $\exp \circ (dq \otimes 1_{\mathbb{C}}) = q \circ \exp$. So we find with respect to that isomorphism

$$G := \ker(q) \cong dq^{-1}(N)/N \cong N/dq(N).$$

Now the homomorphism $q\colon \mathbb{T} \longrightarrow \mathbb{T}$ extends to a morphism

$$\mathbb{T} \subset \mathbb{C}^n \xrightarrow{q} X_\sigma \supset \mathbb{T},$$

since we have $\mathbb{C}^n = X_{\tilde{\sigma}}$ with $\tilde{\sigma} := \mathbb{R}_{\geq 0} e_1 + \ldots + \mathbb{R}_{\geq 0} e_n \subset N_{\mathbb{R}}$ and $dq(\tilde{\sigma}) = \sigma$. (Note that the extended morphism q does **not** respect the $\mathbb{T}$-operations, but instead is q-equivariant, i.e. we have $q(t \cdot z) = q(t) \cdot q(z)$ for $t \in \mathbb{T}, z \in \mathbb{C}^n$).

The group $G = \ker(q) \subset \mathbb{T} \subset \mathrm{GL}_n(\mathbb{C})$ is a group of diagonal matrices acting on $\mathbb{C}^n$ in the standard way. In fact the extended morphism $q\colon \mathbb{C}^n \longrightarrow X_\sigma$ gives rise to an isomorphism

$$X_\sigma = \mathbb{C}^n/G\ . \tag{5}$$

This holds for the underlying topological spaces (with respect to both, the Zariski topologies or the strong topologies on $\mathbb{C}^n$ and X_σ) and also in the sense of algebraic varieties: The regular functions on X_σ are exactly the G-invariant regular functions (i.e., polynomials) on $\mathbb{C}^n$, i.e.,

$$\mathcal{O}(X_\sigma) \cong \mathcal{O}(\mathbb{C}^n)^G\ .$$

For the proof we note that $\mathbb{C}^n/G := \mathrm{Sp}(\mathcal{O}(\mathbb{C}^n)^G)$ is the quotient just described (for a finite group G). Then we have a factorization of the finite (check that!) map q through that quotient

$$q\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n/G \longrightarrow X_\sigma,$$

so the second map is finite too, and birational, since it is an isomorphism on the dense open subset $\mathbb{T} \subset X_\sigma$ and hence, X_σ being normal, an isomorphism.

If, in addition, we assume that the "ray generators" $\alpha_i \in N$ are primitive, i.e., not a multiple $k\beta_i$ with some $k \in \mathbb{N}_{>1}, \beta_i \in N$, then the branch locus of the map $q : \mathbb{C}^n \longrightarrow X_\sigma$ is at most $(n-2)$-dimensional and the group G is determined by the variety X_σ itself.

To see that, note first that $\alpha \in N$ is primitive iff $\alpha\colon \mathbb{C}^* \longrightarrow \mathbb{T}$ is injective. Since $dq(e_i) = \alpha_i$, this implies that $G \cap \mathbb{T}_i$ is trivial for $\mathbb{T}_i := e_i(\mathbb{C}^*)$ and all $i = 1, \ldots, n$, in other words: If a matrix in G has the eigenvalue 1 with a multiplicity $\geq n-1$, it is already the identity (In that case $G \subset GL_n(\mathbb{C})$ is called a "**small**" subgroup of $GL_n(\mathbb{C})$). So the set $Z \hookrightarrow \mathbb{C}^n$ of points where the group G does not act freely is a union of coordinate subspaces such that $\dim Z \leqq n-2$. The finite map $q\colon \mathbb{C}^n \longrightarrow X_\sigma$ is unramified over $X_\sigma \setminus q(Z)$. We now show that $S(X_\sigma) = q(Z)$: The inclusion "$\subset$" is clear, while "$\supset$" follows from the fact that the set of points where a morphism between smooth varieties is not étale (i.e., a local analytic isomorphism) has codimension 1. Eventually we note that $\mathbb{C}^n \setminus Z \longrightarrow X_\sigma \setminus S(X)$ is the universal covering of the regular locus of X_σ – the complement of the "small" subvariety Z (of codimension at least two) in $\mathbb{C}^n$ is simply connected. So, finally, we obtain:

$$G \cong \pi_1(X_\sigma \setminus S(X)).$$

Example 4.3. 1. Let $n=2$. Consider $\sigma = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(ke_1 + \ell e_2) \subset N$ with $\ell > 0$ and $\gcd(k,\ell) = 1$. Then $q : \mathbb{T} \longrightarrow \mathbb{T}$ satisfies $q(t_1,t_2) = (t_1 t_2^k, t_2^\ell)$ and G is the cyclic group of order ℓ generated by (η^{-k}, η) with $\eta := e^{2\pi i/\ell}$.

If $k=-1$ and $\ell = 2$, the semigroup $\check{\sigma} \cap M$ is generated by $e_1^*, e_1^*+e_2^*, e_1^*+2e_2^*$. Denote $X_\sigma \hookrightarrow \mathbb{C}^3$ the corresponding closed embedding. Since $2(e_1^*+e_2^*) = e_1^* + (e_1^*+2e_2^*)$, we find $X_\sigma \subset Y := N(\mathbb{C}^3; z_2^2 - z_1z_3)$. Finally both, X_σ and Y, are irreducible surfaces, so $X_\sigma = Y$, a quadric cone.

2. Let $n=3$. Consider the cone $\sigma \subset N_\mathbb{R}$ which is the sum of the four rays $\mathbb{R}_{\geq 0}e_i, i = 1,2,3$ and $\mathbb{R}_{\geq 0}(e_1+e_2-e_3)$. Then the dual cone $\check{\sigma} \subset M_\mathbb{R}$ is generated by the rays $\mathbb{R}_{\geq 0}e_1^*, \mathbb{R}_{\geq 0}e_2^*$ and $\mathbb{R}_{\geq 0}(e_2^*+e_3^*), \mathbb{R}_{\geq 0}(e_1^*+e_3^*)$. The vectors $e_1^*, e_2^*, e_1^*+e_3^*, e_2^*+e_3^*$ even generate $\check{\sigma} \cap M$, since $\check{\sigma} = \tau_1 \cup \tau_2$ with the regular cones τ_1 and τ_2 generated by the bases $e_1^*, e_2^*, e_1^*+e_3^*$ resp. $e_1^*, e_2^*, e_2^*+e_3^*$.

We consider the corresponding closed embedding $X_\sigma \hookrightarrow \mathbb{C}^4$. Since $e^*1+(e^*2+e^*3) = (e^*1+e^*3)+e^*2$, we obtain $X_\sigma \subset Y := N(\mathbb{C}^4; z_1z_4 - z_2z_3)$. Since Y is irreducible and $\dim X_\sigma = 3 = \dim Y$, we find $X_\sigma = Y$, the "Segre cone".

5 General toric varieties

As a consequence of Sumihiro's theorem (2.4), every toric variety X can be covered by open affine toric subvarieties. We consider the following set of strongly convex polyhedral N-cones:

$$\Delta := \Delta(X) := \{\sigma \subset N_\mathbb{R}; X_\sigma \subset X\},$$

where $X_\sigma \subset X$ means that there is a toric morphism $X_\sigma \longrightarrow X$ which is an open embedding (it is unique then!). Clearly $\tau \preceq \sigma \in \Delta \Longrightarrow \tau \in \Delta$. Furthermore if $\tau, \sigma \in \Delta$, then $X_\sigma \cap X_\tau \subset X$ is a $\mathbb{T}$-invariant open affine subspace of X, hence $X_\sigma \cap X_\tau = X_\gamma$ with a cone $\gamma \in \Delta$. In fact, X being separated, we have $\mathcal{O}(X_\gamma) = \mathcal{O}(X_\sigma) \cdot \mathcal{O}(X_\tau)$ and thus $(\check{\sigma} \cap M) + (\check{\tau} \cap M) = (\check{\gamma} \cap M)$ resp. $\check{\sigma} + \check{\tau} = \check{\gamma}$ resp. $\sigma \cap \tau = \gamma \in \Delta$. Furthermore, from the description of the one-to-one correspondence between faces of a cone σ and orbits in X_σ (see Theorem 3.4 and also Remark 5.4 below) and the fact that different orbits are disjoint, it follows that any two cones can only intersect in a common face. That leads to the following

Definition 5.1. *A* **fan** *in $N_{\mathbb{R}}$ is a non-empty set Δ of strongly convex polyhedral N-cones satisfying*

1. $\tau \preceq \sigma \in \Delta \Longrightarrow \tau \in \Delta$
2. $\sigma, \tau \in \Delta \Longrightarrow \sigma \cap \tau \preceq \sigma, \tau$ *(a common face); in particular,* $\sigma \cap \tau \in \Delta$.

Proposition 5.2. *Every fan Δ in $N_{\mathbb{R}}$ determines a toric variety*

$$X_\Delta := \bigcup_{\sigma \in \Delta} X_\sigma / \sim ,$$

where two points $x' \in X_\sigma, x'' \in X_\tau$ are identified iff there is a point $x''' \in X_\gamma, \gamma = \sigma \cap \tau$, which is mapped to x' resp. x'' by the open embedding $X_\gamma \longrightarrow X_\sigma$ resp. $X_\gamma \longrightarrow X_\tau$. The Zariski topology is obtained from the quotient map p: $\bigcup_{\sigma \in \Delta} X_\sigma \longrightarrow X_\Delta$ as quotient topology; a function on $U \subset X_\Delta$ is defined to be regular iff its pull back is regular on $p^{-1}(U)$.

Example 5.3. Let $N_{\mathbb{R}} \ni e_{n+1} := -(e_1 + \ldots + e_n)$ and

$$\Delta := \{\sigma_A := \sum_{i \in A} \mathbb{R}_{\geq 0} e_i \; ; \;\; A \subsetneqq \{1, \ldots, n+1\}\}.$$

Then $X_\Delta \cong \mathbb{P}_n$.

Proof of 5.3. Let $N_n := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}_n)$, additively written. Furthermore let $\sigma := \mathbb{R}_{\geq 0} f_1 + \ldots + \mathbb{R}_{\geq 0} f_{n+1}$ with the standard basis vectors $f_1, \ldots, f_{n+1}$ of N_{n+1} and

$$\partial\sigma := \{\tau \; ; \; \tau \precneqq \sigma\}$$

the boundary fan of σ, and let Δ the fan in $(N_n)_{\mathbb{R}}$ as described above.

The homomorphism

$$q : \mathbb{T}_{n+1} \longrightarrow \mathbb{T}_n, t := (t_1, \ldots, t_{n+1}) \mapsto (t_1 t_{n+1}^{-1}, \ldots, t_1 t_{n+1}^{-1})$$

induces the map

$$dq : (N_{n+1})_{\mathbb{R}} \longrightarrow (N_n)_{\mathbb{R}}, f_i \mapsto e_i$$

and thus maps cones in $\partial\sigma$ onto the cones in Δ. Hence q extends to a morphism

$$(\mathbb{C}^{n+1})^* = X_{\partial\sigma} \longrightarrow X_\Delta.$$

Now $\ker(q) = D := \{(s, \ldots, s); s \in \mathbb{C}^*\}$, and thus the map q is D-invariant and factorizes over $\mathbb{P}_n = (\mathbb{C}^{n+1})^*/D$. In order to see that $\mathbb{P}_n \cong X_\Delta$, consider a cone $\tau_A := \sum_{i \in A} \mathbb{R}_{\geq 0} f_i \in \partial\sigma$. Then

$$q^{-1}(X_{\sigma_A}) = X_{\tau_A} \cong Z_{\tau_A} \times D \longrightarrow X_{\sigma_A} \cong Z_{\tau_A}$$

is the projection onto the first factor. That proves our claim. $\square$

Remark 5.4. 1. As in the affine case, we can associate to any cone $\sigma \in \Delta$ an orbit O_σ via

$$\Delta \ni \sigma \mapsto O_\sigma := \mathbb{T} \cdot (\lim_{s \to 0} \alpha(s)),$$

where $\alpha \in \mathring{\sigma}$, and again Theorem 3.4 and formula (4) hold. In fact, the orbit $O_\sigma \hookrightarrow X_\sigma$ is the unique closed orbit of the open subvariety $X_\sigma \subset X_\Delta$, and the $\mathbb{T}/\mathbb{T}_\sigma$-toric variety $\overline{O}_\sigma \hookrightarrow X_\Delta$ satisfies

$$\overline{O}_\sigma \cong X_{\Delta/\sigma}$$

with the fan $\Delta/\sigma := \{\tau/\sigma; \tau \in \Delta, \sigma \preceq \tau\}$ in $(N/N_\sigma)_\mathbb{R}$.

2. A toric variety X_Δ is **complete** iff the support $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ of Δ is the entire space $N_\mathbb{R}$.

3. Let Δ, Λ be fans in $N_\mathbb{R}$. Then there is a toric morphism $X_\Delta \longrightarrow X_\Lambda$ iff every cone $\sigma \in \Delta$ is contained in some cone $\tau \in \Lambda$.

4. Assume that in the above situation, every cone in Δ is contained in some cone of Λ. Then the morphism $X_\Delta \longrightarrow X_\Lambda$ is **proper*** iff $|\Lambda| = |\Delta|$. In that case Δ is also called a **subdivision** or **refinement** of Λ.

Multiplicities and divisors. The characters $\chi \in \mathbb{X}(\mathbb{T})$ are regular functions on $\mathbb{T} \subset X_\Delta$ and thus, rational functions on X_Δ, i.e., elements of the function field $\mathbb{C}(X_\Delta)$. We recall that quite generally, in order to study the zeros and poles of a rational function $f \in \mathbb{C}(Z)$ on a normal variety Z, one associates to every 1-codimensional irreducible subvariety $Y \hookrightarrow Z$ a "multiplicity" $v_Y(f) \in \mathbb{Z}$ (vanishing order). Since, for given $f \in \mathbb{C}(Z)$, there are only finitely many such subvarieties with $v_Y(f) \neq 0$, the pertinent information may be encoded in the "divisor" (f) of f, a formal linear combination of the following kind: A divisor D on Z is a finite formal sum

$$D = \sum_{Y \hookrightarrow Z} n_Y \cdot Y$$

where the sum runs over all 1-codimensional irreducible subvarieties $Y \hookrightarrow Z$ and the coefficients n_Y are integers, such that $n_Y \neq 0$ for only finitely many Y.

The divisor (f) of a rational function f is defined as follows:

$$(f) := \sum_{Y \hookrightarrow Z} v_Y(f) \cdot Y.$$

Returning to the toric case, a character $\chi \in \mathbb{X}(\mathbb{T}) \subset \mathbb{C}(X_\Delta)$ has neither zeros nor poles on $\mathbb{T} \subset X_\Delta$, and since $X_\Delta \setminus \mathbb{T} = \bigcup_{i=1}^k \overline{O}_{\varrho_i}$ with the rays (1-dimensional cones) $\varrho_1, \dots, \varrho_k$ of Δ,

*In the strong topology of complex varieties, "proper" means that the inverse image of a compact subset is again compact.

non-trivial multiplicities can only occur along the orbit closures $\overline{O}_{\varrho_i}$. Let $\varrho_i = \mathbb{R}_{\geq 0}\alpha_i, i = 1, \ldots, k$ with primitive vectors (injective one parameter subgroups) $\alpha_i \in N$. Then

$$(\chi) = \sum_{i=1}^{k} \langle \chi, \alpha_i \rangle \, \overline{O}_{\varrho_i} \ .$$

For a proof consider a ray $\varrho = \mathbb{R}_{\geq 0}\alpha$ with primitive $\alpha \in N$. Then $\alpha : \mathbb{C}^* \longrightarrow \mathbb{T}_\varrho$ is an isomorphism, write now, as in 3.6, $X_\varrho = Z_\varrho \times \mathbb{T}_0 = \mathbb{C} \times \mathbb{T}_0$ with $O_\varrho = 0 \times \mathbb{T}_0 =: O$. For $f \in \mathbb{C}(X_\varrho)$ then $v_O(f)$ is the multiplicity of the function $s \mapsto f(s,t)$ at $s = 0$ for almost all $t \in \mathbb{T}$. But $\chi(s,t) = s^{\langle \chi, \alpha \rangle}$.

In general, a "toric" divisor on X_Δ has the form

$$D = \sum_{i=1}^{k} n_i \overline{O}_{\varrho_i} \ .$$

The toric divisor obtained by assigning to each orbit closure $\overline{O}_{\varrho_i}$ the coefficient $n_i = 1$ is naturally distinguished. Without getting into any details, we just mention that its negative

$$K_X = -\sum_{i=1}^{k} \overline{O}_{\varrho_i}$$

is the *canonical divisor* that plays a very important role for studies that are outside of the scope of the present minicourse.

6 Resolution of toric singularities

In general, smooth (i.e., non-singular) varieties are much better understood and usually enjoy much nicer formal properties than singular ones. In studying singular varieties, it is thus a natural attempt to "resolve" the singularities of such a variety X, i.e., to find a non-singular "model" $\widetilde{X}$ together with a proper morphism $\widetilde{X} \to X$ that is an isomorphism on the complement of the singularities. Slightly more generally, allowing some regular points also to be "resolved", we may look for a proper morphism $\widetilde{X} \to X$ from a smooth variety that is an isomorphism between open dense subsets. The general resolution of singularities is rather involved. For complex varieties, it has been achieved by a celebrated result of HIRONAKA.

Resolution of singularities in the case of toric varieties is much more accessible: We recall that a general toric variety X_Δ is smooth if and only if the fan $\Delta = \Delta(X)$ is regular, i.e., it consists only of regular cones (see Prop. 4.1). Furthermore, we recall from Remark 5.4 that a subdivision Δ' of a (general) fan Δ corresponds to a (torus-equivariant) proper morphism $X_{\Delta'} \to X_\Delta$ that induces an isomorphism on the common open invariant subvariety $X_{\Delta''}$ for $\Delta'' := \Delta \cap \Delta'$. Hence, finding a regular subdivision provides an equivariant resolution of singularities.

Theorem 6.1 (Equivariant resolution of toric singularities). *For every toric variety, there is a regular subdivision of the defining fan and thus, an equivariant resolution of singularities.*

We first discuss the proof in the case of toric surfaces since it can be dealt with most explicitely: It is the lowest dimension where singularities can occur; these singularities are necessarily isolated, and since a two-dimensional fan is necessarily simplicial, the singularities are of the nice "quotient" type discussed in §4. At the end of the section, we sketch the main ideas for the proof in the general case.

Theorem 6.2 (Equivariant resolution of toric surface singularities). *For every toric surface, there is a unique minimal regular subdivision of the defining fan and thus, a canonical equivariant resolution of singularities.*

To obtain this canonical subdivision, it clearly suffices to prove the statements for cones.

Lemma 6.3. *For every affine toric surface, there is a unique minimal regular subdivision of the defining cone and thus, a canonical equivariant resolution of the singularity.*

Before starting the proof, we first associate to a simplicial cone σ of arbitrary dimension $d \leqq n$ an integer $m_\sigma \in \mathbb{N}_{\geqq 1}$ called its *multiplicity* that measures its "non-regularity".

Definition and Remark 6.4 (Multiplicity of a simplicial cone). To each d-dimensional simplicial cone $\sigma := \sum_{j=1}^{d} \varrho_j$ spanned by the rays $\varrho_j = \mathbb{R}_{\geqq 0}\alpha_j$ with primitive generators $\alpha_j \in N$, one assigns its *multiplicity* m_σ as follows: Let $N_\sigma := N \cap (\sigma - \sigma)$ be the saturated sublattice cut out by the linear hull of σ, and put $\Gamma_\sigma := \bigoplus_{j=1}^{d} \mathbb{Z}\alpha_j$, the sublattice spanned by the primitive generators of the rays. Then

$$m_\sigma := [N_\sigma : \Gamma_\sigma] \in \mathbb{N}_{\geqq 1} .$$

(For an n-dimensional simplicial cone σ, this multiplicity coincides with the order of the group G studied in §4.) For later use, we note the equivalence

$$m_\sigma = 1 \iff \sigma \text{ is regular.}$$

We now prove Lemma 6.3.

Proof. Let $\sigma = \operatorname{cone}(\alpha_1, \alpha_2) := \mathbb{R}_{\geqq 0}\alpha_1 + \mathbb{R}_{\geqq 0}\alpha_2$ be a two-dimensional cone spanned by primitive lattice vectors $\alpha_i \in N$.

The proof proceeds by induction on m_σ. For $m_\sigma = 1$, we are in the smooth case, so there is nothing to prove. For $m_\sigma > 1$, there is a lattice basis e_1, e_2 of $N \cong \mathbb{Z}^2$ with

$$\alpha_2 = e_2 \qquad \text{and} \qquad \alpha_1 = \lambda e_1 - \kappa e_2 \quad \text{with} \quad 1 \leqq \kappa < \lambda = m_\sigma .$$

Subdividing the cone σ by the ray $\mathbb{R}_{\geqq 0}e_1$ yields the regular cone $\sigma' = \operatorname{cone}(e_1, e_2)$ and the complementary cone $\sigma'' := \operatorname{cone}(\alpha_1, e_1)$ with multiplicity $m_{\sigma''} = \kappa < \lambda$.

By induction hypothesis, this cone σ'' admits a unique minimal regular subdivision. To show uniqueness and minimality of the "total" subdivision of σ thus obtained, it essentially suffices to see that removing the ray $\mathbb{R}_{\geqq 0}e_1$ spoils regularity of the "final" cone that contains e_2. By hypothesis, the slope of the ray through α_1 satisfies the inequality $-1 < -\kappa/\lambda < 0$, so cone(e_1, e_2) is the largest regular subcone that contains e_2. □

Exercise 6.5. 1. Given two primitive lattice vectors $v_1, v_2 \in \mathbb{Z}^2$ with $\det(v_1, v_2) = m > 1$, prove that there exists a (positively oriented) lattice basis (e_1, e_2) and an integer q with $1 \leqq q < m$, $\gcd(m, q) = 1$ such that $v_1 = me_1 - qe_2$ and $v_2 = e_2$.

2. In the proof of the lemma, show that the first subdivision yields a resolution of the singularity if and only if $\kappa = 1$.

3. Show that the maximal number of necessary subdivisions equals $m_\sigma - 1$, and characterize the case when this occurs.

The basic idea of the *proof of the general Toric Resolution Theorem* 6.1 is to proceed in two subdivision steps: In the first step, the fan is made simplicial without introducing new rays. In the second step, the simplicial fan thus obtained is made regular. Both steps rely on the following process called "stellar subdivision".

Remark and Definition 6.6 (Stellar subdivision). Let σ be a cone and let ϱ be a ray contained in σ. (Note that ϱ need not be a face of σ.) Then the set of all cones

$$s_\varrho(\sigma) := \{\tau, \varrho + \tau\,;\ \tau \precneqq \sigma\,,\ \tau \cap \varrho = o\}$$

is a fan subdividing σ. It is called the *stellar subdivision* of σ with respect to ϱ.

If Δ is a fan containing the ray ϱ in its support, then its *stellar subdivision* with respect to ϱ is the fan

$$s_\varrho(\Delta) := \bigcup_{\sigma \in \Delta} s_\varrho(\sigma)\,,$$

where $s_\varrho(\sigma) := \langle\sigma\rangle$, the fan consisting of σ and all its faces, for $\varrho \not\subset \sigma$.

For the "simplicialization" step, we apply stellar subdivision with respect to rays of the fan.

Lemma 6.7 ("Simplicialization"). *Every fan admits a simplicial subdivision without additional rays.*

Outline of the proof. We call a ray $\varrho \in \Delta$ in a fan Δ *free* if in every cone $\sigma \in \Delta$ containing ϱ as an edge, all the other rays lie in a single "complementary" facet $\tau \in \partial\sigma$, i.e., we have $\sigma = \varrho + \tau$. Clearly $\varrho \in \Delta$ is free iff $s_\varrho(\Delta) = \Delta$, so in particular, ϱ becomes a free ray in $s_\varrho(\Delta)$. If another ray $\varrho' \in \Delta$ is already free, then it remains free when regarded as ray of $s_\varrho(\Delta)$. Furthermore, Δ is simplicial iff all its rays are free.

Now successively applying stellar subdivision with respect to the rays $\varrho_1, \dots, \varrho_k$ of the fan Δ yields a sequence $\Delta_0 = \Delta, \Delta_1, \dots, \Delta_k$ of successive subdivisions, where Δ_{i+1} is obtained from Δ_i by stellar subdivision with respect to ϱ_i. Using the properties stated above, it is clear that in the final fan Δ_k, all rays are free, so the final fan is a simplicial subdivision of Δ. □

Whereas the minimal regular subdivision in the surface case is unique, the result of the simplicial subdivision depends on the chosen order of the rays, even if the resulting simplicial fans are regular.

Exercise 6.8. For the three-dimensional toric singularity considered in example 4.3, part 2, show that there are exactly two different simplicial subdivisions without additional rays and that the resulting fans are regular.

Lemma 6.9 ("Regularization"). *Every simplicial fan admits a regular subdivision.*

Outline of the proof. We only sketch key ideas. One works through the "skeleta" of increasing dimension, making these regular by introducing new rays and subdividing the fan using these new rays so that it remains simplicial and that it gets "more regular" at each step. This is measured by the "total" multiplicity $m_\Delta := \max_{\sigma\in\Delta} m_\sigma$: In the case $m_\Delta = 1$, the fan is regular. For $m_\Delta = m > 1$, one takes a non-regular cone σ of smallest dimension d and of maximal multiplicity among all d-dimensional cones, and applies the following procedure: One chooses a new ray $\widehat{\varrho} \notin \Delta$ that passes through the relative interior of σ. Applying the stellar subdivision process with respect to $\widehat{\varrho}$ yields a new simplicial fan. One shows that the new ray may be chosen in such a way that the new d-dimensional cones obtained in the process have lower multiplicity, and that the total multiplicity does not increase. By induction, this yields the proof. □

Exercise 6.10. Consider the three-dimensional singular simplicial cone σ spanned by $v_1 = e_1, v_2 = e_2$, and $v_3 = \sum_{j=1}^{3} j \cdot e_j$.

1. Prove that its boundary fan is *regular*.
2. Prove that the new ray $\widehat{\varrho} := \mathbb{R}_{\geq 0} \cdot w$ spanned by $w := v_3 - e_3$ passes through the relative interior of σ and that the stellar subdivision with respect to it yields cones of smaller multiplicity.

7 References

Standard references for toric varieties are the books *Combinatorial Convexity and Algebraic Geometry* by Günter EWALD (Springer GTM 168, 1996), *Introduction to Toric Varieties* by William FULTON (Princeton Annals of Math. Studies 131, 1993), and *Convex Bodies and Algebraic Geometry* by Tadao ODA (Springer Ergebnisse 15, 1988).

Furthermore, we have used the notes *Vorlesungen über torische Varietäten* by one of us (Ludger KAUP, Konstanzer Schriften in Mathematik und Informatik 130, 2001, web address: `http://www.inf.uni-konstanz.de/Schriften`).

Searching the web unveils quite a few lecture notes on toric varieties. In particular, we have profited from the notes written by David COX (`http://www.amherst.edu/~dacox/`) for the "Toric Summer School" at the Institut Fourier in Grenoble, 2000 (see `http://www-fourier.ujf-grenoble.fr/~bonavero/articles/ecoledete/ecoledete.html`)

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