The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1671/4

Advanced School and Workshop on Singularities in Geometry and Topology

(15 August - 3 September 2005)

On Milnor's Fibration Theorem

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ICTP, Trieste, August 2005
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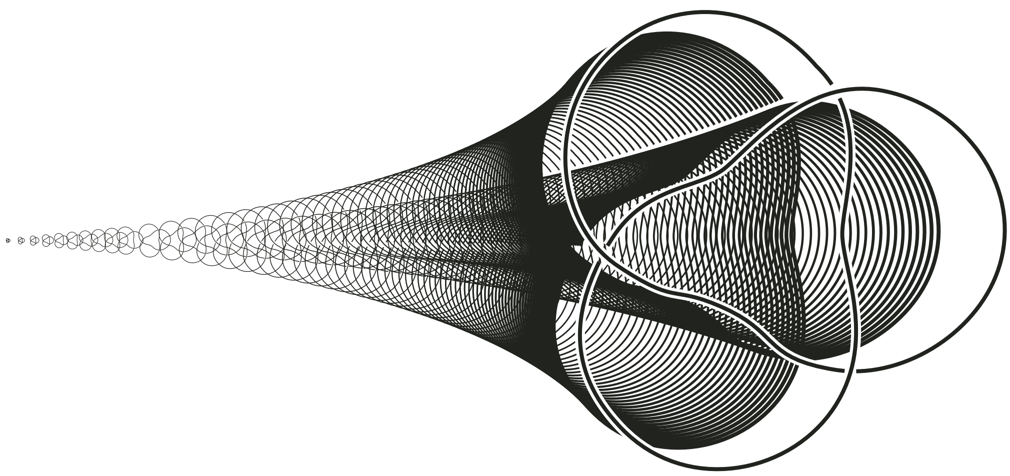

1 Introduction and an example

Milnor's fibration theorem is about the topology of the fibres of holomorphic functions $\mathbb{C}^{n+1} \xrightarrow{f} \mathbb{C}$ near their critical points. Let us illustrate this with an example which was the motivation for outstanding results in this direction in the early 60's, including Milnor's theorem.

Consider the Pham-Brieskorn polynomial

$$f(z_0, ..., z_n) = z_0^{a_0} + \cdots + z_n^{a_n} \quad , \ a_i > 1 \, .$$

It is clear that the origin $0 \in \mathbb{C}^{n+1}$ is the only critical point of f, so the fibres $V_t = f^{-t}$ are all complex n-manifolds for $t \neq 0$ and $V = f^{-1}(0)$ is a complex hypersurface with an isolated singularity at 0.

We want to study the topology of V and of the $V_t's$. For this let d be the least common multiple of the a_i and define an action Γ of the non-zero complex numbers $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by:

$$\lambda \cdot (z_0, \cdots, z_n) \mapsto (\lambda^{d/a_0} z_0, \cdots, \lambda^{d/a_n} z_n) \, .$$

Notice this action satisfies:

$$f(\lambda \cdot (z_0, \cdots, z_n)) \; = \; \lambda^d \cdot f(z_0, \cdots, z_n) \, .$$

Hence V is an invariant set of the action and one has the following properties:

1.1) Restricting the action to $t \in \mathbb{R}^+$ we get a real analytic flow (or a vector field) on $\mathbb{C}^{n+1}$ whose orbits are real lines (arcs) which converge to 0 when t tends to 0, they escape to ∞ when $t \to \infty$ being transversal to all spheres around 0, and they leave V invariant (i.e. if an orbit meets V then it is fully contained in V).

1.2) Restricting the action to the unit circle $\{e^{i\theta}\}$ we get an $\mathbb{S}^1$-action on $\mathbb{C}^{n+1}$ such that each sphere around 0 is invariant and for each $(z_0, ..., z_n) \neq V$ we have

$$e^{id\theta} f(z_0, \cdots, z_n) = f(e^{i\theta} \cdot (z_0, ..., z_n))$$

that is, if we set $\zeta = f(z_0, \cdots, z_n)$, then multiplication by $e^{i\theta}$ in $\mathbb{C}^{n+1}$ transports the fibre $f^{-1}(\zeta)$ into the fibre over $e^{id\theta} \cdot \zeta$.

Each of these two properties has important implications. The first property (1.1) implies:

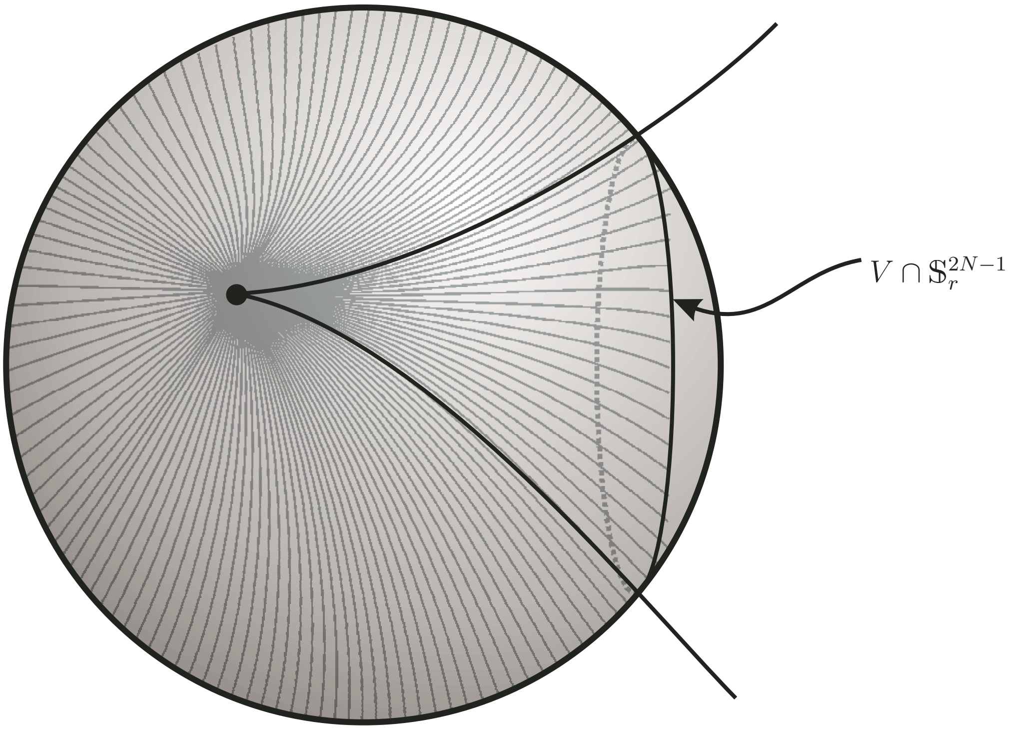

Figure 1: The conical structure

1.3) V intersects transversally every $(2n+1)$-sphere $\mathbb{S}_r$ around the origin; hence the intersection $K_r = V \cap \mathbb{S}_r$ is a smooth manifold of real dimension $2n-1$ embedded as a codimension 2 submanifold of the sphere $\mathbb{S}_r$;

1.4) the flow determines a 1-parameter group of difeomorphisms that preserve K, thus the diffeomorphism type of K_r is independent of the choice of the sphere $\mathbb{S}_r$; and

1.5) the embedded topological type of V in $\mathbb{C}^{n+1}$ is determined by the pair $(\mathbb{S}_r, K_r)$, i.e. the pair $(\mathbb{C}^{n+1}, V)$ is homeomorphic to the (global) cone over the pair $(\mathbb{S}_r, K_r)$.

The manifold $K = K_r$ (for some r) is called the *link* of the singularity. The manifolds that arise in this way have been studied by several authors that have obtained remarkable results. We refer to Chapter 1 in [18] for an overview of the topic, including a large bibliography.

On the other hand, the second property (1.2) above implies that for constant $|t| = \delta > 0$ the fibres $f^{-1}(t)$ are all diffeomorphic and we have a locally trivial fibre bundle over the circle $C_\delta = \{t \in \mathbb{C} \,\big|\, |t| = \delta\}$:

$$f : f^{-1}(C_\delta) \longrightarrow C_\delta \,. \tag{1.6}$$

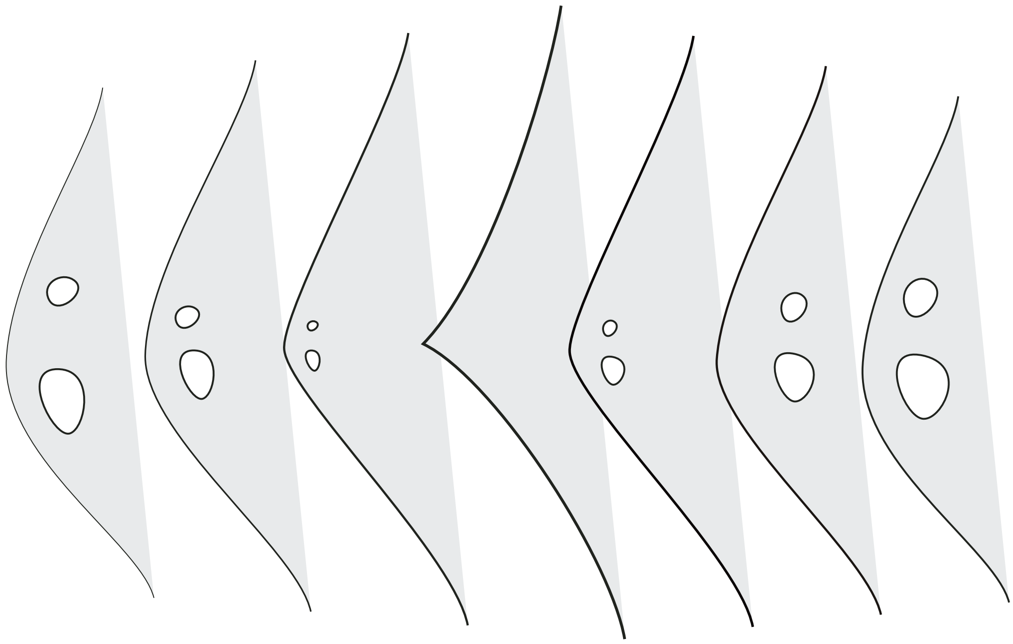

Figure 2: The Milnor fibration.

Furthermore, notice one can restrict the map f to the unit sphere $\mathbb{S}^{2n+1}$ and define

$$\phi := \frac{f}{|f|} : \mathbb{S}^{2n+1} \setminus K \longrightarrow \mathbb{S}^1 \,. \tag{1.7}$$

We see from the previous discussion that the $\mathbb{S}^1$ action Γ on $\mathbb{C}^{n+1}$ preserves the unit sphere and leaves the link K invariant; thus it also carries its complement $\mathbb{S}^{2n+1} \setminus K$ into itself. Furthermore, (1.2) tells us that the orbits of this action are also transversal to the fibres of ϕ and carry fibres of ϕ into fibres of ϕ, showing that (1.7) is also a locally trivial fibre bundle (actually equivalent to that in (1.6)). Both of these are known as *the Milnor fibration* of f, and we will indicate in the following section how they generalize to other situations. But before doing so, let us have a brief look at the topology of the fibres in the above fibrations. Since all fibres are diffeomorphic, it is enough to look at the fibre over 1:

$$f^{-1}(1) \,=\, V_1 \,=\, \{z_0^{a_0} + \cdots + z_n^{a_n} \,=\, 1\}\,.$$

The first significant results regarding the topology of these fibrations were obtained by F. Pham. To explain his results, let G_a denote the finite cyclic group of a^{th} roots of unity. Given the integers $\{a_0, \cdots, a_n\}$, denote by $J = J_{(a_0,\cdots,a_n)}$ the join:

$$J = G_{a_0} * G_{a_1} * \cdots * G_{a_n} \,\subset\, \mathbb{C}^{n+1}\,,$$

which consists of all linear combinations

$$(t_0\,\omega_0, \cdots, t_n\,\omega_n)$$

with the t_i real numbers ≥ 0 such that $t_0 + \cdots + t_n = 1$ and $\omega_j \in G_{a_j}$. Note that J can be identified with the subset $\mathcal{P} = \mathcal{P}_{(a_0,\cdots,a_n)}$ defined by:

$$\mathcal{P} = \{z \in V_1 \mid z_j^{a_j} \in \mathbb{R} \text{ and } z_j^{a_j} \geq 0\,, \text{ for all}\, j = 0, \cdots n\}\,.$$

To see this notice that $\mathcal{P}$ can also be described by the conditions:

$$z_j = u_j|z_j|\,,\, u_j \in G_{a_j}\,,\, t_j = |z_j|^{a_j}, \text{ for all}\, j = 0, \cdots, n\,.$$

Hence $\mathcal{P}$ is contained in the manifold V_1. The set $\mathcal{P}$ is known as *the join of Pham* of the polynomial f. It is not hard to see that V_1 has $\mathcal{P}$ as *a deformation retract* and therefore its homotopy type is that of $\mathcal{P}$. In fact, given a point $z \in V_1$, first deform each coordinate z_j along a path in $\mathbb{C}$ chosen so that the trajectory described by $z_j^{a_j}$ is the straight line to the nearest point on the real axis, that we denote by $\hat{z}_j$. This carries z into a vector $\hat{z} = (\hat{z}_0, \cdots, \hat{z}_n)$ satisfying $\hat{z}_j^{a_j} \in \mathbb{R}$ for each j, and it is clear that this deformation leaves V_1 invariant. Now, whenever one has that $\hat{z}_j^{a_j} < 0$, move $\hat{z}_j$ along a straight line to $0 \in \mathbb{C}$. Hence the point $\hat{z} = (\hat{z}_0, \cdots, \hat{z}_n)$ moves along a straight line towards a point $\check{z} = (\check{z}_0, \cdots, \check{z}_n) \in V_1$ whose coordinates are all ≥ 0 and one has that each coordinate $\check{z}_j$ is necessarily of the form $t_j\,\omega_j$ for some $t_j \geq 0$ and some $\omega_j \in G_{a_j}$. Finally move $\check{z}$ along a straight line to the point in $\mathcal{P}$ given by $\check{z}/(t_0 + \cdots + t_n)$. This gives a deformation of V_1 into $\mathcal{P}$ that leaves this set invariant, so the join $\mathcal{P}$ *is a deformation retract* of V_1. It is now an exercise to show that $\mathcal{P}$ *has the homotopy type of a wedge (or bouquet) of spheres* of real dimension n. Moreover, the number of spheres in this wedge is $(a_0 - 1) \cdot (a_1 - 1) \cdots (a_n - 1)$. Thus we have obtained:

1.8 Theorem (Pham). *The variety*

$$V_{(a_0,\cdots,a_n)} := \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \cdots + z_n^{a_n} \;=\; 1\,\}\,,$$

has the join $J = G_{a_0} * G_{a_1} * \cdots * G_{a_n} \subset \mathbb{C}^{n+1}$ as a deformation retract, where G_a is the finite cyclic group of a^{th}-roots of unity. Thus V_1 has the homotopy type of a bouquet $\bigvee \mathbb{S}^n$ of spheres of dimension n, the number of spheres in this wedge being $[(a_0 - 1) \cdot (a_1 - 1) \cdots (a_n - 1)]$.

2 The classical fibration theorem of Milnor

Consider now, more generally, a holomorphic function

$$(U \subset \mathbb{C}^{n+1}, 0) \xrightarrow{f} (\mathbb{C}, 0)\,,$$

defined on an open neighbourhood U of the origin in $\mathbb{C}^{n+1}$ with a critical value at $0 \in \mathbb{C}$. Assume for simplicity that $0 \in \mathbb{C}^{n+1}$ is the only critical point of f in U. Let V be the singular variety defined by f, i.e. $V := \{f^{-1}(0)\}$; thus $V^* = (V \setminus \{0\})$ is a smooth complex manifold of dimension n. We know (by work of Milnor and others) that one has in this general setting similar properties to those of the Pham-Brieskorn singularities explained in the previous section, the main difference being that in the general setting one must restrict the discussion to a "sufficiently small" neighbourhood of the singular point. Let us explain this briefly. First, one has that V is locally a cone: there exists $\varepsilon > 0$ sufficiently small so that given the ball $\mathbb{B}_\varepsilon$ in $\mathbb{C}^{n+1}$ centred at 0 of radius ε, one can construct a vector field (flow) similar to the one in the previous section given by the $\mathbb{R}^+$-action: its orbits are transversal to all the spheres in $\mathbb{B}_\varepsilon$ centred at 0, and it leaves $V \cap \mathbb{B}_\varepsilon$ invariant. Hence each sphere $\mathbb{S}_{\varepsilon'}$ in $\mathbb{C}^{n+1}$ centred at 0 of radius $\varepsilon' \leq \varepsilon$ meets V transversally. The intersection $K_\varepsilon = V \cap \mathbb{S}_\varepsilon$ is a smooth manifold of real dimension $2n - 1$

embedded as a submanifold of the $(2n+1)$-sphere $\mathbb{S}_\varepsilon$. The diffeomorphism type of the manifold K_ε and the isotopy class of the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$ does not depend on the choice of the sphere $\mathbb{S}_\varepsilon$; the pair $(\mathbb{B}_\varepsilon, K_\varepsilon)$ is homeomorphic to the cone over the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$, where ($\mathbb{B}_\varepsilon$ is the ball bounded by $\mathbb{S}_\varepsilon$, so that the topology of V near 0, and its embedding in $\mathbb{C}^{n+1}$, are determined by the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$. The manifold $K = K_\varepsilon$ is called *the link of the singularity* and the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$ is called *an algebraic knot* (notation introduced by Lê Dũng Tráng in 1971, [6])

One may thus consider the obvious map:

$$\phi = \frac{f}{|f|} : (\mathbb{S}_\varepsilon \setminus K_\varepsilon) \to \mathbb{S}^1 .$$

2.1 Theorem (Milnor, 1968). *This is a (locally trivial) C^∞ fibred bundle.*

Milnor gave two proofs of this theorem; we already had glimpses of both of them in the previous section; each proof brings out different insights and lends itself to different generalizations. Let us sketch the key-points in each of them.

1^{st} **Proof:** This is along the lines of the above proof of (1.7). The idea is simple: first show that the map ϕ has no critical points at all, so the fibres of ϕ are all smooth, codimension-1 submanifolds of $(\mathbb{S}_\varepsilon \setminus K_\varepsilon)$; then construct a tangent vector field on $(\mathbb{S}_\varepsilon \setminus K_\varepsilon)$ which is transversal to the fibres of ϕ and the corresponding flow moves at constant speed with respect to the argument of the complex number $\phi(z)$, so it carries fibres of ϕ into fibres of ϕ. This proves one has a product structure around each fibre of ϕ. For this, to begin, Milnor shows that the critical points of $(\mathbb{S}_\varepsilon \setminus K_\varepsilon) \xrightarrow{\phi} \mathbb{S}^1$ are exactly the points $z = (z_0, \ldots, z_n)$ where the vector $\big(i\,\mathrm{grad}(\log(f))\big)$ is a real multiple of z. For this, setting $f/|f| = e^{i\theta(z)}$, notice that ϕ assigns to each $z \in \mathbb{C}^n$ the argument of the complex number $f(z)$, i.e.

$$\phi(z) = \frac{f}{|f|}(z) := e^{i\theta(z)} ,$$

and therefore the argument of this map satisfies:

$$\theta(z) = Re\,(-i\,log\,f(z)) .$$

An easy computation then shows that given any curve $z = p(t)$ in $\mathbb{C}^n \setminus f^{-1}(0)$, the chain rule implies:

$$d\theta(p(t))/dt \; = \; Re\, \langle \frac{dp}{dt}(t), i\, grad\, log\, f(z) \rangle \,, \qquad (2.1.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual hermitian product in $\mathbb{C}^n$. Hence given a vector $v(z)$ in $\mathbb{C}^n$ based at z, the directional derivative of $\theta(z)$ in the direction of $v(z)$ is:

$$Re\, \langle v(z), i\, grad\, log\, f(z) \rangle \,.$$

Since the real part of the hermitian product is the usual inner product in $\mathbb{R}^{2n}$, it follows that if $v(z)$ is tangent to the sphere $\mathbb{S}_\epsilon^{2n-1}$ then the corresponding directional derivative vanishes whenever $\big(i\, \mathrm{grad}(\log(f))\big)$ is orthogonal to the sphere, *i.e.* when it is a real multiple of z; conversely, if this inner product vanishes for all vectors tangent to the sphere then z is a critical point of ϕ.

Once we know how to characterize the critical points of ϕ and how the argument of the complex number $\phi(z)$ varies as z moves along paths in $\mathbb{S}_\varepsilon \setminus K$, Milnor makes a sharp use of his *Curve Selection Lemma* (see [9]) to conclude that ϕ has no critical points at all. This part is a little technical and we refer to Milnor's book (Chapter 4) for details. It follows that all fibres of ϕ are smooth submanifolds of the sphere $\mathbb{S}_\varepsilon$ of real codimension 1. In order to show that ϕ is actually the projection map of a C^∞ fibre bundle one must prove that one has a local product structure around each fibre. This is achieved in [9] by making a sharper use of (2.1.1) to construct a vector field w on $\mathbb{S}_\epsilon \setminus K$ satisfying:

i) the real part of the hermitian product $\langle w(z), i\, grad\, log\, f(z) \rangle$ is identically equal to 1; recall that by equation (2.1.1) this is the directional derivative of the argument of ϕ in the direction of $w(z)$.

ii) the absolute value of the corresponding imaginary part is less than 1:

$$|Re\, \langle w(z),\, grad\, log\, f(z) \rangle| \; < \; 1 \,.$$

Consider now the integral curves of this vector field, *i.e.* the solutions $p(t)$ of the differential equation $dz/dt = w(t)$. Set $e^{i\theta(z)} = \phi(z)$ as before. Since the directional derivative of $\theta(z)$ in the direction $w(t)$ is identically equal to 1 we have:

$$\theta(p(t)) \; = \; t + constant \,.$$

Therefore the path $p(t)$ projects to a path which winds around the unit circle in the positive direction with unit velocity. In other words, these paths are transversal to the fibres of ϕ and for each t they carry a point $z \in \phi^{-1}(e^{it_o})$ into a point in $\phi^{-1}(e^{it_o+t})$. If there is a real number $t_o > 0$ so that all these paths are defined for, at least, a time t_o then, being solutions of the above differential equation, they will carry each fibre of ϕ diffeomorphically into all the nearby fibres, proving that one has a local product structure and ϕ is the projection of a locally trivial fibre bundle. Milnor proves this by showing that condition (ii) above implies that all these paths are actually defined for all $t \in \mathbb{R}$, so we arrive to Theorem 2.1. □

2^{nd} **Proof:** This is along the lines of (1.6) above and this was the original approach followed by Milnor in [8] to prove (2.1) when f has an isolated critical point at $0 \in \mathbb{C}^{n+1}$; however theorem (2.1) holds also if $0 \in \mathbb{C}^{n+1}$ is a non-isolated critical point, as proved in [9] and sketched above. This second method for proving (2.1) consists of showing that given a map-germ $(\mathbb{C}^{n+1}, 0) \xrightarrow{f} (\mathbb{C}, 0)$ one has the following *Milnor-Lê fibration theorem.*

2.2 Theorem. *Let $\mathbb{S}_e$ be a sufficiently small sphere in $\mathbb{C}^{n+1}$ centred at* 0 *and choose $\delta_o > 0$ small enough with respect to ε so that all the fibres $f^{-1}(t)$ with $|t| \leq \delta$ meet $\mathbb{S}_\varepsilon$ transversally. For each $\delta > 0$, $\delta \leq \delta_o$, let $C_\delta \cong \mathbb{S}^1$ be the circle in $\mathbb{C}$ of radius δ and centred at* 0*, and set $N(\varepsilon, \delta) = f^{-1}(\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$. ($N(\varepsilon, \delta)$ is usually called a Milnor tube for f) Then:*

$$f|_{N(\varepsilon,\delta)} : N(\varepsilon, \delta) \longrightarrow C_\delta \cong \mathbb{S}^1 ,$$

is a fibre bundle, C^∞-equivalent to the bundle in (2.1).

In his book Milnor proved in the general that the fibres of the map ϕ in (2.1) are diffeomorphic to the fibres in 2.2, but he only proved that 2.2 is a fibre bundle when f has an isolated critical point. The general case was proved by Lê Dũng Tráng using the fact that in this situation (when the target is 1-dimensional) the maps satisfy the a_f-condition of Thom. For simplicity, here we restrict to the isolated singularity case.

Given $\varepsilon > 0$ as above, then the fact that

$$f : N(\varepsilon, \delta) \setminus f^{-1}(0) \rightarrow C_\delta ,$$

is a fibre bundle is essentially a consequence of Ehresmann's fibration lemma; this follows by Thom's transversality together with the fact that, since $0 \in \mathbb{C}$

is an isolated critical point, the map f in (2.2) is a surjection and we can lift a vector field on the circle to a vector field on the tube $N(\varepsilon,\delta)$, transversal to the fibres of f. That this fibration is equivalent to the one in (2.1) is proved by showing that there exists a vector field on $\mathbb{B}_\varepsilon \setminus f^{-1}(0)$ whose solutions move away from the origin being transversal to all the spheres around 0, transversal to all the tubes $f^{-1}(C'_\delta)$, $d' \leq d_o$, and preserving for all times the argument of the complex number $f(z)$. This allows us to "inflate" the Milnor tube $f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$ to become the complement of a neighbourhood of the link K in the sphere $\mathbb{S}_\varepsilon$, taking the fibres of the fibration (2.2) into the fibres of the map ϕ in Theorem 2.1.

Notice (2.2) implies that the fibres of Milnor's fibration are diffeomorphic to complex Stein manifolds of dimension n in $\mathbb{C}^{n+1}$, hence the theorem of Andreotti-Frankel implies they have the homotopy type of a CW-complex of middle-dimension n. When f has an isolated critical point at 0 Milnor proves more: the fibres of ϕ have the homotopy type of a bouquet $\wedge\mathbb{S}^n$ of n-spheres. The number of spheres in this wedge is by definition *the Milnor number* of f, an important invariant of f.

For instance the Milnor fibre of the Morse singularity $z_o^2 + z_1^2 + \cdots z_n^2$ is diffeomorphic to the total space of the tangent bundle of the unit sphere $\mathbb{S}^n$, so it has Milnor number 1.

3 Generalizations

Several natural generalizations of this theorem have been considered by various authors. We mention here some of them.

i) The target is not $\mathbb{C}$ but $\mathbb{C}^m$, i.e. one considers holomorphic functions $\mathbb{C}^n \longrightarrow \mathbb{C}^m$. The case $n \leq m$ is interesting for several reasons but for our viewpoint here the relevant case is $n > m$. In 1971 H. Hamm [3] considered complete intersection germs

$$f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)\,,$$

and proved a fibration theorem in this context. Notice that in this case, since the critical points of f are the points where the rank of the jacobian matrix drops down, if $k > 1$ and $0 \in \mathbb{C}^k$ is a critical value, then the critical values of f are necessarily an analytic subset $\Delta \subset \mathbb{C}^k$ of dimension > 0 containing 0;

the set Δ is called *the discriminant* of f. Hence one can not possibly expect to have a fibration over a small punctured disc $\mathbb{D}_\delta$ around $0 \in \mathbb{C}^k$ as one does when $k = 1$. However, if $0 \in \mathbb{C}^{n+k}$ is an isolated singularity of $V = f^{-1}(0)$, then Hamm proved that one has a fibration over $\mathbb{D}_\delta \setminus \Delta$ analogous to that in (2.2), and the fibres have the homotopy type of a bouquet $\wedge \mathbb{S}^n$ of n-spheres, just as in the case of hypersurface singularities.

We remark however that (as noticed by Lê Dũng Tráng) this theorem is false in general if the critical points of f are non-isolated in V (see *Lê's example* in [18]).

ii) Given $\tilde{f} : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ holomorphic and an analytic singular variety $X \subset \mathbb{C}^N$, one may consider the restriction f of $\tilde{f}$ to X. The concept of *critical points* of f on X makes perfect sense once we equip X with a Whitney stratification (Goreski-MacPherson, Lê and others). It is proved in [5] that one has in this case a fibration theorem as in (2.2); these are called *Milnor-Lê fibrations* and they have given rise to a vast literature.

iii) Consider real analytic germs $f : (U \subset \mathbb{R}^{n+k}, 0) \to (\mathbb{R}^k, 0)$. This situation was first considered by Milnor himself in his book [9] and several authors have worked on this topic afterwards. This is explained in the following section.

A specially interesting case is when the real analytic map f is of the form $h\bar{g}$ with h and g holomorphic, or a sum of functions of this type. These situations are also discussed below.

4 On real analytic germs with a Milnor fibration

Our basic references for this section are [15, 17] (see also [13] and chapters VI to VIII in [18] for more on the subject).

For real analytic germs, Milnor's fibration theorem in [8, 9] states:

4.1 Theorem. *Let* $(\mathbb{R}^{n+k}, 0) \xrightarrow{f} (\mathbb{R}^k, 0)$ *be the germ of a real analytic map with an isolated critical point at the origin. Then for every sufficiently small sphere* $\mathbb{S}_\varepsilon = \partial \mathbb{B}_\varepsilon$ *around* $0 \in \mathbb{R}^{n+k}$ *one has that the complement* $\mathbb{S}_\varepsilon \setminus K$ *of the link* $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ *fibres over the sphere* $\mathbb{S}^{k-1}$.

The proof of this result is by noticing first that for $\delta > 0$ sufficiently small the tube $f^{-1}(\mathbb{S}^1_\delta) \cap \mathbb{B}_\varepsilon$ fibres over the circle $\mathbb{S}^1_\delta \subset \mathbb{C}$ of radius δ (just as in (2.2) above), and then constructing a vector field that "inflates" this tube taking it into the complement of (a regular neighbourhood of) the link in the sphere; see [9] or [18] for details.

When the map f is from $\mathbb{C}^{n+1}$ into $\mathbb{C}$ and is holomorphic, Milnor shows that one actually has a much richer structure, as indicated before:

i) first, one does not actually need to have an isolated critical point of f to have such a fibration: here the critical value is automatically isolated and this is enough in this case to have a fibration. For real analytic germs, isolated critical value is not enough in general (see [13]).

ii) for holomorphic germs the projection map $\phi : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^1$ can be taken to be the obvious map $\phi = f/|f|$; as Milnor shows in his book, this statement is false in general when f is not holomorphic, even if one does have a fibration;

iii) the fibres F of ϕ are parallelizable manifolds with the homotopy type of a CW-complex of dimension n, and the link K is $(n-2)$-connected.

The geometry of these fibrations associated to holomorphic singularity germs has given rise to a rich literature, as for instance the theory of fibred knots and links, open book decompositions, the results of Lawson and others about codimension 1 foliations, etc. This is not the case for the real analytic germs. There are various reasons for this, in particular because it is difficult to find examples of real analytic singularities with an isolated critical point, and it is even harder to study their underlying geometry, such as the topology of the link and of the fibres, the monodromy, etc.

Several natural -related- problems arise, as for instance: i) find examples of real analytic germs with isolated singularities and describe their underlying geometry; ii) relax the conditions in Milnor's fibration theorems in order to include larger families.

These questions have been addressed by several authors in various ways; we begin by discussing here briefly some general facts about real analytic germs with a Milnor fibration as above, and giving examples of such singularities. For simplicity we restrict the discussion to real analytic functions $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, $n > 2$.

4.2 Definition. We say that the map f *satisfies the Milnor condition at* 0 if the derivative $Df(x)$ has rank 2 at every point $x \in U - 0$, where U is an open neighbourhood of $0 \in \mathbb{R}^n$, i.e. if f is a local submersion at every point in a punctured neighbourhood of $0 \in \mathbb{R}^n$.

One has the above theorem (4.1) of Milnor for functions satisfying 4.2. For instance, every complex valued holomorphic function with an isolated critical point in its domain satisfies these conditions, and so does if we compose such function with a real analytic diffeomorphism of the source $\mathbb{C}^{n+1}$ or of the target $\mathbb{C}$, but these are somehow "fake" examples of real analytic functions. The interesting point here is to find examples which are honestly real analytic. As Milnor pointed out in his book, the hypothesis of Df having maximal rank everywhere near 0 is too strong and it is difficult to find such examples since the generic case is to have real curves in $\mathbb{R}^2$ converging to $(0,0)$, whose inverse image contains points where the Jacobian matrix has rank less than 2. Milnor actually asked whether there exist "non-trivial" examples satisfying the condition of 4.2. This question was answered positively by Looijenga [7] for n even and by Church and Lamotke [2] for n odd, using Looijenga's technique. However, no explicit examples of such singularities are given in those articles. The first explicit non-trivial example of a real analytic singularity satisfying the Milnor condition at 0, other than those of Milnor, was given by A'Campo [1]. This is given by the map $\mathbb{C}^{m+2} \to \mathbb{C}$ defined by

$$(u, v, z_1, ..., z_m) \longmapsto uv(\bar{u} + \bar{v}) + z_1^2 + ... + z_m^2 \,, \tag{4.3}$$

which is not holomorphic due to the presence of complex conjugation.

In [17, 15] there are given infinite families of singularities satisfying Milnor's condition, which are somehow in the same vein as (4.3). Before explaining these examples, let us look at a more subtle question for which we introduce the following notation from [15]:

4.4 Definition. Let $f = (f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be analytic and satisfy the Milnor condition at 0. Let K be the link; f satisfies *the strong Milnor condition at* 0 if for every sufficiently small sphere $\mathbb{S}_\varepsilon$ around 0, the map

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon - K \to \mathbb{S}^1$$

is the projection of a fibre bundle.

As shown by Milnor himself in [8, p. 99], there exist examples satisfying the condition 4.2 but not the stronger condition 4.4. So the question is: given a real analytic map-germ f satisfying the Milnor condition (4.1), when does it satisfy the strong Milnor condition? This question was first studied by Jacquemard in [4] where he gave two conditions that were sufficient to guarantee that a map f that satisfies condition (4.2) satisfies also the strong Milnor condition. The first condition (A) is geometric: that the angle between the gradient vector fields of its components $(f_1, f_2) = f$ be bounded; the second condition (B) is algebraic: that the integral closures of the jacobian ideals of f_1 and f_2 in the ring of local analytic map-germs coincide. The second condition (B) was relaxed in [15] and later Araujo dos Santos and Ruas [14] improved these results giving a condition in terms of Bekkas's c-regularity for a map-germ to satisify the strong Milnor condition. They also proved that if a map-germ satisfies Jacquemard's conditions (A) and (B) (or the weaker condition in [15]) then it satisfies c-regularity, but not conversely. As an application they proved that for quasi-homogeneous germs, conditions 4.2 and 4.4 are equivalent.

The first explicit families of examples of real singularities satisfying the Milnor conditions were given in [17] (see also [15, 18]). These singularities satisfy (4.4) and they are quasi-homogeneous; in fact the results in [15, 18] for these singularities were the starting point for the interesting work in [14]. These are defined in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by:

$$f(z) \,=\, z_1^{a_1} \bar{z}_{\sigma_1} + \cdots + z_n^{a_n} \bar{z}_{\sigma_n} \quad , \; a_i > 1 \,, \tag{4.5}$$

where $(\sigma_1, \cdots, \sigma_n)$ is any permutation of $(1, \cdots, n)$. Notice these are reminiscent of the Pham-Brieskorn singularities $f(z) = z_1^{a_1} + \cdots + z_n^{a_n}$ and they are called in [18] *twisted Pham-Brieskorn singularities.*

The proof that these singularities define Milnor fibrations

$$\phi = \frac{f}{|f|} : \mathbb{S}^{2n-1} \setminus K \longrightarrow \mathbb{S}^1 \,,$$

is similar to that of (1.7): first prove that ϕ has no critical points at all, so that all its fibres are smooth, codimension 1, submanifolds of $\mathbb{S}^{2n-1} \setminus K$, and then use the weights $a_1, \cdots, a_n$ and the permutation $(\sigma_1, \cdots, \sigma_n)$ to construct an explicit $\mathbb{S}^1$ action on $\mathbb{S}^{2n-1}$ that leaves the link K invariant and transports fibres of ϕ into fibres of ϕ.

The obvious problem now is to study the topology of these singularities, and this is essentially an open problem when $n > 2$. The simplest case is when the permutation $(\sigma_1, \cdots, \sigma_n)$ is the identity, so that the singularities are of the form:

$$f(z) \,=\, z_1^{a_1} \bar{z}_1 + \cdots + z_n^{a_n} \bar{z}_n \quad , \; a_i > 1 \,.$$

This case was studied and answered in [15] by proving that these singularities are topologically (not analytically) equivalent to the Pham-Brieskorn singularities:

$$z_1^{a_1-1} + \cdots + z_n^{a_n-1} \,,$$

whose topology is well understood.

For $n = 2$ the remaining case is when f is of the form:

$$f(z_1, z_2) \;=\; z_1^p \bar{z}_2 \;+\; z_2^q \bar{z}_1 \quad , \; p, q > 1 \,.$$

This case was studied and answered in [12]. The results in that article, together with [10], imply that the link $K = f^{-1}(0) \cap \mathbb{S}^3$ of these singularities is isotopic to the link of the complex singularity defined by

$$\widehat{h}(z_1, z_2) \;=\; z_1\, z_2\, (z_1^{p+1} \;+\; z_2^{q+1}) \,, \tag{4.6}$$

but their corresponding Milnor fibrations are not equivalent. Actually one has that the real analytic singularity

$$h(z_1, z_2) \;=\; \bar{z}_1\, \bar{z}_2\, (z_1^{p+1} \;+\; z_2^{q+1}) \,, \tag{4.7}$$

also satisfies condition (4.4) and its Milnor fibration is equivalent to that of f, by [10, 12]. The components of K corresponding to the axes $z_1 z_2 = 0$ get different orientations in (4.6) and (4.7), in a sense that can be made precise, and this implies (by [10]) that the corresponding Milnor fibrations:

$$\mathbb{S}^3 \setminus K \xrightarrow{\frac{h}{|h|}} \mathbb{S}^1 \qquad \text{and} \qquad \mathbb{S}^3 \setminus K \xrightarrow{\frac{\widehat{h}}{|\widehat{h}|}} \mathbb{S}^1 \,,$$

are not equivalent: the fibres have different Euler characteristic and the monodromy maps have different period (see [12]).

5 Singularities $f\bar{g}$ and Milnor fibrations for meromorphic germs

We notice that the singularities (4.6), as well as A'Campo's example (4.3) for $m = 0$, are all of the form $f\bar{g}$ with f, g being holomorphic functions in two complex variables. This type of singularities also appeared already in Lee Rudolph's work [16]. This motivated the study in [11, 13] of singularities in $\mathbb{C}^N$ of the form $f\bar{g}$. Notice that the zero-set of $f\bar{g}$ is $\{f = 0\} \cup \{g = 0\}$, so for $N > 2$ the link $K = \{f\bar{g} = 0\} \cap \mathbb{S}_\varepsilon^{2n-1}$ is necessarily a singular variety and $f\bar{g}$ must have non-isolated critical points. Yet, the following theorem is proved in [13]. This says that the full fibration theorem of Milnor for complex singularities (in the case of non-isolated critical point) remains valid for singularities $f\bar{g}$, including the statements about the topology of the fibres.

5.1 Theorem. *Let $f, g : (U \subset \mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be holomorphic maps on an open neighbourhood U of the origin in $\mathbb{C}^{n+1}$ such that the real analytic map*

$$f\overline{g} : (U, 0) \to (\mathbb{R}^2, 0)$$

has an isolated critical value at $0 \in \mathbb{R}^2$. Let $K = (fg)^{-1}(0) \cap \mathbb{S}_\varepsilon^{2n+1}$ be the link of fg. Then the map:

$$\phi := \frac{f\overline{g}}{|f\overline{g}|} : \mathbb{S}_\varepsilon^{2n+1} \setminus K \longrightarrow \mathbb{S}^1 \subset \mathbb{C}$$

is the projection of a $\mathbb{C}^\infty$ (locally trivial) fibre bundle, whose fibres F_θ are parallelizable manifolds, diffeomorphic to the complex manifolds $(f/g)^{-1}(t) \cap \mathring{\mathbb{D}}_\varepsilon$, where $t \in \mathbb{C}$ is a regular value of the meromorphic function f/g, with $|t|$ small, and $\mathring{\mathbb{D}}_\varepsilon$ is the interior of the disc in $\mathbb{C}^N$ whose boundary is $\mathbb{S}_\varepsilon$. Hence each fibre has the homotopy type of a CW-complex of dimension n.

The proof of this result essentially follows step by step Milnor's proof (with some extra work sometimes). A key step for proving this result is the observation that away from the link K the map $\phi = \frac{f\overline{g}}{|f\overline{g}|}$ equals the meromorphic map $\widehat{\phi} = \frac{f/g}{|f/g|}$. Thus Theorem 5.1 can be regarded as a theorem for meromorphic germs, with essentially the same proof (see [13] for details).

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