The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1671/5

Advanced School and Workshop on Singularities in Geometry and Topology

(15 August - 3 September 2005)

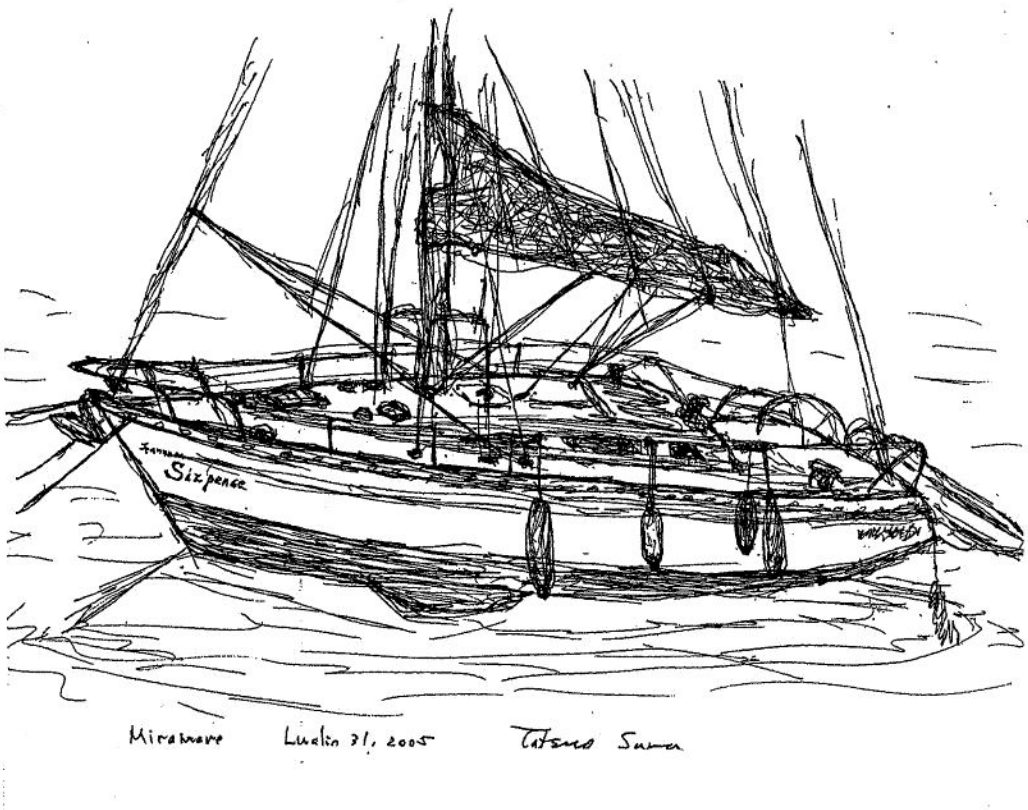
Introduction to Complex Analytic Geometry

Tatsuo Suwa
Department of Information Engineering
Niigata University
Niigata, Japan

INTRODUCTION TO COMPLEX ANALYTIC GEOMETRY

TATSUO SUWA

ICTP Trieste, August 2005



Miramare Luglio 31, 2005 Tatsuo Suwa

Contents

Chapter I. Analytic functions of several complex variables and analytic varieties

1. Analytic functions of one complex variable

Let D be an open set in the complex plane $\mathbb{C}$ and f a complex valued function on D.

Definition 1.1. We say that f is *analytic* at a point a in D if there is a power series $\sum_{n\geq 0} c_n(z-a)^n$, which converges at each point z in a neighborhood of a, such that

$$f(z) = \sum c_n(z-a)^n$$

in a neighborhood of a. We say that f is analytic in D if it is analytic at every point of D.

Definition 1.2. We say that f is *holomorphic* at a point a in D if the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. We say that f is *holomorphic* in D if it is holomorphic at every point of D.

The above limit, if it exists, is denoted by $\frac{df}{dz}(a)$ and is called the derivative of f at a. If f is holomorphic in D, then we may think of $\frac{df}{dz}$ as a function on D.

Let $z = x + \sqrt{-1}y$ with x and y the real and imaginary partrs, respectively. We may think of f as a function of (x, y). We write $f = u + \sqrt{-1}v$ with u and v the real and imaginary partrs.

In general, we say that a function of real variables is (of class) C^r, if the partial derivatives exist up to order r and are continuous. If all the partial derivatives exist we say it is C^∞.

Theorem 1.3. *The following are equivalent:*
(1) *f is analytic in D,*
(2) *f is holomorphic in D,*
(3) *f is C^1 in (x, y) and satisfies the "Cauchy-Riemann equations" in D;*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that, if we introduce the orerators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right),$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

we may write the Cauchy-Riemann equation as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

If this is the case, we have $\frac{\partial f}{\partial z} = \frac{df}{dz}$.

We finish this section by recalling the *Cauchy integral formula*. Let f be an analytic function in a neighborhood of a and γ the boundary of a small disk about a, oriented counterclockwise. Then we have

$$\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{f(z)\,dz}{z-a} = f(a).$$

2. Analytic functions of several complex variables

Let $\mathbb{C}^n = \{ z = (z_1, ..., z_n) \mid z_i \in \mathbb{C} \}$ be the product of n copies of $\mathbb{C}$. For an n-tuple $\nu = (\nu_1, ..., \nu_n)$ of non-negative integers, we set $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$, $|\nu| = \nu_1 + \cdots + \nu_n$ and $\nu! = \nu_1! \cdots \nu_n!$.

Let D be an open set in $\mathbb{C}^n$ and f a complex valued function on D.

Definition 2.1. We say that f is *analytic* at a point a in D if there is a power series $\sum_{|\nu|\geq 0} c_\nu (z-a)^\nu = \sum_{\nu_1,\ldots,\nu_n \geq 0} c_{\nu_1\ldots\nu_n} (z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{\nu_n}$, which converges absolutely at each point z in a neighborhood of a, such that

$$f(z) = \sum c_\nu (z-a)^\nu$$

in a neighborhood of a. We say that f is analytic in D if it is analytic at every point of D.

The following can be proved by a repeated use of the Cauchy integral formula:

Theorem 2.2. *The following conditions are equivalent :*
(1) *f is analytic in D.*
(2) *f is continuous and is analytic in each variable z_i in D, for $i = 1, \ldots, n$.*

It is known that we may remove the continuity condition in (2) above (Hartogs' theorem). From Theorems 1.3 and 2.2, we have :

Theorem 2.3. *The following are equivalent :*
(1) *f is analytic in D.*
(2) *f is C^1 and satisfies the Cauchy-Riemann equation $\dfrac{\partial f}{\partial \bar{z}_i} = 0$ in D, for $i = 1, \ldots, n$.*

In the sequel, we call analytic function also a holomorphic function and use the words "analytic" and "holomorphic" interchangeably.

Note that, if f is holomorphic, for arbitrary ν, the partial derivative

$$\frac{\partial^\nu f}{\partial z^\nu} = \frac{\partial^{|\nu|} f}{\partial z_1^{\nu_1} \cdots \partial z_n^{\nu_n}}$$

exists and is holomorphic in D. If $f(z) = \sum c_\nu (z-a)^\nu$ is a power series expansion of f, then each coefficient c_ν is given by

$$c_\nu = \frac{1}{\nu!} \frac{\partial^\nu f}{\partial z^\nu}(a).$$

This series is called the Taylor series of f at a.

Let D be an open set in $\mathbb{C}^n$ and $f : D \longrightarrow \mathbb{C}^m$ a map. We say that f is holomorphic if, when we write f componentwise as $f = (f_1, \ldots, f_m)$, each f_i is holomorphic. Let D and D' be two open sets in $\mathbb{C}^n$ and $f : D \longrightarrow D'$ a map. We say that f is biholomorphic, if f is bijective and if both f and f^{-1} are holomorphic.

It is not difficult to see that the composition of holomorphic maps is holomorphic.

For a holomorphic map $f = (f_1, \ldots, f_m)$ from an open set D in $\mathbb{C}^n$ into $\mathbb{C}^m$, we set

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_n)} = \begin{pmatrix} \dfrac{\partial f_1}{\partial z_1} & \cdots & \dfrac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial z_1} & \cdots & \dfrac{\partial f_m}{\partial z_n} \end{pmatrix}$$

and call it the Jacobian matrix of f with respect to z.

Definition 2.4. We say that a point a in D is a *regular point* of f, if the rank of the Jacobian matrix $\left(\partial(f_1, \ldots, f_m)/\partial(z_1, \ldots, z_n)\right)(a)$, evaluated at a, is maximal possible, i.e., $\min(n, m)$. Otherwise we say that a is a *critical* (or *singular*) point of f.

When $n = m$, the determinant of the Jacobian matrix is called the Jacobian of f with respect to z. Thus, in this case, a is a regular point of f if and only if $\det\left(\partial(f_1, \ldots, f_n)/\partial(z_1, \ldots, z_n)\right)(a) \neq 0$. If we denote by u_i and v_i the real and the imaginary parts of f_i, we compute :

$$\det \frac{\partial(u_1, v_1, \ldots, u_n, v_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)} = \left| \det \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)} \right|^2. \tag{2.5}$$

The following two theorems show how a holomorphic map looks like in a neighborhood of a regular point. Without loss of generality, we may only consider maps f from a neighborhood of the origin 0 in $\mathbb{C}^n$ into $\mathbb{C}^m$ with $f(0) = 0$.

Theorem 2.6 (Inverse mapping theorem). *Let f be a holomorphic map from a neighborhood of 0 in $\mathbb{C}^n$ into $\mathbb{C}^n$ with $f(0) = 0$. If 0 is a regular point of f, then there are open neighborhoods U and V of 0 such that f is a biholomorphic map from U onto V.*

This theorem follows from (2.5), the inverse mapping theorem in the real case and the Cauchy-Riemann equation. From this theorem, we get the following theorem as in the real variable case.

Theorem 2.7 (Implicit function theorem). *Let f be a holomorphic map from a neighborhood of the origin 0 in $\mathbb{C}^n$ into $\mathbb{C}^m$ with $f(0) = 0$. We assume that 0 is a regular point of f.*
(I) *Suppose $n \geq m$. Thus the rank of the Jacobian matrix is m and, by renumbering the functions and the variables, if necessary, we may assume that*

$$\det \frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_m)}(0) \neq 0.$$

In this case, there exist neighborhoods U and V of 0 in $\mathbb{C}^n$ and a biholomorphic map h from U onto V with $h(0) = 0$ such that

$$(f \circ h)(z_1, \ldots, z_m, \ldots, z_n) = (z_1, \ldots, z_m)$$

for $(z_1, \ldots, z_n)$ in a neighborhood of 0.
(II) *Suppose $n \leq m$. Thus the rank of the Jacobian matrix is n and we may assume that*

$$\det \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}(0) \neq 0.$$

In this case, there exist neighborhoods U and V of 0 in $\mathbb{C}^m$ and a biholomorphic map h from U onto V with $h(0) = 0$ such that

$$(h \circ f)(z_1, \ldots, z_n) = (z_1, \ldots, z_n, 0, \ldots, 0)$$

for $(z_1, \ldots, z_n)$ in a neighborhood of 0.
(III) *Suppose $n > m$. Thus the rank of the Jacobian matrix is m and we may assume as in* (I) *that*

$$\det \frac{\partial(f_1, \ldots, f_m)}{\partial(z_1, \ldots, z_m)}(0) \neq 0.$$

In this case, there is a holomorphic map g from a neighborhood of 0 in $\mathbb{C}^{n-m}$ into $\mathbb{C}^m$ with $g(0) = 0$ such that

$$f(g_1(z_{m+1}, \ldots, z_n), \ldots, g_m(z_{m+1}, \ldots, z_n), z_{m+1}, \ldots, z_n) = 0$$

for $(z_{m+1}, \dots, z_n)$ *in a neighborhood of* 0.

Remark 2.8. In the case (I) above, f is a submersion in a neighborhood of 0, in the case (II), f is an embedding in a neighborhood of 0 and in the case (III), we may solve the equation

$$f(z_1, \dots, z_m, \dots, z_n) = 0$$

for $z_1, \dots, z_m$ as functions $g_1, \dots, g_m$ of $(z_{m+1}, \dots, z_n)$ in a neighborhood of 0 and the set $f^{-1}(0)$ is the graph of the map $g = (g_1, \dots, g_m)$.

The following theorem can be proved as in the one variable case.

Theorem 2.9 (Uniqueness of analytic continuation). *Let D be an open connected subset of $\mathbb{C}^n$ and let f and g be holomorphic functions in D. If there is a non-empty open set U in D such that $f = g$ on U, then $f = g$ on D.*

The following can be proved using the corresponding result in the case of one variable.

Theorem 2.10 (Maximum principle). *Let D be a connected open set in $\mathbb{C}^n$ and let f be a holomorphic function in D. If there is a point a in D such that $|f(a)| \geq |f(z)|$ for all z in a neighborhood of a, then f is a constant function on D.*

3. Germs of holomorphic functions

We list , for example, [GR] and [Mat] as references for this section. Let H be the set of functions holomorphic in some neighborhood of 0. We define a relation $\sim$ in H as follows. For two elements f and g in H, $f \sim g$ if there is a neighborhood U of 0 such that the restrictions of f and g to U are identical. Then it is easily checked that $\sim$ is an equivalence relation in H. The equivalence class of a function$f$ is called the germ of f at 0, which we also denote by f, if there is no fear of confusion. We let $\mathcal{O}_n$ be the quotient set of H by this equivalence relation. The set $\mathcal{O}_n$ has the structure of a commutative ring with respect to the operations induced from the addition and the multiplication of functions. It has the unity which is the equivalence class of the function constantly equal to 1.

If we denote by $\mathbb{C}\{z_1, \dots, z_n\}$ the set of power series which converge absolutely in some neighborhood of 0, this set also has the structure of a ring. Since, as in the one variable case, $f \sim g$ if and only if f and g have the same power series expansion, we may identify $\mathcal{O}_n$ with $\mathbb{C}\{z_1, \dots, z_n\}$.

In what follows we denote by R a commutative ring with unity 1. A *zero divisor* in R is an element a in R such that there is an element $b \neq 0$ in R with $ab = 0$. A ring $R \neq 0$ is an *integral domain* if there are no non-zero zero divisors, i.e, if $ab = 0$, for $a,\ b \in R$, then $a = 0$ or $b = 0$. As a consequence of Theorem 2.9, the ring $\mathcal{O}_n$ is an integral domain. Thus we may form the quotient field of $\mathcal{O}_n$, which we denote by $\mathcal{M}_n$. Each element in $\mathcal{M}_n$ can be expressed as f/g and two expressions f/g and f'/g' stand for the same element if and only if $fg' = f'g$. We call an element of $\mathcal{M}_n$ a germ of *meromorphic function* at 0 in $\mathbb{C}^n$.

We say that an element u in a ring R is a *unit* if there is an element v in R such that $uv = 1$. it is not difficult to see that a germ u in $\mathcal{O}_n$ is a unit if and only if it is the germ of a function u with $u(0) \neq 0$.

We say that an ideal I in a ring R is *maximal* if $I \neq R$ and if there are no ideals J with $I \subsetneqq J \subsetneqq R$. This is equivalent to saying that the quotient R/I is a field. Let $\mathfrak{m}$ denote the set of non-units in $\mathcal{O}_n$. Then it is an ideal in $\mathcal{O}_n$. Moreover, we have the following proposition.

Proposition 3.1. *The ideal $\mathfrak{m}$ is the unique maximal ideal in $\mathcal{O}_n$.*

A ring with a unique maximal ideal is called a local ring.

We analyze the structure of the ring $\mathcal{O}_n$ by induction on n. First, for a germ f in $\mathcal{O}_n$, we write $f = \sum_{|\nu| \geq 0} a_\nu z^\nu$. We say that the order of f is k, if $a_\nu = 0$ for all ν with $|\nu| < k$ and $a_{\nu_0} \neq 0$ for some ν_0 with $|\nu_0| = k$. We define the order of the germ 0 to be $+\infty$. We say that the order of f in z_n is k, if the order of $f(0, \ldots, 0, z_n)$, as a power series in z_n, is k. In this case, if k is finite, we also say that f is regular in z_n (of order k). Then we have;

Lemma 3.2. *If the order of f is k, then we may find a suitable coordinate system $(\zeta_1, \ldots, \zeta_n)$ of $\mathbb{C}^n$ such that the order of f in ζ_n is k.*

We consider the ring $\mathcal{O}_{n-1}[z_n]$ of polynomials in z_n with coefficients in $\mathcal{O}_{n-1}$:

$$\mathcal{O}_{n-1}[z_n] = \{\, f(z) = a_0 + a_1 z_n + \cdots + a_k z_n^k \ \mid a_i \in \mathcal{O}_{n-1} \,\}.$$

Definition 3.3. A *Weierstrass polynomial* in z_n of degree k is an element h of $\mathcal{O}_{n-1}[z_n]$ of the form

$$h = a_0 + a_1 z_n + \cdots + a_{k-1} z_n^{k-1} + z_n^k,$$

where k is a positive integer and $a_0, a_1, \ldots, a_{k-1}$ are non-units in $\mathcal{O}_{n-1}$.

Note that in the above, $h(0, \ldots, 0, z_n) = z_n^k$. Hence the order of h in z_n is k. In general, any germ f in $\mathcal{O}_n$ is written as

$$f(z) = a_0 + a_1 z_n + \cdots + a_k z_n^k + \cdots$$

with $a_i \in \mathcal{O}_{n-1}$. The order of f in z_n is k if and only if $a_0, a_1, \ldots, a_{k-1}$ are non-units in $\mathcal{O}_{n-1}$ and a_k is a unit in $\mathcal{O}_{n-1}$. In this case, $a_k^{-1}(a_0 + a_1 z_n + \cdots + a_k z_n^k)$ is a Weierstrass polynomial in z_n of degree k. The Weierstrass preparation theorem stated below shows that such an f is essentially equal to a Weierstrass polynomial of degree k.

Theorem 3.4 (Weierstrass division theorem). *If h is a Weierstrass polynomial in z_n of degree k, then for any germ f in $\mathcal{O}_n$, there exist uniquely determined elements q in $\mathcal{O}_n$ and r in $\mathcal{O}_{n-1}[z_n]$ with* deg $r < k$ *such that*

$$f = qh + r.$$

Moreover, if f is in $\mathcal{O}_{n-1}[z_n]$, so is q. Thus we also have a division theorem in the ring $\mathcal{O}_{n-1}[z_n]$.

Theorem 3.5 (Weierstrass preparation theorem). *Let f be a germ in $\mathcal{O}_n$ which is regular in z_n of order k. Then there is a unique Weierstrass polynomial h in z_n of degree k such that $f = uh$ with u a unit in $\mathcal{O}_n$.*

Next we discuss some important properties of the ring $\mathcal{O}_n$ which follow from the above theorems. First we recall some more terms from algebra. Let R be an integral domain. An element a in R is *irreducible* if a is not a unit and if the identity $a = bc$ for elements b and c in R implies that either b or c is a unit. Note that 0 is not irreducible. We say that R is a *unique factorization domain*, or simply a *UFD*, if every element a in R which is not 0 or a unit can be expressed as a product of irreducible elements in R and the expression is unique up to the order and multiplications by units. It is known that if R is a UFD, so is the polynomial ring $R[X]$ in the variable X (Gauss' Theorem).

Theorem 3.6. *The ring $\mathcal{O}_n$ is a unique factorization domain.*

Let R be a UFD. For elements a and b in R, there is always the greatest common divisor $\gcd(a, b)$, which is unique up to multiplication by units. We say that a and b are *relatively prime* if $\gcd(a, b)$ is a unit. For a point z in $\mathbb{C}^n$, let $\mathcal{O}_{n,z}$ be the ring of germs of holomorphic functions at z, which is naturally isomorphic with $\mathcal{O}_n$. Using Theorem 3.5, we can also prove that if f and g are relatively prime in $\mathcal{O}_n$, then they are relatively prime in $\mathcal{O}_{n,z}$ for all z sufficiently close to 0.

We say that a ring R is a *Noetherian ring* if every ideal in R has a finite number of generators, namely, if I is an ideal in R, there exist a finite number of elements $a_1, \ldots, a_r$ in I such that every element a in I is written as $a = \sum_{i=1}^{r} x_i a_i$ with $x_i \in R$. It is known that if R is Noetherian, so is $R[X]$ (Hilbert basis theorem).

Theorem 3.7. *The ring $\mathcal{O}_n$ is a Noetherian ring.*

The following is a consequence of the "Riemann extension theorem", which is proved using the Weierstrass preparation theorem.

Theorem 3.8. *Let D be an open set in $\mathbb{C}^n$ and f a function holomorphic and not identically* 0 *in D. We set $V = \{ z \in D \mid f(z) = 0 \}$. If D is connected, then $D \setminus V$ is also connected.*

4. Complex manifolds and analytic varieties

References for this section will be [GH] and [Ko]. The notion of complex manifold is obtained by replacing C^∞ maps by holomorphic maps in the definition of a C^∞ manifold;

Definition 4.1. Let M be a Hausdorff topological space with countable basis. We say that M is a *complex manifold* if it admits an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ with the following properties :
(1) for each α, there is a homeomorphism φ_α from U_α onto an open set D_α in $\mathbb{C}^n$, for some n,
(2) for each pair (α, β), the map $\varphi_\alpha \circ {\varphi_\beta}^{-1}$ is biholomorphic from $\varphi_\beta(U_\alpha \cap U_\beta)$ onto $\varphi_\alpha(U_\alpha \cap U_\beta)$.

The natural number n, which is uniquely determined on each connected component of M, is called the (complex) dimension of the component. If all the components have dimension n, we say the dimension of M is n.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering as above. We call $(U_\alpha, \varphi_\alpha)$ a (holomorphic) local coordinate system on M. For a point p in U_α, we call U_α a coordinate neighborhood of p and

$$\varphi_\alpha(p) = (z_1^\alpha(p), \dots, z_n^\alpha(p))$$

the local coordinates of p (with respect to φ_α). Sometimes we identify U_α with D_α by the homeomorphism φ_α and identify p with the point $(z_1^\alpha(p), \dots, z_n^\alpha(p))$ in $D_\alpha \subset \mathbb{C}^n$. In this case we call $(z_1^\alpha, \dots, z_n^\alpha)$ a coordinate system on U_α. The collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of pairs $(U_\alpha, \varphi_\alpha)$ as above is called a system of (holomorphic) coordinate neighborhoods on M.

Examples 4.2. 1. A (non-empty) open subset in $\mathbb{C}^n$ is an n dimensional complex manifold.

2. The complex *projective space* $\mathbb{CP}^n$. We introduce a relation $\sim$ in $\mathbb{C}^{n+1} \setminus \{0\}$ by setting, for $\zeta = (\zeta_0, \dots, \zeta_n)$ and $\zeta' = (\zeta'_0, \dots, \zeta'_n)$ in $\mathbb{C}^{n+1} \setminus \{0\}$, $\zeta \sim \zeta'$ if and only if $\zeta' = t\zeta$ for some non-zero complex number t. Obviously, $\sim$ is an equivalence relation and the equivalence class of $(\zeta_0, \dots, \zeta_n)$ is denoted by $[\zeta_0, \dots, \zeta_n]$. We may give a complex structure on the quotient set $M = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ as follows. First, we give the quotient topology on M. The space M is covered by $n+1$ open sets U_i, $i = 0, 1, \dots, n$, defined by

$$U_i = \{\, [\zeta_0, \dots, \zeta_n] \in M \mid \zeta_i \neq 0 \,\}.$$

Then the map $\varphi_i : U_i \to \mathbb{C}^n$ defined by

$$\varphi_i([\zeta_0, \dots, \zeta_n]) = (\zeta_0/\zeta_i, \dots, \zeta_{i-1}/\zeta_i, \zeta_{i+1}/\zeta_i, \dots, \zeta_n/\zeta_i)$$

is a homeomorphism. Moreover, it is not difficult to check that for each pair (i,j), the map $\varphi_i \circ \varphi_j{}^{-1}$ is a biholomorphic map from $\varphi_j(U_i \cap U_j)$ onto $\varphi_i(U_i \cap U_j)$. Thus M becomes a (connected) complex manifold of dimension n, which we denote by $\mathbb{CP}^n$ and call the n dimensional complex projective space. We call $[\zeta_0, \ldots, \zeta_n]$ homogeneous coordinates on $\mathbb{CP}^n$. Note that $\mathbb{CP}^1$ is the Riemann sphere.

From the construction, the projective space $\mathbb{CP}^{n-1}$ is interpreted as the set of complex lines through 0 (one dimensional subspaces) in $\mathbb{C}^n$. Likewise the *Grassmannian* $G_p(n)$ is defined to be the set of p dimensional subspaces of $\mathbb{C}^n$. It admits also naturally the structure of a compact complex manifold of dimension $p(n-p)$ (cf. [GH] Ch.1, 5).

3. If M and M' are complex manifolds of dimensions n and n', respectively, the product $M \times M'$ has naturally the structure of a complex manifold of dimension $n + n'$.

Exercise 4.3. Let $S^{2n+1} = \{ (\zeta_0, \ldots, \zeta_n) \in \mathbb{C}^{n+1} \mid |\zeta_0|^2 + \cdots + |\zeta_n|^2 = 1 \}$ be the $2n+1$ dimensional unit sphere and π the restriction of the canonical surjection $\mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$ to S^{2n+1}. Show that π is surjective (thus $\mathbb{CP}^n$ is compact) and find the inverse image $\pi^{-1}(p)$ for each point p in $\mathbb{CP}^n$.

A complex valued function f on an open set U in a complex manifold M is said to be holomorphic if, for each local coordinate system $(U_\alpha, \varphi_\alpha)$, the function $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha)$. Also, a map $f : M \longrightarrow M'$ from a complex manifold M into another M' is said to be holomorphic if, for local coordinate systems $(U_\alpha, \varphi_\alpha)$ on M and $(V_\lambda, \psi_\lambda)$ on M', the map $\psi_\lambda \circ f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_\alpha \cap f^{-1}(V_\lambda))$. A biholomorphic map is a bijective holomorphic map f such that f^{-1} is also holomorphic.

If M is a complex manifold of dimension n, since we may identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and a holomorphic map is of class C^∞, M has the structure of a C^∞ manifold of real dimension $2n$. If $(z_1, \ldots, z_n)$ is a coordinate system on a neighborhood U of a point p in M, then writing $z_i = x_i + \sqrt{-1}y_i$ with x_i and y_i the real and the imaginary parts of z_i, we see that $(x_1, y_1, \ldots, x_n, y_n)$ is a C^∞ coordinate system on U. Let $T_{\mathbb{R},p}M$ denote the tangent space of M at p as a C^∞ manifold. We may think of the vectors

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$$

as being in the complexification $T^{c}_{\mathbb{R},p}M = T_{\mathbb{R},p}M \otimes_{\mathbb{R}} \mathbb{C}$ of $T_{\mathbb{R},p}M$. It is not difficult to see that, if we denote by T_pM and $\overline{T}_pM$ the subspaces of the $\mathbb{C}$-vector space $T^{c}_{\mathbb{R},p}M$ spanned, respectively, by $\partial/\partial z_1, \ldots, \partial/\partial z_n$ and $\partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_n$, then they do not depend on the choice of the coordinates $(z_1, \ldots, z_n)$. Thus we have:

Proposition 4.4. *For a complex manifold M we have a decomposition*

$$T^c_{\mathbb{R},p} M = T_pM \oplus \overline{T}_pM.$$

We call T_pM and $\overline{T}_pM$, respectively, the holomorphic and antiholomorphic parts of $T^c_{\mathbb{R},p} M$.

There is an important class of subsets in a complex manifold, namely, analytic varieties.

Definition 4.5. Let D be an open subset of a complex manifold M and V a subset of D. We say that V is an (*analytic*) *variety* in D if, for any point p in D, there exist a neighborhood U of p and a finite number of holomorphic functions $f_1, \dots, f_r$ on U such that

$$V \cap U = \{\, q \in U \mid f_1(q) = \cdots = f_r(q) = 0 \,\}.$$

We call $(f_1, \dots, f_r)$ a system of local defining functions of V and $f_1 = \cdots = f_r = 0$ local equations for V near p.

A variety in D is sometimes called a subvariety of D. Note that a variety in D is a closed subset of D. If V is a closed subset of D, it is a variety in D if (and only if) each point p_0 in V admits a neighborhood U with the properties in Definition 4.5. A non-empty variety which is locally defined by a single (not identically zero) holomorphic function is called a *hypersurface* (cf. Theorem 5.11 below).

The first part of the following is obvious from the uniqueness of analytic continuation and the second part follows from Theorem 3.8.

Theorem 4.6. *Let V be a variety in a connected open set D. If it is a proper subset of D, it does not have interior points. Moreover, $D \setminus V$ is connected.*

Definition 4.7. Let V be a variety. A point p in V is called a *regular point* of V if there is a system of local defining functions $(f_1, \dots, f_r)$ of V in a neighborhood of p such that p is a regular point of the map $f = (f_1, \dots, f_r)$. We say p is a *singular point* of V if it is not a regular point.

Note that if p is a regular point of V, by the inverse mapping theorem, we may assume without loss of generality that $r \leq n$ in the above.

Exercises 4.8. In what follows, let p be a regular point of a variety V.
(1) Show that, if $(f_1, \dots, f_r)$ is a system as in 4.7, then there is a neighborhood U of p such that $V \cap U$ has the structure of a complex manifold of dimension $n - r$ so that the inclusion map $\iota : V \cap U \to U$ is holomorphic.
(2) Show that, in this situation, the differential $\iota_* : T_pV \to T_pU = T_pM$ is injective. Thus we may identify T_pV with a subspace of T_pM. We call the quotient space T_pM/T_pV the (holomorphic) normal space of V in M at p.

(3) Let $(z_1, \ldots, z_n)$ be a coordinate system in a neighborhood of p in M. We identify T_pM with $\mathbb{C}^n = \{(\zeta_1, \ldots, \zeta_n)\}$ by taking $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ as its basis. Show that, in $\mathbb{C}^n$, T_pV is given by

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial z_j}(p) \cdot \zeta_j = 0, \qquad i = 1, \ldots, r.$$

For a variety V, we denote by $\mathrm{Reg}(V)$ and $\mathrm{Sing}(V)$, respectively, the sets of regular and singular points of V. By 4.8, $\mathrm{Reg}(V)$ is a complex manifold. It is shown that $\mathrm{Sing}(V)$ is again an analytic variety (cf. Ch.III, Proposition 5.3 and Remark 5.4). Hence $\mathrm{Sing}(V)$ is a closed set in V and $\mathrm{Reg}(V)$ is an open set in V. An analytic set V in D is said to be a (closed) submanifold of D if $V = \mathrm{Reg}(V)$. In this case, it is a locally closed submanifold of M.

Examples 4.9. 1. Let M be $\mathbb{C}^2$ with coordinates (z_1, z_2). We set $f(z_1, z_2) = z_1 z_2$ and $V = \{\, (z_1, z_2) \mid f(z_1, z_2) = 0 \,\}$. Thus V consists of two "complex lines" (z_1 and z_2 "axes") intersecting in $\mathbb{C}^2$ at one point (the origin 0). By definition we see that $V \setminus \{0\} \subset \mathrm{Reg}(V)$, while by looking at the neighborhood structure of 0, we see that 0 is a singular point of V (cf. Exercise 4.10, (1) below). This can be also checked by studying the behavior of the tangent spaces of the regular part. See also Ch.III, Proposition 5.3. Thus $\mathrm{Reg}(V) = V \setminus \{0\}$, which has two connected components each being a one dimensional complex manifold biholomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

2. Again let M be $\mathbb{C}^2$. We set $f(z_1, z_2) = z_1^3 - z_2^2$ and let V be the variety defined by f. By definition we see that $V \setminus \{0\} \subset \mathrm{Reg}(V)$, while 0 is a singular point of V (cf. Exercise 4.10, (2)). Thus $\mathrm{Reg}(V) = V \setminus \{0\}$, which has one component biholomorphic to $\mathbb{C}^*$. Note that V is homeomorphic to $\mathbb{C}$.

3. Let M be $\mathbb{C}^3$ with coordinates (z_1, z_2, z_3). We set $f(z_1, z_2, z_3) = z_1 z_2^2 - z_3^2$ and let V be the variety defined by f. Then $\mathrm{Reg}(V)$ is a two dimensional complex manifold and $\mathrm{Sing}(V)$ is the z_1-axis. The set V is called the Whitney umbrella.

4. Let M be $\mathbb{C}^3$. We set $f(z_1, z_2, z_3) = z_2^2 - z_1^2 z_3^2 - z_1^3$ and let V be the variety defined by f. Then $\mathrm{Reg}(V)$ is a two dimensional complex manifold and $\mathrm{Sing}(V)$ is the z_3-axis.

Exercises 4.10. (1) Let $S^3 = \{\, (z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1 \,\}$ be the three dimensional unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$. Show that, in Example 4.9, 1, the intersection $K = V \cap S^3$ consists of two circles which are unknotted but link with each other.

(2) Show that, in Example 4.9, 2, $K = V \cap S^3$ is the "torus knot of type $(2, 3)$".

(3) In Example 4.9, 2, find an explicit (holomorphic) homeomorphism from $\mathbb{C}$ onto V.

(4) Let V be the variety in $\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4)\}$ defined by the three equations :

$$z_1 z_4 - z_2 z_3 = 0, \quad z_2^2 - z_1 z_3 = 0 \quad \text{and} \quad z_3^2 - z_2 z_4 = 0.$$

Find $\mathrm{Reg}(V)$ and $\mathrm{Sing}(V)$. What is the dimension of each connected component of $\mathrm{Reg}(V)$?

Consider the n dimensional complex projective space $\mathbb{CP}^n$ with homogeneous coordinates $[\zeta_0, \ldots, \zeta_n]$ and let, for each $j = 1, \ldots, r$, $P_j(\zeta_0, \ldots, \zeta_n)$ be a homogeneous polynomial in $(\zeta_0, \ldots, \zeta_n)$ of degree d_j. Then the set

$$V = \{\, [\zeta_0, \ldots, \zeta_n] \in \mathbb{CP}^n \mid P_j(\zeta_0, \ldots, \zeta_n) = 0,\ j = 1, \ldots, r \,\}$$

is a well-defined subset of $\mathbb{CP}^n$ and is, moreover, a variety in $\mathbb{CP}^n$. In fact, in each open set $U_i = \{\, \zeta_i \neq 0 \,\}$, V is defined by the holomorphic functions $f_j = P_j(\zeta_0, \ldots, \zeta_n)/\zeta_i^{d_j}$, $j = 1, \ldots, r$. Such a variety is called a (projective) algebraic variety. It is known that every variety in $\mathbb{CP}^n$ is algebraic (Chow's theorem). In particular, the "hyperplane" defined by $\zeta_0 = 0$ is an $n-1$ dimensional submanifold of $\mathbb{CP}^n$ which may be identified with $\mathbb{CP}^{n-1}$. Thus we may express $\mathbb{CP}^n$ as a disjoint union $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$, which leads to a cellular decomposition of $\mathbb{CP}^n$; $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0$. Using this we may compute the homology of $\mathbb{CP}^n$;

$$H_p(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } p = 0, 2, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

Exercise 4.12. For complex numbers α, β and γ, let $V_{\alpha,\beta,\gamma}$ be the variety in $\mathbb{CP}^2$ defined by

$$V_{\alpha,\beta,\gamma} = \{\, [\zeta_0, \zeta_1, \zeta_2] \in \mathbb{CP}^2 \mid \zeta_0 \zeta_2^2 - (\zeta_1 - \alpha\zeta_0)(\zeta_1 - \beta\zeta_0)(\zeta_1 - \gamma\zeta_0) = 0 \,\}$$

(1) Show that, if α, β and γ are mutually distinct, $V_{\alpha,\beta,\gamma}$ has no singular points.
(2) Show that, if $\gamma \neq 0$, $V_{0,0,\gamma}$ has only one singular point at $p = [1, 0, 0]$, which is equivalent to the one in Example 4.9, 1.
(3) Show that $V_{0,0,0}$ has only one singular point at $p = [1, 0, 0]$, which is equivalent to the one in Example 4.9, 2.

5. Germs of varieties

In this section, we consider the germs of varieties and the relation between these germs and the ideals in $\mathcal{O}_n$. See, e.g, [Har] for the corresponding theory in Algebraic Geometry.

We first introduce a relation $\sim$ in the set of subsets of $\mathbb{C}^n$. Let A and B be two subsets of $\mathbb{C}^n$. We define $A \sim B$ if there is a neighborhood U of 0 such that

$A \cap U = B \cap U$. It is easily checked that this is an equivalence relation. We call the equivalence class of A the germ of A at 0 and we denote it also by A unless it is necessary to distinguish the two. Usual operations of sets induce those of germs. Thus for two germs A and B at 0, $A \cap B$, $A \cup B$ and $A \setminus B$ are well-defined. The relation $A \subset B$ is also well-defined.

Let $f_1, \ldots, f_r$ be germs in $\mathcal{O}_n$. We choose a neighborhood U of 0 such that these germs are represented by holomorphic functions on U, which we also denote by $f_1, \ldots, f_r$. We set

$$V(f_1, \ldots, f_r) = \text{the germ at 0 of } \{\, z \in U \mid f_1(z) = \cdots = f_r(z) = 0 \,\}$$

and call it the germ of the variety defined by $f_1, \ldots, f_r$. More generally, let I be an ideal in $\mathcal{O}_n$. By the Noetherian property of $\mathcal{O}_n$ (Theorem 3.7), there exist a finite number of germs $f_1, \ldots, f_r$ such that $I = (f_1, \ldots, f_r)$ (the ideal generated by $f_1, \ldots, f_r$). We set $V(I) = V(f_1, \ldots, f_r)$ and call it the germ of the variety defined by I. It is easily checked that it does not depend on the choice of generators of I. Thus each ideal in $\mathcal{O}_n$ defines a germ of variety at 0. Conversely suppose we are given a germ V of variety at 0. We choose a neighborhood U of 0 such that the germ is represented by a variety in U, which we also denote by V. We set

$$I(V) = \{\, f \in \mathcal{O}_n \mid f(z) = 0 \text{ for all } z \text{ in } V \text{ and near } 0 \,\}.$$

It is easily checked that this is an ideal in $\mathcal{O}_n$. For an ideal I in $\mathcal{O}_n$, we set

$$\sqrt{I} = \{\, f \in \mathcal{O}_n \mid f^k \in I \text{ for some positive integer } k \,\}$$

and call it the radical of I. This is again an ideal in $\mathcal{O}_n$ and it contains I.

Exercise 5.1. For $k = 1, \ldots, n$, we consider the "coordinate functions" $z_1, \ldots, z_k$ as germs in $\mathcal{O}_n$. Show that

$$I(V(z_1, \ldots, z_k)) = (z_1, \ldots, z_k).$$

There are various relations between germs of varieties and ideals, most of which follow rather straightforward from definition. The most important and deep fact will be the following theorem. We refer to [GR], for example, for the proof.

Theorem 5.2 (Hilbert Nullstellensatz). *For any ideal I in $\mathcal{O}_n$,*

$$I(V(I)) = \sqrt{I}.$$

Exercise 5.3. Show that, for an ideal I $(\neq \mathcal{O}_n)$ in $\mathcal{O}_n$, the complex vector space $\mathcal{O}_n/I$ is finite dimensional if and only if $V(I) = \{0\}$.

In general, let R be a commutative ring with identity. For an ideal I in R, its radical $\sqrt{I}$ is defined similarly as for the ones in $\mathcal{O}_n$. An ideal $\mathfrak{p}$ in R is said to be *prime* if $R/\mathfrak{p}$ is an integral domain, i.e., $\mathfrak{p} \neq R$ and $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. If $\mathfrak{p}$ is prime, then $\sqrt{\mathfrak{p}} = \mathfrak{p}$.

Definition 5.4. Let V be a germ of variety at 0. We say that V is *irreducible* if $V \neq \emptyset$ and if $V = V_1 \cup V_2$ implies $V_1 = V$ or $V_2 = V$.

Theorem 5.5. *A germ of variety V is irreducible if and only if the ideal $I(V)$ is prime.*

Corollary 5.6. *For a germ f, which is not 0 or a unit, in $\mathcal{O}_n$, the following are equivalent :*
(i) *$V(f)$ is irreducible.*
(ii) *$I(V(f))$ $(= \sqrt{(f)})$ is a prime ideal.*
(iii) *There is an irreducible element p in $\mathcal{O}_n$ such that $f = p^m$ for some positive integer m.*

The following is a consequence of the "primary decomposition theorem":

Theorem 5.7. *Any (non-empty) germ V of variety can be written as*

$$V = V_1 \cup \cdots \cup V_r,$$

where $V_1, \ldots, V_r$ are germs of varieties such that each V_i is irreducible and that $V_i \not\subset V_j$, if $i \neq j$. Moreover, $V_1, \ldots, V_r$ are uniquely determined by V up to order.

The proof of the following is not difficult.

Theorem 5.8. *Let f be a germ, which is not 0 or a unit, in $\mathcal{O}_n$. If $f = p_1^{m_1} \cdots p_r^{m_r}$ is the irreducible decomposition of f, then*

$$V(f) = V(p_1) \cup \cdots \cup V(p_r)$$

is the irreducible decomposition of $V(f)$.

Let f be a germ in $\mathcal{O}_n$, which is not 0 or a unit. We say that f is *reduced* if the irreducible decomposition of f has no multiple factors, i.e., in the irreducible decomposition $f = p_1^{m_1} \cdots p_r^{m_r}$, we have $m_i = 1$ for all i. We represent f by a holomorphic function f in a neighborhood of 0. Then, it can be proved that, if f is reduced at 0, the germ f_z in $\mathcal{O}_{n,z}$ is reduced for all z sufficiently close to 0. Note that, on the other hand, even if f is irreducible at 0, f_z may not be irreducible. For example, consider the "Whitney umbrella" (Example 4.9, 3).

Exercise 5.9. Show that f is reduced if and only if $I(V(f)) = (f)$.

We define the dimension of a variety on the basis of the following theorem. For the proof we refer to [GR].

Theorem 5.10. *Let V be an irreducible germ of variety. We may find a representative V of V such that* $\mathrm{Reg}(V)$ *is connected and dense in V.*

For a germ of variety V at 0, we define its *dimension* (at 0), denoted by $\dim V$, as follows. If V is irreducible, then we define $\dim V$ to be the dimension of the complex manifold $\mathrm{Reg}(V)$. In general, if $V = V_1 \cup \cdots \cup V_r$ is the irreducible decomposition of V, we set $\dim V = \max_{1 \leq i \leq r} \dim V_i$. We also define the codimension (denoted by codim V) by codim $V = n - \dim V$. Note that in this case we have the corresponding decomposition $\mathrm{Reg}(V) = C_1 \cup \cdots \cup C_r$ of $\mathrm{Reg}(V)$ into its connected components C_i. Each C_i is a complex manifold whose closure coincides with V_i. However, in general, C_i does not coincide with $\mathrm{Reg}(V_i)$. We say that V is pure dimensional if all the components V_i have the same dimension.

The "if" part of the following theorem follows from Theorem 5.8. For the "only if" part, we refer to [GR].

Theorem 5.11. *A germ V of variety is pure $n-1$ dimensional if and only if there is a germ f in $\mathcal{O}_n$, not 0 or a unit, such that $I(V) = (f)$.*

Let D be an open set in a complex manifold M. A variety V in D is said to be (globally) irreducible if it cannot be expressed as the union of two varieties V_1 and V_2 in D with $V_1, V_2 \neq V$. This notion should be distinguished from the "local" irreducibility (Definition 5.4). For example the variety $V_{0,0,\gamma}$ of Exercise 4.12, (2) is globally irreducible, but locally not irreducible at p. Note that every variety is written as a union of irreducible varieties. Note also that V is irreducible if and only if the regular part $\mathrm{Reg}(V)$ is connected. Hence, for an irreducible variety V and a point p in V, the dimension of V at p remains constant. We call it the dimension of V. In general, we say that V is pure dimensional, if all the irreducible components of V have the same dimension.

Chapter II. Differential forms and Čech-de Rham cohomology

1. Vector bundles

In what follows, we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ and $M_r(\mathbb{K})$ the sets of $r \times r$ matrices with entries in $\mathbb{K}$. It is naturally identified with $\mathbb{K}^{r^2}$. Also we set

$$GL(r,\mathbb{K}) = \{\, A \in M_r(\mathbb{K}) \mid \det A \neq 0\}.$$

It has the structure of a real or complex Lie group. Namely, it is a group with respect to the multiplication of matrices and, moreover, it is a C^∞ or a complex manifold according as $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, since it is an open set of $M_r(\mathbb{K})$, and the group operation is C^∞ or holomorphic.

Definition 1.1. Let M be a C^∞ manifold. A (C^∞) *vector bundle* of rank r over M is a topological space E together with a continuous map $\pi : E \to M$ such that there exists an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M with the following properties :
(1) for each α, there is a homeomorphism

$$\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$$

with $\varpi \circ \psi_\alpha = \pi$, where ϖ denotes the projection $U_\alpha \times \mathbb{K}^r \to U_\alpha$,
(2) for each pair (α,β), there is a C^∞ map

$$h^{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,\mathbb{K})$$

with

$$\psi_\alpha \circ \psi_\beta^{-1}(p,\zeta) = (p, h^{\alpha\beta}(p)\zeta) \qquad \text{for} \qquad (p,\zeta) \in U_\alpha \cap U_\beta \times \mathbb{K}^r.$$

We say that E is a real or complex vector bundle according as $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Thus if $\pi : E \to M$ is a vector bundle of rank r over a C^∞ manifold M of dimension m, then E has the structure of a C^∞ manifold of dimension $m + r$ or $m + 2r$, according as $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, so that π is a C^∞ surjective submersion (surmersion) and each *fiber* $E_p = \pi^{-1}(p)$, $p \in M$, has the structure of a vector space of dimension r over $\mathbb{K}$. We call ψ_α a trivialization of E on U_α. We also call $h^{\alpha\beta}$ the transition matrix of E on $U_\alpha \cap U_\beta$ and the collection $\{h^{\alpha\beta}\}$ the system of transition matrices of E. For each point p in $U_\alpha \cap U_\beta \cap U_\gamma$, we have the identity

$$h^{\alpha\beta}(p)h^{\beta\gamma}(p) = h^{\alpha\gamma}(p). \tag{1.2}$$

Thus, in particular, $h^{\alpha\alpha}(p) = I$ (the identity matrix) and $h^{\beta\alpha}(p) = (h^{\alpha\beta}(p))^{-1}$. We may think of the system $\{(U_\alpha, \psi_\alpha, h^{\alpha\beta})\}$ as defining a vector bundle structure on E.

If we are given an open covering $\{U_\alpha\}$ of M and a collection $\{h^{\alpha\beta}\}$ of C^∞ maps

$$h^{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r, \mathbb{K})$$

satisfying (1.2) for p in $U_\alpha \cap U_\beta \cap U_\gamma$, we may construct a vector bundle as follows. For (p_α, ζ_α) in $U_\alpha \times \mathbb{K}^r$ and (p_β, ζ_β) in $U_\beta \times \mathbb{K}^r$, we define $(p_\alpha, \zeta_\alpha) \sim (p_\beta, \zeta_\beta)$ if and only if

$$\begin{cases} p_\alpha = p_\beta (= p) \\ \zeta^\alpha = h^{\alpha\beta}(p)\zeta^\beta. \end{cases}$$

Then it is easy to see that this is an equivalence relation in the disjoint union $\bigsqcup_\alpha (U_\alpha \times \mathbb{K}^r)$. Let E be the quotient space. Then, since

$$(U_\alpha \times \mathbb{K}^r)/\sim = U_\alpha \times \mathbb{K}^r,$$

E has a vector bundle structure with $\{h^{\alpha\beta}\}$ as a system of transition matrices.

Let E and F be two vector bundles on M. A C^∞ map $\varphi : E \longrightarrow F$ is said to be a vector bundle homomorphism if it commutes with the projections and if the induced map $\varphi_p : E_p \longrightarrow F_p$ on each fiber is $\mathbb{K}$-linear. We say that φ is an isomorphism if it is a C^∞ diffeomorphism. In this case φ induces a $\mathbb{K}$-isomorphism on each fiber. We also say that E and F are isomorphic (or E is isomorphic to F), and write $E \simeq F$, if there is an isomorphism of E onto F. A vector bundle is called *trivial* if it is isomorphic to the product $M \times \mathbb{K}^r$.

Exercise 1.3. Let E and F be two vector bundles on M with systems of transition matrices $\{h^{\alpha\beta}\}$ and $\{g^{\alpha\beta}\}$, respectively, on an open covering $\{U_\alpha\}$. Show that E and F are isomorphic if and only if there exists a C^∞ map $h^\alpha : U_\alpha \longrightarrow GL(r, \mathbb{K})$, for each α, such that

$$h^{\alpha\beta}(p) = h^\alpha(p)^{-1} g^{\alpha\beta}(p) h^\beta(p),$$

for p in $U_\alpha \cap U_\beta$.

We say that a sequence of vector bundle homomorphisms

$$E \xrightarrow{\varphi} F \xrightarrow{\psi} G$$

is exact if, for each p in M, the induced sequence $E_p \xrightarrow{\varphi_p} F_p \xrightarrow{\psi_p} G_p$ is exact, i.e., $\operatorname{Ker} \psi_p = \operatorname{Im} \varphi_p$.

Let $\pi : E \longrightarrow M$ be a vector bundle of rank r. A subset E' of E is said to be a subbundle of E, if there is a system $\{(U_\alpha, \psi_\alpha, h^{\alpha\beta})\}$ as in Definition 1.1 such that each ψ_α maps ${\pi'}^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{K}^{r'}$, where π' denotes the restriction of π to E' and $\mathbb{K}^{r'}$ is identified with the subspace of $\mathbb{K}^r$ consisting of (column) vectors

$\zeta = {}^t(\zeta_1, \ldots, \zeta_r)$ (the transposed of $(\zeta_1, \ldots, \zeta_r)$) with $\zeta_{r'+1} = \cdots = \zeta_r = 0$. In this case, each $h^{\alpha\beta}$ is of the form

$$(1.4) \qquad h^{\alpha\beta} = \begin{pmatrix} h'^{\alpha\beta} & * \\ 0 & h''^{\alpha\beta} \end{pmatrix},$$

where $h'^{\alpha\beta}$ and $h''^{\alpha\beta}$ are C^∞ maps from $U_\alpha \cap U_\beta$ into $GL(r', \mathbb{K})$ and $GL(r'', \mathbb{K})$, $r'' = r - r'$, respectively. Note that each of the systems $\{h'^{\alpha\beta}\}$ and $\{h''^{\alpha\beta}\}$ satisfies (1.2). Thus E' has the structure of a vector bundle of rank r' with $\{h'^{\alpha\beta}\}$ as a system of transition matrices. The vector bundle of rank r'' defined by the system $\{h''^{\alpha\beta}\}$ is called the quotient bundle of E by E' and is denoted by E/E'. Note that there is a surjective vector bundle homomorphism $\varphi : E \longrightarrow E/E'$ so that the sequence

$$0 \longrightarrow E' \xrightarrow{\iota} E \xrightarrow{\varphi} E/E' \longrightarrow 0$$

is exact, where ι denotes the inclusion.

In general, if we may choose a system $\{h^{\alpha\beta}\}$ of transition matrices of a vector bundle E so that each $h^{\alpha\beta}$ is of the form (1.4), then E admits a subbundle with $\{h'^{\alpha\beta}\}$ as a system of transition matrices.

Exercise 1.5. Let $\varphi : E \longrightarrow F$ be a homomorphism of vector bundles. Show that, if the rank of the restriction φ_p of φ to each fiber E_p, $p \in M$, is constant, then the kernel $\operatorname{Ker}\varphi = \bigsqcup_{p\in M} \operatorname{Ker}\varphi_p$ and the image $\operatorname{Im}\varphi = \bigsqcup_{p\in M} \operatorname{Im}\varphi_p$ of φ are subbundles of E and F, respectively. Show also that the quotient bundle $E/\operatorname{Ker}\varphi$ is isomorphic to $\operatorname{Im}\varphi$. The quotient $F/\operatorname{Im}\varphi$ is called the cokernel of φ and is denoted by $\operatorname{Coker}\varphi$.

If $f : M' \longrightarrow M$ is a C^∞ map of C^∞ manifolds and if $\pi : E \longrightarrow M$ is a vector bundle over M, we define the pull-back f^*E of E by f by

$$f^*E = \{\, (p, e) \in M' \times E \mid f(p) = \pi(e) \,\}.$$

It is a vector bundle over M' with projection the restriction of the projection onto the first factor. Note that $(f^*E)_p = E_{f(p)}$. In particular, if V is a submanifold of M with inclusion map i and if E is a vector bundle on M, the pull-back i^*E is called the restriction of E to V and is denoted by $E|_V$.

A complex vector bundle over a complex manifold M is said to be holomorphic if E admits a system of transition matrices $\{h^{\alpha\beta}\}$ such that each $h^{\alpha\beta}$ is holomorphic. Note that in this case, E has the structure of a complex manifold so that the projection $E \longrightarrow M$ is a holomorphic submersion.

Let $\pi : E \longrightarrow M$ be a vector bundle of rank r and U an open set in M. A (C^∞) *section* of E on U is a C^∞ map $s : U \longrightarrow E$ such that $\pi \circ s = 1_U$, the identity map of U. A vector bundle E always admits the "zero section", i.e., the map $M \longrightarrow E$ which assigns to each point p in M the zero of the vector space E_p. The set of C^∞ sections of E on U is denoted by $C^\infty(U, E)$. This has a natural structure of vector space by the operations defined by $(s_1 + s_2)(p) = s_1(p) + s_2(p)$ and $(cs)(p) = cs(p)$ for s_1, s_2 and s in $C^\infty(U, E)$, c in $\mathbb{K}$ and p in U. If E is a holomorphic vector bundle over a complex manifold M, a section over U is said to be holomorphic if it is a holomorphic map from U into E. The set of holomorphic sections of E over U is denoted by $\Gamma(U, E)$. This has the structure of a complex vector space.

A section s on U can be described as follows. We fix a system of transition matrices $\{h^{\alpha\beta}\}$ of E on an open covering $\{U_\alpha\}$. Using the C^∞ diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$, we may write

$$\psi_\alpha(s(p)) = (p, s^\alpha(p)) \quad \text{for} \quad p \in U \cap U_\alpha,$$

where s^α is a C^∞ map from $U \cap U_\alpha$ into $\mathbb{K}^r$. For each point p in $U \cap U_\alpha \cap U_\beta$, we have

$$s^\alpha(p) = h^{\alpha\beta}(p) s^\beta(p). \tag{1.6}$$

Conversely suppose we have a system $\{s^\alpha\}$ of C^∞ maps satisfying (1.6). Then by setting $s(p) = \psi_\alpha^{-1}(p, s^\alpha(p))$ for p in $U \cap U_\alpha$, we have a section s over U.

For $k = 1, \ldots, r$, a k-frame of E on an open set U in M is a collection $\mathbf{s} = (s_1, \ldots, s_k)$ of k sections s_i of E on U linearly independent at each point in U. An r-frame is simply called a *frame*. Note that a frame of E on U determines a trivialization of E over U.

Example 1.7. Let M be a C^∞ manifold of dimension m. We may give naturally a vector bundle structure on the (disjoint) union $T_\mathbb{R} M = \bigsqcup_{p \in M} T_{\mathbb{R},p} M$ of the tangent spaces of M. First, define $\pi : T_\mathbb{R} M \longrightarrow M$ by assigning to each tangent vector its base point. Then let $\{U_\alpha\}$ be a covering of M by coordinate neighborhoods U_α with coordinates $(x_1^\alpha, \ldots, x_m^\alpha)$. By taking $(\partial/\partial x_1^\alpha, \ldots, \partial/\partial x_m^\alpha)$ as a basis of $T_{\mathbb{R},p} M$ for each p in U_α, we have a bijection $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^m$. Since we have the relation

$$\frac{\partial}{\partial x_j^\beta} = \sum_{i=1}^m \frac{\partial x_i^\alpha}{\partial x_j^\beta}(p) \frac{\partial}{\partial x_i^\alpha}, \qquad j = 1, \ldots, m,$$

for p in $U_\alpha \cap U_\beta$, we see that $\psi_\alpha \circ \psi_\beta^{-1}(p, \xi) = (p, t^{\alpha\beta}(p)\xi)$ for $(p, \xi) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^m$, where

$$t^{\alpha\beta} = \frac{\partial(x_1^\alpha, \ldots, x_m^\alpha)}{\partial(x_1^\beta, \ldots, x_m^\beta)}.$$

Hence we see that $T_\mathbb{R} M$ admits the structure of a real vector bundle of rank m with $\{t^{\alpha\beta}\}$ as a system of transition matrices. We call it the (real) *tangent bundle* of

M. A (C^∞) *vector field* v on an open set U is a (C^∞) section of $T_{\mathbb{R}}M$. Thus it is expressed as, on each $U \cap U_\alpha$,

$$v = \sum_{i=1}^{m} f_i^\alpha(x^\alpha) \frac{\partial}{\partial x_i^\alpha},$$

where the f_i^α's are (C^∞) functions on $U_\alpha \cap U$. In $U \cap U_\alpha \cap U_\beta$, we have $f^\alpha = t^{\alpha\beta} f^\beta$, $f^\alpha = {}^t(f_1^\alpha, \dots, f_m^\alpha)$. Note that $(\partial/\partial x_1^\alpha, \dots, \partial/\partial x_m^\alpha)$ is a frame of $T_{\mathbb{R}}M$ on U_α.

If V is a submanifold of dimension ℓ of M, then we may cover V with coordinate neighborhoods U_α on M with coordinates $(x_1^\alpha, \dots, x_m^\alpha)$ such that

$$V \cap U_\alpha = \{\, p \in U_\alpha \mid x_{\ell+1}^\alpha(p) = \cdots = x_m^\alpha(p) = 0 \,\}.$$

Then the restriction $t^{\alpha\beta}|_V$ of $t^{\alpha\beta}$ to $V \cap U_\alpha \cap U_\beta$ is of the form

$$t^{\alpha\beta}|_V = \begin{pmatrix} t'^{\alpha\beta} & * \\ 0 & t''^{\alpha\beta} \end{pmatrix},$$

where $t'^{\alpha\beta}$ and $t''^{\alpha\beta}$ denote the Jacobian matrices $\partial(x_1^\alpha, \dots, x_\ell^\alpha)/\partial(x_1^\beta, \dots, x_\ell^\beta)$ and $\partial(x_{\ell+1}^\alpha, \dots, x_m^\alpha)/\partial(x_{\ell+1}^\beta, \dots, x_m^\beta)$, respectively, both restricted to V. Since the restriction of $(x_1^\alpha, \dots, x_\ell^\alpha)$ to V form a coordinate system on $V \cap U_\alpha$, we see that $T_{\mathbb{R}}M|_V$ admits $T_{\mathbb{R}}V$ as a subbundle. We call the quotient bundle the *normal bundle* of V in M and denote it by $N_{\mathbb{R},V}$.

It is known that there exist a neighborhood U of V in M, a neighborhood W of (the image of) the zero section Z in $N_{\mathbb{R},V}$ and a diffeomorphism ψ of U onto W such that $\psi(V) = Z$ ([GP] p.76). Such a neighborhood U is called a *tubular neighborhood.* Usually we take an open disk bundle (or $N_{\mathbb{R},V}$ itself) as W so that V is deformation retract of U with a C^∞ retraction $\rho : U \to V$.

Example 1.8. Let M be a complex manifold of dimension n and $\{U_\alpha\}$ a covering of M by coordinate neighborhoods U_α with complex coordinates $(z_1^\alpha, \dots, z_n^\alpha)$. Then, as in Example 1.6, the union $TM = \bigsqcup_{p \in M} T_pM$ of the holomorphic parts of the complexified tangent spaces of M admits the structure of a complex vector bundle of rank n with $\{\tau^{\alpha\beta}\}$,

$$\tau^{\alpha\beta} = \frac{\partial(z_1^\alpha, \dots, z_n^\alpha)}{\partial(z_1^\beta, \dots, z_n^\beta)},$$

as a system of transition matrices. Since, for each pair (α, β), $\tau^{\alpha\beta}$ is a holomorphic map from $U_\alpha \cap U_\beta$ into $GL(n, \mathbb{C})$, TM is a holomorphic bundle. We call it the *holomorphic tangent bundle* of M. Note that, as a real bundle, TM is isomorphic to $T_{\mathbb{R}}M$ (see Proposition 2.2 of the following section). A holomorphic section v of TM is called a *holomorphic vector field.* On U_α, we may write as

$$v = \sum_{i=1}^{n} f_i^\alpha(z^\alpha)\frac{\partial}{\partial z_i^\alpha},$$

where the f_i^α's are holomorphic functions on U_α.

If V is a complex submanifold of M, then as in 1.6, $TM|_V$ admits TV as a subbundle. We call the quotient the *holomorphic normal bundle* of V in M and denote it by N_V so that we have the exact sequence

$$0 \longrightarrow TV \longrightarrow TM|_V \longrightarrow N_V \longrightarrow 0.$$

For each point p in V, we have the situation considered in Ch. I, Exercise 4.8 (2). Note that, again by Proposition 2.2, N_V is is isomorphic to $N_{\mathbb{R},V}$ as a real bundle.

Example 1.9. Let M be a complex manifold of dimension n and V a hypersurface (possibly with singularity) in M. We cover M by open sets U_α so that in each U_α, V is defined by a "reduced equation" $f^\alpha = 0$, i.e., the germ of f^α at each point in $V \cap U_\alpha$ is reduced (see Ch.I, section 5). Note that if $V \cap U_\alpha = \emptyset$, then we may take a non-zero constant as f^α. Then, for each pair (α, β), $f^{\alpha\beta} = f^\alpha/f^\beta$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$ and the system $\{f^{\alpha\beta}\}$ defines a complex vector bundle of rank one (a line bundle) on M. We call this bundle the line bundle defined by V and denote it by $L(V)$. Note that $L(V)$ is a holomorphic bundle and admits a natural holomorphic section whose zero set is exactly V, i.e., the section determined by the collection $\{f^\alpha\}$.

In particular, the line bundle on the projective space $\mathbb{CP}^n$ defined by the "hyperplane" $\mathbb{CP}^{n-1}$ is called the hyperplane bundle and denoted by H_n. If we use the notation of Ch.I, Example 4.2, 2, the bundle H_n is defined by the system transition functions $\{h^{ij}\}$ with $h^{ij} = \zeta_j/\zeta_i$ on the covering $\{U_i\}$.

Exercise 1.10. Show that, if V is a non-singular hypersurface of M, then there is a natural isomorphism $L(V)|_V \simeq N_V$.

If we are given some vector bundles, we may construct new ones by algebraic operations. Thus we let E and F be vector bundles on M. We may construct the direct sum $E \oplus F$, the homomorphism $\mathrm{Hom}(E, F)$ and the tensor product $E \otimes F$. Note that there is a natural isomorphism

$$\mathrm{Hom}(E, F) \simeq E^* \otimes F.$$

We also have the k-th exterior power $\bigwedge^k E$. For a complex vector bundle E, we have the complex conjugate $\overline{E}$ and for a real vector bundle E, the complexification $E^c = E \otimes_{\mathbb{R}} \mathbb{C}$.

The complex vector space $\mathbb{C}^r$ is naturally considered as a real vector space of dimension $2r$ and this defines a natural homomorphism

$$GL(r, \mathbb{C}) \to GL(2r, \mathbb{R}).$$

Thus if E is a complex vector bundle of rank r, it has the structure of a real vector bundle of rank $2r$.

2. Vector fields and differential forms

We denote by TM the holomorphic tangent bundle of a complex manifold M as in section 1. The following two propositions are consequences of the Cauchy-Riemann equation.

Proposition 2.1. *If M is a complex manifold, there is a natural isomorphism*

$$T^c_{\mathbb{R}}M \simeq TM \oplus \overline{T}M.$$

Proposition 2.2. *We have $TM \simeq T_{\mathbb{R}}M$ as real bundles.*

The following shows how a complex vector field (a section of TM) and a real vector field (a section of $T_{\mathbb{R}}M$) correspond in the above isomorphism, when they are expressed using local coordinates :

$$\sum_{i=1}^{n} f_i(z)\frac{\partial}{\partial z_i} \longleftrightarrow \sum_{i=1}^{n} u_i(x,y)\frac{\partial}{\partial x_i} + \sum_{i=1}^{n} v_i(x,y)\frac{\partial}{\partial y_i},$$

where $f_i = u_i + \sqrt{-1}v_i$ with u_i and v_i real valued functions.

Example 2.3. In $\mathbb{C} = \{z\}$ the complex vector field $z\frac{\partial}{\partial z}$ corresponds to the real vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $z^2\frac{\partial}{\partial z}$ to $(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$.

Let M be a C^∞ manifold of dimension m. We call a C^∞ section ω of the bundle $\bigwedge^p(T^c_{\mathbb{R}}M)^*$ on an open set U in M a (complex valued) *differential p-form* of class C^∞ (simply, a C^∞ p-form) on U. We denote by $A^p(U)$ the set of C^∞ p-forms on U, which has naturally the structure of a $\mathbb{C}$-vector space. The set $A^0(U)$ is thought of as the set of C^∞ functions on U. We have the exterior product

$$A^p(U) \times A^q(U) \longrightarrow A^{p+q}(U), \qquad (\omega, \theta) \mapsto \omega \wedge \theta.$$

It is bilinear in ω and θ and satisfies $\omega \wedge \theta = (-1)^{pq}\theta \wedge \omega$.

We also have the exterior derivative

$$d = d^p : A^p(U) \longrightarrow A^{p+1}(U),$$

which is a $\mathbb{C}$-linear map satisfying $d^{p+1} \circ d^p = 0$ and

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta \tag{2.4}$$

for $\omega \in A^p(U)$ and $\theta \in A^q(U)$.

For a complex vector bundle E on M and an open set U in M, we set $A^p(U,E) = C^\infty(U, \bigwedge^p (T^c_{\mathbb{R}}M)^* \otimes E)$. An element σ in $A^p(U,E)$, called a differential p-form with coefficients in E, is expressed locally as a finite sum $\sum \omega_i \otimes s_i$ with ω_i p-forms and s_i sections of E. The exterior product induces a bilinear map

$$A^p(U) \times A^q(U,E) \longrightarrow A^{p+q}(U,E).$$

Now let M be a complex manifold. Recall that the holomorphic cotangent bundle is the vector bundle T^*M dual to the holomorphic tangent bundle TM. By Proposition 2.1, we have a natural isomorphism

$$(T^c_{\mathbb{R}}M)^* \simeq T^*M \oplus \overline{T}^*M. \tag{2.5}$$

Hence we have an isomorphism

$$\bigwedge^r (T^c_{\mathbb{R}}M)^* \simeq \bigoplus_{p+q=r} \bigwedge^p T^*M \otimes \bigwedge^q \overline{T}^*M.$$

We call a section of $\bigwedge^p T^*M \otimes \bigwedge^q \overline{T}^*M$ a differential form of type (p,q) (simply, a (p,q)-form). Thus a differential r-form is expressed as a sum of (p,q)-forms with $p+q=r$. Suppose that a point z in M is in a coordinate neighborhood U_α with coordinates $(z^\alpha_1, \ldots, z^\alpha_n)$. We write $z^\alpha_i = x^\alpha_i + \sqrt{-1}y^\alpha_i$ and identify T^*_zM and $\overline{T}^*_zM$ with subspaces of $(T_{\mathbb{R}},^c_z M)^*$ by the isomorphism (2.5). Then, if we set

$$dz^\alpha_i = dx^\alpha_i + \sqrt{-1}dy^\alpha_i \quad \text{and} \quad d\bar{z}^\alpha_i = dx^\alpha_i - \sqrt{-1}dy^\alpha_i,$$

a straightforward computation shows that $dz^\alpha_1, \ldots, dz^\alpha_n$ are in T^*_zM and form a basis dual to the basis $(\partial/\partial z^\alpha_1, \ldots, \partial/\partial z^\alpha_n)$ of T_zM and that $d\bar{z}^\alpha_1, \ldots, d\bar{z}^\alpha_n$ are in $\overline{T}^*_zM$ and form a basis dual to the basis $(\partial/\partial \bar{z}^\alpha_1, \ldots, \partial/\partial \bar{z}^\alpha_n)$ of $\overline{T}_zM$. Since $dz^\alpha_{i_1} \wedge \cdots \wedge dz^\alpha_{i_p}$, where $(i_1, \ldots, i_p)$ runs through p-tuples of integers with $1 \le i_1 < \cdots < i_p \le n$, form a basis of $\bigwedge^p T^*_xM$ and $d\bar{z}^\alpha_{j_1} \wedge \cdots \wedge d\bar{z}^\alpha_{j_q}$, where $(j_1, \ldots, j_q)$ runs through q-tuples of integers with $1 \le j_1 < \cdots < j_q \le n$, form that of $\bigwedge^q \overline{T}^*_xM$, a (p,q)-form ω is written as, on U_α,

$$\omega = \sum_{\substack{1 \le i_1 < \cdots < i_p \le n \\ 1 \le j_1 < \cdots < j_q \le n}} f^\alpha_{i_1, \ldots, i_p, j_1, \ldots, j_q}(z^\alpha) dz^\alpha_{i_1} \wedge \cdots \wedge dz^\alpha_{i_p} \wedge d\bar{z}^\alpha_{j_1} \wedge \cdots \wedge d\bar{z}^\alpha_{j_q}, \tag{2.6}$$

where $f^{\alpha}_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are C^∞ functions on U_α. By setting $I = (i_1,\ldots,i_p)$ and $J = (j_1,\ldots,j_q)$ we may write (2.6) simply as

$$\omega = \sum f^{\alpha}_{IJ}(z) dz^{\alpha}_{I} \wedge d\bar{z}^{\alpha}_{J}.$$

In particular, a $(p,0)$-form ω can be written as, on U_α,

$$\omega = \sum_{1\le i_1<\cdots<i_p\le n} f^{\alpha}_{i_1,\ldots,i_p}(z^\alpha) dz^{\alpha}_{i_1} \wedge \cdots \wedge dz^{\alpha}_{i_p}.$$

When each $f^{\alpha}_{i_1,\ldots,i_p}$ is holomorphic, we say that ω is a holomorphic p-form. It is nothing but a holomorphic section of $\bigwedge^p T^*M$.

We denote by $A^{p,q}(U)$ the set of (p,q)-forms on an open set U in M. For each (p,q), it is an $A^0(U)$-module and we have the decomposition

$$A^r(U) = \bigoplus_{p+q=r} A^{p,q}(U).$$

Thus we may express the exterior derivative d as a sum $d = \partial + \bar{\partial}$ with

$$\partial : A^{p,q}(U) \longrightarrow A^{p+1,q}(U) \qquad \text{and} \qquad \bar{\partial} : A^{p,q}(U) \longrightarrow A^{p,q+1}(U).$$

From $d \circ d = 0$,

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0 \quad \text{and} \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0.$$

3. Stokes' theorem

Let M be an oriented C^∞ manifold of dimension m. Recall that, for a C^0 m-form ω with compact support, we may define the integral $\int_M \omega$.

Let D be an open set in M and assume that the boundary ∂D of D is C^∞, i.e., for any point p of ∂D there is a coordinate neighborhood U with coordinates $(x_1,\ldots,x_m)$ such that

$$R \cap U = \{\, q \in U \mid x_1(q) \le x_1(p) \,\},$$

where $R = \overline{D}$ (the closure of D in M). In this case, $\partial R = \partial D$ is an $m-1$ dimensional C^∞ submanifold of M. In fact if $(x_1,\ldots,x_m)$ is a coordinate system as above, then $(x_2,\ldots,x_m)$ is a coordinate system on $\partial R \cap U$. Moreover, if M is orientable, so is ∂R. If M is oriented so that a coordinate system $(x_1,\ldots,x_m)$ as above is positive, we orient ∂R so that $(x_2,\ldots,x_m)$ is positive. Suppose M is oriented and R as above is compact. Then we may define, for a C^0 m-form ω on a neighborhood of R, the integral $\int_R \omega$.

Theorem 3.1 (Stokes' theorem). *Let D be a relatively compact open set in M with C^∞ boundary (which may be empty). For a C^0 $(m-1)$-form ω in a neighborhood of $R = \overline{D}$,*

$$\int_R d\omega = \int_{\partial R} \iota^* \omega,$$

where $\iota : \partial R \hookrightarrow M$ denotes the inclusion.

Note that the above formula makes sense if the boundary ∂R is only piecewise C^∞.

More generally, let $\sigma = \sum n_i \sigma_i$ be a (C^∞) singular p-chain in M. Thus each σ_i is a C^∞ map from (a neighborhood of) the standard p-simplex Δ^p into M. For a p-form ω on M, we define

$$\int_\sigma \omega = \sum n_i \int_{\Delta^p} \sigma_i^* \omega.$$

If M is a complex manifold, M is orientable (cf. Ch.I, (2.5)). We orient M so that, if $(z_1, \dots, z_n)$ is a complex coordinate system on M, $(x_1, y_1, \dots, x_n, y_n)$ is a positive coordinate system, where $z_i = x_i + \sqrt{-1} y_i$, $i = 1, \dots, n$.

4. de Rham cohomology

We list [BT] as a basic reference for this section. Let M be a C^∞ manifold of dimension m. For an open set U in M, we denote by $A^p(U)$ the space of complex valued C^∞ p-forms on U.

The exterior derivative d defines the de Rham complex of M :

$$0 \to A^0(M) \xrightarrow{d^0} A^1(M) \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} A^m(M) \to 0.$$

The p-th de Rham cohomology $H_d^p(M; \mathbb{C})$ is the p-th cohomology of this complex; $H_d^p(M; \mathbb{C}) = \text{Ker } d^p / \text{Im } d^{p-1}$. For a close p-form ω, we denote its class in $H_d^p(M; \mathbb{C})$ by $[\omega]$. If M is connected, we easily see that $H_d^0(M; \mathbb{C}) \simeq \mathbb{C}$.

Lemma 4.1 (Poincaré lemma). *The de Rham complex of $\mathbb{R}^n$ is acyclic, i.e.,*

$$H_d^p(\mathbb{R}^n; \mathbb{C}) = 0 \quad \textit{for} \quad p > 0.$$

This is a special case of the following de Rham theorem, in fact it is a key ingredient in the proof of the theorem.

Let $H_p(M; \mathbb{C})$ and $H^p(M; \mathbb{C})$ denote the singular (or simplicial) homology and cohomology of M. The integration of a p-form on a (piecewise C^∞) singular p-chain of M induces a homomorphism

$$H_d^p(M; \mathbb{C}) \to H^p(M; \mathbb{C}),$$

which is shown to be an isomorphism ;

Theorem 4.2 (de Rham theorem). *For a C^∞ manifold M,*

$$H^p_d(M;\mathbb{C}) \simeq H^p(M;\mathbb{C}).$$

Now, by (2.4), the pairing

$$H^p_d(M;\mathbb{C}) \times H^q_d(M;\mathbb{C}) \longrightarrow H^{p+q}_d(M;\mathbb{C})$$

given by $([\omega],[\theta]) \mapsto [\omega\wedge\theta]$ is well-defined and it corresponds to the cup product in the isomorphism of Theorem 4.2. We write $[\omega\wedge\theta] = [\omega] \smile [\theta]$. This product makes the direct sum $H^*_d(M;\mathbb{C}) = \bigoplus_p H^p_d(M;\mathbb{C})$ a graded ring.

If M is compact, connected and oriented then, by the Stokes theorem, the integration on M induces a linear map

$$\int_M : H^m_d(M;\mathbb{C}) \longrightarrow \mathbb{C}.$$

Then it is proved that the bilinear form

$$H^p_d(M;\mathbb{C}) \times H^{m-p}_d(M;\mathbb{C}) \xrightarrow{\smile} H^m_d(M;\mathbb{C}) \xrightarrow{\int_M} \mathbb{C}$$

is non-degenerate ([BT] Ch.I, §5) :

Theorem 4.3 (Poincaré duality). *For a compact, connected and oriented C^∞ manifold M of dimension m, the above pairing induces an isomorphism*

$$P : H^p_d(M;\mathbb{C}) \xrightarrow{\sim} H^{m-p}(M;\mathbb{C})^* = H_{m-p}(M;\mathbb{C}).$$

In the isomorphism of Theorem 4.3, a class $[\omega]$ in $H^p(M;\mathbb{C})$ corresponds to the class of a (piecewise C^∞) singular $(m-p)$-cycle C in M satisfying

$$\int_M \omega\wedge\theta = \int_C \theta \tag{4.4}$$

for all closed $(m-p)$-form θ on M. In particular,

$$H^m(M;\mathbb{C}) \simeq H_0(M;\mathbb{C}) \simeq \mathbb{C}$$

and, for the class $[\omega]$ of a closed m-form ω, the corresponding homology class may be thought of as a complex number, which is given by $\int_M \omega$. Also

$$H_m(M;\mathbb{C}) \simeq H^0(M;\mathbb{C}) \simeq \mathbb{C}$$

and the homology class in $H_m(M;\mathbb{C})$ corresponging to the class [1] of the function constantly equal to 1 is represented by an m-cycle C such that $\int_M \theta = \int_C \theta$ for all closed m-form θ. Thus this homology class coincides with the class $[M]$ of M considered as an m-cycle, the *fundamental class* of the compact oriented manifold M. Note that it is the canonical generator of the integral homology $H_m(M;\mathbb{Z}) \simeq \mathbb{Z}$.

Remark 4.5. Let M be a complex manifold and V a compact analytis variety of dimension ℓ in M. Then we may think of V as a 2ℓ-cycle, for example, by triangulation, in which case we have the class $[V]$ in $H_{2\ell}(V,\mathbb{Z})$, or by integration of 2ℓ-forms on M ([GH] Ch.0), in which case we have the class $[V]$ in $H_{2\ell}(M,\mathbb{C})$. Moreover, if V is (globally) irreducible, then $H_{2\ell}(V,\mathbb{Z}) \simeq \mathbb{Z}$ and $[V]$ is the fundamental class (e.g., [Br]).

Recall that the Poincaré isomorphism P in Theorem 4.3 is given by the "cap product" with the fundamental class;

$$P([\omega]) = [\omega] \frown [M].$$

5. Čech-de Rham cohomology

The Čech-de Rham cohomology is defined for arbitrary covering of a manifold M, however for simplicity here we only consider coverings of M consisting of only two open sets.

Let M be a C^∞ manifold of dimension m and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. We set $U_{01} = U_0 \cap U_1$. Define a vector space $A^p(\mathcal{U})$ as follows:

$$A^p(\mathcal{U}) = A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

Therefore an element $\sigma \in A^p(\mathcal{U})$ is given by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ with σ_0 a p-form on U_0, σ_1 a p-form on U_1 and σ_{01} a $(p-1)$-form on U_{01}.

We define the operator $D : A^p(\mathcal{U}) \longrightarrow A^{p+1}$ by

$$D\sigma = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

Then it is not difficult to see that $D \circ D = 0$. This allows us to define a cohomological complex, the Čech-de Rham complex:

$$\cdots \longrightarrow A^{p-1}(\mathcal{U}) \xrightarrow{D^{p-1}} A^p(\mathcal{U}) \xrightarrow{D^p} A^{p+1}(\mathcal{U}) \longrightarrow \cdots$$

Set $Z^p(\mathcal{U}) = \operatorname{Ker} D^p$, $B^p(\mathcal{U}) = \operatorname{Im} D^{p-1}$ and

$$H^p_D(\mathcal{U}) = Z^p(\mathcal{U})/B^p(\mathcal{U}),$$

which is called the p-th Čech-de Rham cohomology of $\mathcal{U}$. We denote the image of σ by the canonical surjection $Z^p(\mathcal{U}) \to H^p_D(\mathcal{U})$ by $[\sigma]$.

Theorem 5.1. *The map $A^p(M) \to A^p(\mathcal{U})$ given by $\omega \mapsto (\omega, \omega, 0)$ induces an isomorphism*

$$\alpha : H^p_d(M) \xrightarrow{\sim} H^p_D(\mathcal{U}).$$

Proof. It is not difficult to show that α is well-defined. To prove that α is surjective, let $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ be such that $D\sigma = 0$. Let $\{\rho_0, \rho_1\}$ be a partition of unity subordinated to the covering $\mathcal{U}$. Define $\omega = \rho_0\sigma_0 + \rho_1\sigma_1 - d\rho_0 \wedge \sigma_{01}$. Then it is easy to see that $d\omega = 0$ and $[(\omega, \omega, 0)] = [\sigma]$. The injectivity of α is not difficult to show. □

We define the "cup product"

$$A^p(\mathcal{U}) \times A^q(\mathcal{U}) \to A^{p+q}(\mathcal{U})$$

by assigning to σ in $A^p(\mathcal{U})$ and τ in $A^q(\mathcal{U})$ the element $\sigma \smile \tau$ in $A^{p+q}(\mathcal{U})$ given by

$$(5.2) \qquad (\sigma \smile \tau)_i = \sigma_i \wedge \tau_i, \ i = 0,\ 1, \quad (\sigma \smile \tau)_{01} = (-1)^p \sigma_0 \wedge \tau_{01} + \sigma_{01} \wedge \tau_1.$$

Then we have $D(\sigma \smile \tau) = D\sigma \smile \tau + (-1)^p \sigma \smile D\tau$. Thus it induces the cup product

$$H^p_D(\mathcal{U}) \times H^q_D(\mathcal{U}) \to H^{p+q}_D(\mathcal{U})$$

compatible, via the isomorphism of 5.1, with the cup product in the de Rham cohomology.

Now we recall the integration on the Čech-de Rham cohomology (cf. [Leh]). Suppose that the m-dimensional manifold M is oriented and compact and let $\mathcal{U} = \{U_0, U_1\}$ be a covering of M. Let $R_0, R_1 \subset M$ be two compact manifolds of dimension m with C^∞ boundary with the following properties:
(1) $R_j \subset U_j$ for $j = 0, 1$,
(2) $\operatorname{Int} R_0 \cap \operatorname{Int} R_1 = \emptyset$ and
(3) $R_0 \cup R_1 = M$.

Let $R_{01} = R_0 \cap R_1$ and give R_{01} the orientation as the boundary of R_0; $R_{01} = \partial R_0$, equivalently give R_{01} the orientation opposite to that of the boundary of R_1; $R_{01} = -\partial R_1$. We define the integration

$$\int_M : A^m(\mathcal{U}) \to \mathbb{C} \quad \text{by} \quad \int_M \sigma = \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}.$$

Then by the Stokes theorem, we see that if $D\sigma = 0$ then $\int_M \sigma$ is independent of $\{R_0, R_1\}$ and that if $\sigma = D\tau$ for some $\tau \in A^{p-1}(\mathcal{U})$ then $\int_M \sigma = 0$. Thus we may define the integration

$$\int_M : H^m_D(\mathcal{U}) \to \mathbb{C},$$

which is compatible with the integration on the de Rham cohomology via the isomorphism of 5.1.

Next we define the relative Čech-de Rham cohomology and describe the Alexander duality. Let M is an m-dimensional oriented manifold (not necessarily compact) and S a compact subset of M. Let $U_0 = M \setminus S$ and let U_1 be an open neighborhood of S. We consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$A^p(\mathcal{U}, U_0) = \{ \sigma = (\sigma_0, \sigma_1, \sigma_{01}) \in A^p(\mathcal{U}) \mid \sigma_0 = 0 \}.$$

Then we see that if σ is in $A^p(\mathcal{U}, U_0)$, $D\sigma$ is in $A^p(\mathcal{U}, U_0)$. This gives rise to another complex, called the relative Čech-de Rham complex and we may define the p-th relative Čech-de Rham cohomology of the pair $(\mathcal{U}, U_0)$ as

$$H^p_D(\mathcal{U}, U_0) = \operatorname{Ker} D^p / \operatorname{Im} D^{p-1}.$$

By the five lemma, we see that there is a natural isomorphism

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, M \setminus S; \mathbb{C}).$$

Let R_1 be a compact manifold of dimension m with C^∞ boundary such that $S \subset \operatorname{Int} R_1 \subset R_1 \subset U_1$. Let $R_0 = M \setminus \operatorname{Int} R_1$. Note that $R_0 \subset U_0$. The integral operator $\int_M$ (which is not defined in general for $A^m(\mathcal{U})$ unless M is compact) is well defined on $A^m(\mathcal{U}, U_0)$:

$$\int_M : A^m(\mathcal{U}, U_0) \to \mathbb{C}, \qquad \int_M \sigma = \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01},$$

and induces an operator $\int_M : H^m_D(\mathcal{U}, U_0) \to \mathbb{C}$.

In the cup product $A^p(\mathcal{U}) \times A^{m-p}(\mathcal{U}) \longrightarrow A^m(\mathcal{U})$ given as (5.2), we see that if $\sigma_0 = 0$, the right hand side depends only on σ_1, σ_{01} and τ_1. Thus we have a pairing $A^p(\mathcal{U}, U_0) \times A^{m-p}(U_1) \longrightarrow A^m(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^p(\mathcal{U}, U_0) \times A^{m-p}(U_1) \longrightarrow \mathbb{C}.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality

$$A : H^p(M, M \setminus S; \mathbb{C}) \simeq H^p(A^\bullet(\mathcal{U}, U_0)) \xrightarrow{\sim} H^{m-p}(U_1, \mathbb{C})^* \simeq H_{m-p}(S, \mathbb{C}). \tag{5.3}$$

Proposition 5.4. *If M is compact, we have the commutative diagram*

$$\begin{array}{ccc} H^p(M, M\setminus S;\mathbb{C}) & \xrightarrow{j^*} & H^p(M;\mathbb{C}) \\ \wr\big\downarrow A & & \wr\big\downarrow P \\ H_{m-p}(S,\mathbb{C}) & \xrightarrow{i_*} & H_{m-p}(M,\mathbb{C}), \end{array}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M,\emptyset) \hookrightarrow (M, M\setminus S)$.

Example 5.5. Let $M = \mathbb{R}^m$ and $S = \{0\}$ with $m \geq 2$. Then $U_0 = \mathbb{R}^m \setminus \{0\}$, which retracts to S^{m-1}. Let $U_1 = \mathbb{R}^m$. In this situation, we compute $H^p_D(\mathcal{U}, U_0)$. For $p = 0$, each element σ in $A^0(\mathcal{U}, U_0)$ can be written as $\sigma = (0, f, 0)$ for some C^∞ function f on U_1. If $D\sigma = 0$, we have $f \equiv 0$ and therefore $H^0_D(\mathcal{U}, U_0) = \{0\}$. Next, an element σ in $A^1(\mathcal{U}, U_0)$ can be written as $\sigma = (0, \sigma_1, f)$ with σ_1 a 1-form on U_1 and f a C^∞ function on $U_0 \cap U_1$. If σ is a cocycle then $d\sigma_1 = 0$ on U_1 and $df = \sigma_1$ on $U_0 \cap U_1$. By the Poincaré lemma the first condition implies that $\sigma_1 = dg$ for some C^∞ function g on U_1 and the second condition implies that $f \equiv g + c$ for some $c \in \mathbb{C}$. Therefore f has a C^∞ extension, still denoted by f, over $\{0\}$ and $\sigma = (0, df, f) = D(0, f, 0)$. Hence $H^1_D(\mathcal{U}, U_0) = \{0\}$. For $p \geq 2$ the map

$$H^{p-1}_d(U_0) \to H^p_D(\mathcal{U}, U_0) \qquad \text{given by} \quad [\omega] \mapsto [(0, 0, -\omega)]$$

can be shown to be an isomorphism (we leave the details to the reader) and we have

$$H^p_D(\mathcal{U}, U_0) \simeq H^{p-1}_d(U_0) \simeq H^{p-1}(S^{m-1}) = \begin{cases} \mathbb{C} & \text{for } p = m \\ 0 & \text{for } p = 2, \dots, m-1. \end{cases}$$

An explicit generator of $H^{m-1}(S^{m-1})$ is given as follows ([GH] p.370). For $x = (x_1, \dots, x_m)$ in $\mathbb{R}^m$, we set $\Phi(x) = dx_1 \wedge \cdots \wedge dx_m$ and

$$\Phi_i(x) = (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m.$$

Also, let C_m be the constant given by

$$C_m = \begin{cases} \frac{(\ell-1)!}{2\pi^\ell}, & \text{for } m = 2\ell \\ \frac{(2\ell)!}{2^{2\ell+1}\pi^\ell \ell!}, & \text{for } m = 2\ell + 1. \end{cases}$$

Then the form

$$\psi_m = C_m \frac{\sum_{i=1}^m \Phi_i(x)}{\|x\|^m}$$

is a closed $(m-1)$-form on $\mathbb{R}^m \setminus 0$ whose integral on the unit sphere S^{m-1} (in fact a sphere of arbitrary radius) is 1. Now we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, then $\psi_{2n} = (\beta_n + \overline{\beta_n})/2$, where

$$\beta_n = C'_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)} \wedge \Phi(z)}{\|z\|^{2n}}, \qquad C'_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n}.$$

Then β_n is a closed $(n, n-1)$-form on $\mathbb{C}^n \setminus 0$, real on S^{2n-1} and $\int_{S^{2n-1}} \beta_n = 1$. We call β_n the Bochner-Martinelli kernel on $\mathbb{C}^n$. Note that

$$\beta_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z},$$

the Cauchy kernel on $\mathbb{C}$.

Chapter III. Chern-Weil theory of characteristic classes and some more complex analytic geometry

1. Chern classes via connections

Let M be a C^∞ manifold of dimension m. For an open set U in M, we denote by $A^0(U)$ the $\mathbb{C}$-algebra of C^∞ functions on U. Also, for a C^∞ complex vector bundle E of rank r on M, we let $A^p(U, E)$ be the vector space of C^∞ sections of $\bigwedge^p(T^c_{\mathbb{R}}M)^* \otimes E$ on U. Thus $A^0(U, E)$ is the $A^0(U)$-module of C^∞ sections of E.

Definition 1.1. A connection for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \to A^1(M, E)$$

satisfying

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

Example 1.2. The exterior derivative

$$d : A^0(M) \to A^1(M)$$

is a connection for the trivial line bundle $M \times \mathbb{C}$.

From the definition we have the following :

Lemma 1.3. *A connection ∇ is a local operator, i.e., if a section s is identically 0 on an open set U, so is $\nabla(s)$.*

Thus the restriction of ∇ to an open set U makes sense and it is a connection for $E|_U$. From the definition we also have the following lemma.

Lemma 1.4. *Let $\nabla_1, \ldots, \nabla_k$ be connections for E and $f_1, \ldots, f_k$ C^∞ functions on M with $\sum_{i=1}^k f_i \equiv 1$. Then $\sum_{i=1}^k f_i \nabla_i$ is a connection for E.*

Exercises 1.5. (1) Prove Lemmas 1.3 and 1.4.
(2) Show that every vector bundle admits a connection.

If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M, E) \to A^2(M, E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s) \quad \text{for} \quad \omega \in A^1(M) \text{ and } s \in A^0(M, E).$$

The composition

$$K = \nabla \circ \nabla : A^0(M,E) \longrightarrow A^2(M,E)$$

is called the curvature of ∇. It is not difficult to see that

$$K(fs) = fK(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M,E).$$

The fact that a connection is a local operator allows us to get local representations of it and its curvature by matrices whose entries are differential forms. Thus suppose that ∇ is a connection for a vector bundle E of rank r and that E is trivial on U ; $E|_U \simeq U \times \mathbb{C}^r$. If $\mathbf{s} = (s_1, \ldots, s_r)$ is a frame of E on U, then we may write, for $i = 1, \ldots, r$,

$$\nabla(s_i) = \sum_{j=1}^{r} \theta_{ij} \otimes s_j, \qquad \theta_{ij} \in A^1(U).$$

We call $\theta = (\theta_{ij})$ the connection matrix with respect to $\mathbf{s}$. For an arbitrary section s on U, we may write $s = \sum_{i=1}^{r} f_i s_i$ with f_i C^∞ functions on U and we compute

$$\nabla(s) = \sum_{i=1}^{r} (df_i + \sum_{j=1}^{r} f_j \theta_{ji}) \otimes s_i.$$

The connection ∇ is trivial with respect to $\mathbf{s}$, if and only if $\theta = 0$. Thus in this case we have $\nabla(s) = \sum_{i=1}^{r} df_i \otimes s_i$. Also, from the definition we compute to get

$$K(s_i) = \sum_{j=1}^{r} \kappa_{ij} \otimes s_j, \qquad \kappa_{ij} = d\theta_{ij} - \sum_{k=1}^{r} \theta_{ik} \wedge \theta_{kj}.$$

We call $\kappa = (\kappa_{ij})$ the curvature matrix with respect to $\mathbf{s}$. If $\mathbf{s}' = (s'_1 \ldots, s'_r)$ is another frame of E on U', we have $s'_i = \sum_{j=1}^{r} a_{ij} s_j$ for some C^∞ functions a_{ij} on $U \cap U'$. The matrix $A = (a_{ij})$ is non-singular at each point of $U \cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $\mathbf{s}'$,

$$\theta' = dA \cdot A^{-1} + A\theta A^{-1} \quad \text{and} \quad \kappa' = A\kappa A^{-1} \quad \text{in} \quad U \cap U'. \tag{1.6}$$

Let $n = [m/2]$ and, for each $i = 1, \ldots, n$, let σ_i denote the i-th elementary symmetric function in n variables $X_1, \ldots, X_n$, i.e., $\sigma_i(X_1, \ldots, X_n)$ is a polynomial of degree i defined by

$$\prod_{i=1}^{n} (1 + X_i) = 1 + \sigma_1(X_1, \ldots, X_n) + \cdots + \sigma_n(X_1, \ldots, X_n).$$

Since differential forms of even degrees commute with one another with respect to the exterior product, we may treat κ as an ordinary matrix whose entries are numbers. Thus we define a $2i$-form $\sigma_i(\kappa)$ on U by

$$\det(I_r + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_n(\kappa),$$

where I_r denotes the identity matrix of rank r. Note that $\sigma_i(\kappa) = 0$ for $i = r+1, \dots, n$ and, in particular, $\sigma_1(\kappa) = \mathrm{tr}(\kappa)$ and $\sigma_r(\kappa) = \det(\kappa)$. Although $\sigma_i(\kappa)$ depends on the connection ∇, by (1.6), it does not depend on the choice of the frame of E and it defines a global $2i$-form on M, which we denote by $\sigma_i(\nabla)$. It is shown that the form is closed ([GH] Ch.3, 3 Lemma, [MS] Appendix C, Fundamental Lemma). We set

$$c_i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla)$$

and call it the i-th Chern form.

If we have two connections ∇ and ∇' for E, there is a $(2i-1)$-form $c_i(\nabla, \nabla')$ with $c_i(\nabla, \nabla') = -c_i(\nabla', \nabla)$ and satisfying

$$d\, c_i(\nabla, \nabla') = c_i(\nabla') - c_i(\nabla). \tag{1.7}$$

In fact the form $c_i(\nabla, \nabla')$ is constructed as follows ([Bo] p. 65). We consider the vector bundle $E \times \mathbb{R} \to M \times \mathbb{R}$ and let $\tilde{\nabla}$ be the connection for it given by

$$\tilde{\nabla} = (1-t)\nabla + t\nabla',$$

where t denotes a coordinate on $\mathbb{R}$. Denoting by $[0,1]$ the unit interval and by $\pi : M \times [0,1] \to M$ the projection, we have the integration along the fiber

$$\pi_* : A^{2i}(M \times [0,1]) \to A^{2i-1}(M).$$

Then we set $c_i(\nabla, \nabla') = \pi_*(c_i(\tilde{\nabla}))$.

From the above, we see that the class $[c_i(\nabla)]$ of the closed $2i$-form $c_i(\nabla)$ in the de Rham cohomology $H^{2i}(M; \mathbb{C})$ depends only on E and not on the choice of the connection ∇. We denote this class by $c_i(E)$ and call it the i-th Chern class $c_i(E)$ of E. We call

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$$

the total Chern class of E, which is considered as an element in the cohomology ring $H^*(M; \mathbb{C})$. Note that the class $c(E)$ is invertible in $H^*(M; \mathbb{C})$.

Remarks 1.8. 1°. It can be shown that the top Chern class $c_r(E)$ is equal to the Euler class $e(E)$ of the underlying real bundle.

$2°$. It is known that $c_i(E)$ is in the image of the canonical homomorphism

$$H^{2i}(M;\mathbb{Z}) \to H^{2i}(M;\mathbb{C}).$$

In fact it is possible to define $c_i(E)$ in $H^{2i}(M;\mathbb{Z})$ using the obstruction theory; it is the primary obstruction to constructing $r-i+1$ sections linearly independent everywhere [St].

$3°$. For the hyperplane bundle H_n on $\mathbb{CP}^n$ (Ch.II, Example 1.9),

$$c(H_n) = 1 + h_n,$$

where h_n denotes the canonical generator of $H^2(\mathbb{CP}^n;\mathbb{C})$ (the Poincaré dual of the homology class $[\mathbb{CP}^{n-1}]$.

More generally, if we have a symmetric polynomial φ, we may write $\varphi = P(\sigma_1, \sigma_2, \dots)$ for some polynomial P. We define, for a connection ∇ for E, the characteristic form $\varphi(\nabla)$ for φ by $\varphi(\nabla) = P(c_1(\nabla), c_2(\nabla), \dots)$, which is a closed form and defines the characteristic class $\varphi(E)$ of E for φ in the de Rham cohomology. We may also define the difference form $\varphi(\nabla, \nabla')$ by a similar construction.

2. Virtual bundles

For simplicity, we consider only virtual bundles involving only two vector bundles.

If we have two complex vector bundles E and F, the total Chern class of the "virtual bundle" $F - E$ is defined by

$$c(F-E) = c(F)/c(E). \tag{2.1}$$

Let ∇^E and ∇^F be connections for E and F, respectively. We write the degree i term in the right hand side of (2.1) as

$$c_i(F-E) = \sum_j \varphi_j^{(i)}(E) \cdot \psi_j^{(i)}(F)$$

with $\varphi_i^{(i)}(E)$ and $\psi_i^{(i)}(F)$ polynomials in the Chern classes of E and F, respectively. Then the i-th Chern class of $F - E$ is represented by the differential form

$$c_i(\nabla^\bullet) = \sum_j \varphi_j^{(i)}(\nabla^E) \wedge \psi_j^{(i)}(\nabla^F),$$

where $\nabla^\bullet$ denotes the pair (∇^E, ∇^F).

Also, for a polynomial φ in the Chern classes of E, we may define a closed form $\varphi(\nabla^\bullet)$ which represents the class $\varphi(F - E)$.

If we have two pairs of connections $\nabla_0^\bullet, \nabla_1^\bullet$ for E and F, there is a form $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ satisfying an identity similar to (1.7).

Now let

$$0 \to E \xrightarrow{\varphi} F \xrightarrow{\psi} G \to 0 \tag{2.2}$$

be a sequence of vector bundles on M, and ∇^E, ∇^F and ∇^G connections for E, F and G, respectively. We say that the family $(\nabla^E, \nabla^F, \nabla^G)$ is compatible with the sequence if the following diagram is commutative :

$$\begin{array}{ccccc}
A^0(M,E) & \xrightarrow{\varphi} & A^0(M,F) & \xrightarrow{\psi} & A^0(M,G) \\
\downarrow{\scriptstyle \nabla^E} & & \downarrow{\scriptstyle \nabla^F} & & \downarrow{\scriptstyle \nabla^G} \\
A^1(M,E) & \xrightarrow{1\otimes\varphi} & A^1(M,F) & \xrightarrow{1\otimes\psi} & A^1(M,G).
\end{array}$$

If the above sequence is exact, there is always a family $(\nabla^E, \nabla^F, \nabla^G)$ of connections compatible with the sequence and for such a family we have ([BB] (4.22) Lemma)

$$c(\nabla^\bullet) = c(\nabla^E).$$

3. Characteristic classes in the Čech-de Rham cohomology and a vanishing theorem

Let M be a C^∞ manifold and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. For a vector bundle $E \to M$, we take a connection ∇_j on U_j, $j = 0, 1$. Then let $c_i(\nabla_*)$ be the element of $A^{2i}(\mathcal{U})$ given by

$$c_i(\nabla_*) = (c_i(\nabla_0), c_i(\nabla_1), c_i(\nabla_0, \nabla_1)). \tag{3.1}$$

Then we see that $Dc_i(\nabla_*) = 0$ and this defines a class $[c_i(\nabla_*)] \in H_D^{2i}(\mathcal{U})$.

Theorem 3.2. *The class $[c_i(\nabla_*)] \in H_D^{2i}(\mathcal{U})$ corresponds to the Chern class $c_i(E) \in H_d^{2i}(M)$ under the isomorphism of Ch.II, Theorem 5.1.*

By a similar construction, we may define the characteristic class $\varphi(E)$ for a polynomial φ in the Chern polynomials in the Čech-de Rham cohomology. It can be done also for virtual bundles.

Let E be a complex vector bundle of rank r on a C^∞ manifold M. Let $\mathbf{s} = (s_1, \ldots, s_\ell)$ be an ℓ-frame of E on an open set U, i.e., ℓ sections linearly independent everywhere on U. We say that a connection ∇ for E on U is $\mathbf{s}$-trivial, if $\nabla(s_i) = 0$ for $i = 1, \ldots, \ell$.

Proposition 3.3. *If ∇ is* $\mathbf{s}$*-trivial, then*

$$c_i(\nabla) \equiv 0 \qquad \textit{for} \quad i \geq r - \ell + 1.$$

Proof. For simplicity, we prove the proposition when $\ell = 1$. Let $U \subset M$ be an open set such that $E|_U \simeq U \times \mathbb{C}^r$. Since $s_1 \neq 0$ everywhere on M, we may take a fram $\mathbf{e} = (e_1, \dots, e_r)$ on U so that $e_1 = s_1$. Then all the entries of the first row of the curvature matrix κ of ∇ with respect to $\mathbf{e}$ are zero. Since $c_r(\nabla) = \det \kappa$, up to a constant, we have $c_r(\nabla) = 0$. $\square$

4. Divisors

Let M be a complex manifold of dimension n. A *meromorphic function* φ on M is defined by a data $\{(U_\alpha, f^\alpha, g^\alpha)\}$, where $\{U_\alpha\}$ is a covering of M, and f^α and g^α are holomorphic functions on U_α such that the germ g_z^α at z is non-zero for all z in U_α, the germs f_z^α and g_z^α are relatively prime for all z in U_α (cf. the phrase after Ch.I, Theorem 3.6) and that $f^\alpha g^\beta = f^\beta g^\alpha$ in $U_\alpha \cap U_\beta$. We write $\varphi = f^\alpha / g^\alpha$ on U_α.

A *divisor* D on M is a finite formal sum $D = \sum_{i=1}^r n_i V_i$, where the V_i's are irreducible hypersurfaces in M and the n_i's are integers. Thus, if we cover M by open sets $\{U_\alpha\}$ so that V_i is defined by f_i^α on U_α (cf. Ch.II, Example 1.9), the meromorphic function $\varphi^\alpha = \prod_{i=1}^r (f_i^\alpha)^{n_i}$ defines D on U_α. For each pair (α, β), $f^{\alpha\beta} = \varphi^\alpha / \varphi^\beta$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$ and the system $\{f^{\alpha\beta}\}$ defines a line bundle on M. We call this bundle the line bundle defined by D and denote it by $L(D)$. We may write $L(D) = \bigotimes_{i=1}^r L(V_i)^{n_i}$, where $L(V_i)^{n_i}$ denotes the tensor product of n_i copies of $L(V_i)$, for $n_i > 0$, and the tensor product of $-n_i$ copies of $L(V_i)^*$, for $n_i < 0$. A divisor $D = \sum_{i=1}^r n_i V_i$ is called effective if $n_i \geq 0$ for all i. Thus an effective divisor is defined locally by a holomorphic function.

If φ is a meromorphic function on M given by $\varphi = f^\alpha / g^\alpha$ on U_α, taking the irreducible decompositions of f^α and g^α, we may consider a divisor, which we call the divisor of φ and denote by (φ). We may write $(\varphi) = D_0 - D_\infty$, where D_0 and D_∞ are defined, respectively, by f^α and g^α on U_α. Clearly the bundle $L(\varphi)$ is trivial. Conversely, it is shown that, if $L(D)$ is trivial for a divisor D, then $D = (\varphi)$ for some meromorphic function φ ([GH] Ch.1, 1, [Hi] §15). We say that two divisors D_1 and D_2 are linearly equivalent if $D_1 - D_2 = (\varphi)$ for some meromorphic function φ. Thus this is equivalent to saying $L(D_1) = L(D_2)$.

For a divisor $D = \sum_{i=1}^r n_i V_i$, we set $|D| = \bigcup_{i=1}^r V_i$ and call it the support of D. If each V_i is compact, the divisor D defines a homology class $[D] = \sum_{i=1}^r n_i [V_i]$ in $H_{2n-2}(M; \mathbb{Z})$ (or in $H_{2n-2}(|D|; \mathbb{Z})$). It is known that, if M is compact, $[D]$ is the Poincaré dual of $c_1(L(D))$ ([GH] Ch.1, 1 Proposition, see [Su3] for more "precise" duality). Thus, if D_1 and D_2 are linearly equivalent, then $[D_1] = [D_2]$. Also, for n divisors $D_1, \dots, D_n$ the "global" intersection number $(D_1 \cdots D_n)$ is given by $\int_M c_1(L(D_1)) \cdots c_1(L(D_n))$, where the product is the cup product. If the intersection $\bigcap_{i=1}^n |D_i|$ consists of isolated points, then this number is the sum of the intersection numbers at the points of intersection. See Example 6.2 below for this "local" intersection number when the divisors are effective.

Example 4.1. Let V be the algebraic variety in $\mathbb{CP}^n = \{[\zeta_0, \ldots, \zeta_n]\}$ defined by a homogeneous polynomial P of degree d (Ch.I, 4). The function $\varphi = P/\zeta_0^d$ is a well-defined meromorphic function on $\mathbb{CP}^n$, which is given as the quotient of P/ζ_i^d by ζ_0^d/ζ_i^d on each affine open set $U_i = \{\zeta_i \neq 0\}$. Thus, if we denote by D_∞ the hyperplane defined by $\zeta_0 = 0$, then V is linearly equivalent to $d\,D_\infty$ and $[V] = d\,[D_\infty]$. Recall that $[D_\infty] = [\mathbb{CP}^{n-1}]$ is the generator of $H_{2n-2}(\mathbb{CP}^n; \mathbb{Z}) \simeq \mathbb{Z}$. Also, the intersection of k copies of $[D_\infty]$ generates $H_{2n-2k}(\mathbb{CP}^n; \mathbb{Z}) \simeq \mathbb{Z}$, for $k = 1, \ldots, n$ (cf. Ch.I, (4.11)).

5. Complete intersections and local complete intersections

We start with the local situation. Let $\mathcal{O}_{n+k}$ be the ring of convergent power series in $(z_1, \ldots, z_{n+k})$.

Definition 5.1. Let V be a germ of variety at 0 of pure dimension n in $\mathbb{C}^{n+k}$. We call V a *complete intersection* if the ideal $I(V)$ is generated by k germs of holomorphic functions. In particular, if $k = 1$, V is a (germ of) hypersurface.

In general, let V be a germ of variety at 0 in $\mathbb{C}^{n+k}$. Take germs $f_1, \ldots, f_r$ in $\mathcal{O}_{n+k}$ and set $g_i = \sum_{j=1}^r a_{ij} f_j$ with $a_{ij} \in \mathcal{O}_{n+k}$, for $i = 1, \ldots, s$. Then

$$\operatorname{rank} \frac{\partial(g_1, \ldots, g_s)}{\partial(z_1, \ldots, z_{n+k})}(0) \leq \operatorname{rank} \frac{\partial(f_1, \ldots, f_r)}{\partial(z_1, \ldots, z_{n+k})}(0).$$

Thus we have

Lemma 5.2. *If the germs $f_1, \ldots, f_r$ generate $I(V)$, then the rank of the Jacobian matrix $\partial(f_1, \ldots, f_r)/\partial(z_1, \ldots, z_{n+k})$ at 0 does not depend on $f_1, \ldots, f_r$.*

Let V be a complete intersection of dimension n and $f_1, \ldots, f_k$ generators of $I(V)$. We take a neighborhood U of 0 in $\mathbb{C}^{n+k}$ such that the germs V and $f_1, \ldots, f_k$ have representatives on U. We may assume that the germs $f_{1,x}, \ldots, f_{k,x}$ generate $I(V_x)$ for all x in U ("coherence of the ideal sheaf", e.g., [GR]) and hence we may write

$$V = \{\, x \in U \mid f_1(x) = \cdots = f_k(x) = 0 \,\}.$$

We call $f_1, \ldots, f_r$ as above "reduced defining functions" for V. With these functions, we may describe the singular set of V as follows:

Proposition 5.3. *If V is a k codimensional complete intersection,*

$$\operatorname{Sing}(V) = \left\{ x \in V \mid \operatorname{rank} \frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_{n+k})}(x) < k \right\}.$$

Remark 5.4. If V is a pure k-codimensional subvariety, which may not be a complete intersection, the set $\operatorname{Sing}(V)$ may be expressed similarly as above, replacing $f_1, \dots, f_k$ by (arbitrary number of) generators $f_1, \dots, f_r$ of $I(V)$ (e.g., [Ok] Ch.I, §1). Thus, for an analytic variety V, $\operatorname{Sing}(V)$ is also an analytic variety.

In general, let V be a variety in a neighborhood U of 0 in $\mathbb{C}^{n+k}$ which has 0 as its only singular point. Let $B_\varepsilon = \{ (z_1, \dots, z_{n+k}) \mid |z_1|^2 + \cdots + |z_{n+k}|^2 \le \varepsilon^2 \}$ be the closed disk of radius ε and S_ε the $(2(n+k)-1)$-sphere of radius ε, which is the boundary of B_ε. It is known that, for sufficiently small ε, the pair $(B_\varepsilon, B_\varepsilon \cap V)$ is homeomorphic to the cone over $(S_\varepsilon, S_\varepsilon \cap V)$ ([Mi] Theorem 2.10, see also [Ok] Ch.I, §1). In this case, S_ε and V are transverse and $K = S_\varepsilon \cap V$ is a $(2n-1)$ dimensional C^∞ manifold, which is called the *link* of the singularity of V at 0.

Now let V be a (germ of) complete intersection of dimension n and $f_1, \dots, f_k$ generators of $I(V)$. We suppose that these germs have representatives in U and we think of $f = (f_1, \dots, f_k)$ as a holomorphic map from U onto a neighborhood W of 0 in $\mathbb{C}^k$. Let $C(f)$ be the set of critical points of f. Then, by Proposition 5.3, $\operatorname{Sing}(V) = V \cap C(f)$. Note that, when $k = 1$, $\operatorname{Sing}(V) = C(f)$ (cf. [Lo] Proof of (1.2) Proposition).

We have the following "fibration theorem", which is due to [Mi] when $k = 1$ and to [Ham] for general k, see also [HL], [Lê2], [Lo] and [Ok].

Theorem 5.5. *Let V be a complete intersection of dimension n with isolated singularity at 0 in $\mathbb{C}^{n+k}$. Then there exist small disks B_ε about 0 in U and B'_δ about 0 in W such that $D(f) = B'_\delta \cap f(C(f))$ is a hypersurface in B'_δ and that f induces a fiber bundle structure $B_\epsilon \cap f^{-1}(B'_\delta \setminus D(f)) \longrightarrow B'_\delta \setminus D(f)$. Moreover, the (typical) fiber F of this bundle has the homotopy type of a bouquet of n-spheres.*

The fiber F is called the *Milnor fiber* and the number of spheres appearing in the above is called the *Milnor number* of V at 0 and is denoted by $\mu(V, 0)$. The number $\mu(V, 0)$ does not depend on the choice of generators of $I(V)$. There is an algebraic formula for this number ([Lê1], [Gr], see also [Lo]). We set, for $i = 1, \dots, k$,

$$a_i = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(f_1, \dots, f_i), f_1, \dots, f_{i-1}),$$

where the denominator in the right hand side is the ideal generated by the Jacobians $\det(\partial(f_1, \dots, f_i)/\partial(z_{\nu_1}, \dots, z_{\nu_i}))$, $1 \le \nu_1 < \cdots < \nu_i \le n+k$, and $f_1, \dots, f_{i-1}$. Then

$$\mu(V, 0) = \sum_{i=1}^{k} (-1)^{k-i} a_i.$$

In particular, when $k = 1$,

$$\mu(V, 0) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/J(f), \tag{5.6}$$

where $J(f) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_{n+1})$, $f = f_1$ (cf. [Or]).

Now we consider the global situation. Let W be a complex manifold of dimension $n+k$ and V an analytic variety in W. Suppose V is pure n dimensional. We call V a *local complete intersection* (LCI) if the germ of V at each point of V is a complete intersection. Thus each point of V has a neighborhood, where V admits k reduced defining functions.

In this case, there is a vector bundle N_V over V of rank k, which extends the normal bundle $N_{V'}$ of V' in W. We have a commutative diagram with an exact row (e.g., [LS] Proposition 1)

$$\begin{tikzcd}
 & & TW|_V \arrow[r, "\pi"] & N_V & \\
0 \arrow[r] & TV' \arrow[r] & TW|_{V'} \arrow[r, "\pi"] \arrow[u, "\text{incl.}"'] & N_{V'} \arrow[r] \arrow[u, "\text{incl.}"'] & 0.
\end{tikzcd} \tag{5.7}$$

If $f_1, \ldots, f_k$ are local reduced defining functions of V, then there is a frame of N_V which extends the frame $(\pi(\partial/\partial f_1), \ldots, \pi(\partial/\partial f_k))$ of $N_{V'}$. (Note that near a regular point of $f = (f_1, \ldots, f_k)$, i.e., a regular point of V, we may take $(f_1, \ldots, f_k)$ as a part of a local coordinate system on W.) We call it the frame of N_V associated to $f = (f_1, \ldots, f_k)$.

For an LCI V in W, we call $TW|_V - N_V$ the *virtual tangent bundle* of V.

Now let N be a holomorphic vector bundle over W of rank k and s a holomorphic section of N. We call the zero set V of s in W an *LCI defined by s* if it is an LCI with local components of s (with respect to some local holomorphic frame of N) as its reduced defining functions. In this case, we have $N_V = N|_V$.

Examples 5.8. As examples of LCIs defined by a section of a holomorphic vector bundle, we have the following :

1. V a hypersurface in W ($k = 1$). In this case, we may take as N the line bundle $L(V)$ defined by V and as s the natural section described in Ch.II, Example 1.9.

2. V a complete intersection. In this case we may take as N the trivial bundle and as s a system of generators of the ideal of holomorphic functions vanishing on V.

3. V a (projective algebraic) complete intersection in the projective space $\mathbb{CP}^{n+k}$. This means that the ideal of homogeneous polynomials vanishing on V is generated by k homogeneous polynomials $P_1, \ldots, P_k$. Let d_i denote the degree of P_i for $i = 1, \ldots, k$ and U_j the affine coordinate $\zeta_j \neq 0$ for $j = 0, \ldots, n+k$. Then, in U_j, V is defined by $f_1 = \cdots = f_k = 0$, $f_i = P_i/\zeta_j^{d_i}$. Note that it is only locally a complete intersection. In this case, we may take as N the bundle $H^{d_1} \oplus \cdots \oplus H^{d_k}$, where H denotes the hyperplane bundle (Ch.II, Example 1.9).

6. Grothendieck residues

For details on this subject, we refer to [GH]. Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n$ and $f_1, \dots, f_n$ germs in $\mathcal{O}_n$ such that $V(f_1, \dots, f_n) = \{0\}$ (Ch.I, 5). For a germ ω at 0 of holomorphic n-form we choose a neighborhood U of 0 in $\mathbb{C}^n$ where $f_1, \dots, f_n$ and ω have representatives and let Γ be the n-cycle in U defined by

$$\Gamma = \{ z \in U \mid |f_1(z)| = \cdots = |f_n(z)| = \varepsilon \},$$

where, ε is a small positive number. We orient Γ so that the form $d\theta_1 \wedge \cdots \wedge d\theta_n$ is positive, $\theta_i = \arg f_i$. Then we set

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix} = \frac{1}{(2\pi\sqrt{-1})^n} \int_\Gamma \frac{\omega}{f_1 \cdots f_n}.$$

Note that this residue is alternating in $(f_1, \dots, f_n)$.

Example 6.1. When $n = 1$, the above residue is the usual Cauchy residue at 0 of the meromorphic 1-form ω / f_1.

Example 6.2. If $\omega = df_1 \wedge \cdots \wedge df_n$, then

$$\mathrm{Res}_0 \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}$$

is a positive integer which is simultaneously equal to
(i) the intersection number $(D_1 \cdots D_n)_0$ at 0 of the divisors D_i defined by f_i (see section 4 and [GH] Ch.5, 2, [Su3]),
(ii) $\dim_{\mathbb{C}} \mathcal{O}_n / (f_1, \dots, f_n)$ and
(iii) the (Poincaré-Hopf) index at 0 in $\mathbb{C}^n$ of the vector field $v = \sum_{i=1}^n f_i \cdot \partial / \partial z_i$ (the mapping degree of $f = (f_1, \dots, f_n)$), see Ch.IV, sections 2 and 3.

Example 6.3. In particular, if $f_i = \partial f / \partial z_i$ for some f in $\mathcal{O}_n$, then it is the Milnor number $\mu(V, 0)$ of the hypersurface V defined by f at 0 (see section 5) :

$$\mathrm{Res}_0 \begin{bmatrix} d\left(\frac{\partial f}{\partial z_1}\right) \wedge \cdots \wedge d\left(\frac{\partial f}{\partial z_n}\right) \\ \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \end{bmatrix} = \mu(V, 0).$$

We also call this number the multiplicity of f at 0 and denote it by $m(f, 0)$ (cf. Ch.IV, 3 below).

Chapter IV. Localization of Chern classes and associated residues

1. Localization of the top Chern class

Let $\pi : E \longrightarrow M$ be a C^∞ complex vector bundle of rank r over an oriented C^∞ manifold M of dimension m. Let s be a non-vanishing section of E on some open set U. Recall that a connection ∇ for E on U is s-trivial, if $\nabla(s) = 0$. If ∇ is an s-trivial connection, we have the vanishing (Ch.III, Proposition 3.3)

$$c_r(\nabla) = 0. \tag{1.1}$$

Let S be a closed set in M and suppose we have a C^∞ non-vanishing section s of E on $M \setminus S$. Then, from the above fact, we will see that there is a natural lifting $c_r(E, s)$ in $H^{2r}(M, M \setminus S; \mathbb{C})$ of the top Chern class $c_r(E)$ in $H^{2r}(M, \mathbb{C})$.

Letting $U_0 = M \setminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall the Chern class $c_r(E)$ is represented by the cocycle $c_r(\nabla_*)$ in $A^{2r}(\mathcal{U})$ given by

$$c_r(\nabla_*) = (c_r(\nabla_0), c_r(\nabla_1), c_r(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an s-trivial connection, then $c_r(\nabla_0) = 0$ by (1.1) and thus the cocycle is in $A^{2r}(\mathcal{U}, U_0)$ and it defines a class in the relative cohomology $H^{2r}(M, M \setminus S; \mathbb{C})$, which we denote by $c_r(E, s)$. It is sent to the class $c_r(E)$ by the canonical homomorphism $j^* : H^{2r}(M, M \setminus S; \mathbb{C}) \longrightarrow H^{2r}(M; \mathbb{C})$. It does not depend on the choice of the connection ∇_1 or on the choice of the s-trivial connection ∇_0. We call $c_r(E, s)$ the localization of $c_r(E)$ with respect to the section s at S.

In the above situation, suppose that S is a compact set, with a finite number of connected components $(S_\lambda)_\lambda$, admitting a regular neighborhood. Then we have the Alexander duality Ch. II (5.3) :

$$A : H^{2r}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{m-2r}(S, \mathbb{C}) = \bigoplus_\lambda H_{m-2r}(S_\lambda, \mathbb{C}).$$

Thus the class $c_r(E, s)$ defines a class in $H_{m-2r}(S_\lambda; \mathbb{C})$, which we call the residue of $c_r(E)$ at S_λ with respect to s and denote by $\mathrm{Res}_{c_r}(s, E; S_\lambda)$. This residue corresponds to what is called the "localized top Chern class" of E with respect to s in [Fu] §14.1.

For each λ, we choose a neighborhood U_λ of S_λ in U_1, so that the U_λ's are mutually disjoint. Let R_λ be an m-dimensional manifold with C^∞ boundary in U_λ containing S_λ in its interior. We set $R_{0\lambda} = -\partial R_\lambda$. Then the residue $\mathrm{Res}_{c_r}(s, E; S_\lambda)$ is represented by an $(m - 2r)$-cycle C in S_λ such that

$$\int_C \tau = \int_{R_\lambda} c_r(\nabla_1) \wedge \tau + \int_{R_{0\lambda}} c_r(\nabla_0, \nabla_1) \wedge \tau \tag{1.2}$$

for any closed $(m-2r)$-form τ on U_λ. In particular, if $2r = m$, the residue is a complex number given by

$$\operatorname{Res}_{c_r}(s, E; S_\lambda) = \int_{R_\lambda} c_r(\nabla_1) + \int_{R_{0\lambda}} c_r(\nabla_0, \nabla_1). \tag{1.3}$$

By Ch.II, Proposition 5.4, we have the following "residue formula".

Proposition 1.4. *In the above situation, if M is compact,*

$$\sum_\lambda (i_\lambda)_* \operatorname{Res}_{c_r}(s, E; S_\lambda) = c_r(E) \frown [M] \qquad \text{in } H_{m-2r}(M; \mathbb{C}),$$

where i_λ denotes the inclusion $S_\lambda \hookrightarrow M$.

2. Residues at an isolated zero

Let $\pi : E \longrightarrow M$ be a holomorphic vector bundle of rank n over a complex manifold M of dimension n. Suppose we have a section s with an isolated zero at p in M. In this situation, we have $\operatorname{Res}_{c_n}(s, E; p)$ in $H_0(\{p\}; \mathbb{C}) = \mathbb{C}$. In the following, we give explicit expressions of this residue.

Let U be an open neighborhood of p where the bundle E is trivial with holomorphic frame $(e_1, \dots, e_n)$. We write $s = \sum_{i=1}^n f_i \, e_i$ with f_i holomorphic functions on U.

(I) Analytic expression

Theorem 2.1. *In the above situation, we have*

$$\operatorname{Res}_{c_n}(s, E; p) = \operatorname{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}.$$

Proof. We indicate the proof for the case $n = 1$ (for $n > 1$, we use the Čech-de Rham cohomology theory for n open sets, see [Su3], [Su5]). Thus $s = f e_1$ for some holomorphic function f on U. Let R be a closed disk about p in U. In the expression (1.3) of the residue, we may take as ∇_1 an e_1-trivial connection on U, thus $c_1(\nabla_1) \equiv 0$ and

$$\operatorname{Res}_p(s, E, p) = -\int_{\partial R} c_1(\nabla_0, \nabla_1)$$

with ∇_0 an s-trivial connection on $U' = U \setminus \{p\}$. Now we recall how the Bott difference form $c_1(\nabla_0, \nabla_1)$ is defined (Ch.III, 1). Consider the bundle $\tilde{E} = E \times \mathbb{R}$ over $U \times \mathbb{R}$, and let t be a coordinate on $\mathbb{R}$. Define a connection for $\tilde{E}$ on $U' \times \mathbb{R}$ by

$\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let $\pi : U' \times [0,1] \to U'$ be the canonical projection and let π_* be the integration along the fibers of π. Then we define $c_1(\nabla_0, \nabla_1) = \pi_* c_1(\tilde{\nabla})$.

Let θ_i be the connection matrix of ∇_i, $i = 0, 1$ with respect to the frame e_1. Therefore $\theta_1 = 0$. To find θ_0, we use Ch.III, (1.6). Since the connection matrix with respect to s is zero, we get

$$\theta_0 = -\frac{df}{f}.$$

Hence $\tilde{\theta} = (1-t)\theta_0 = (t-1)\frac{df}{f}$ and the curvature matrix $\tilde{\kappa}$ is given by

$$\tilde{\kappa} = d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta} = dt \wedge \frac{df}{f}.$$

Thus

$$c_1(\nabla_0, \nabla_1) = \pi_* c_1(\tilde{\nabla}) = \frac{\sqrt{-1}}{2\pi}\pi_*(dt \wedge \frac{df}{f}) = -\frac{1}{2\pi\sqrt{-1}}\frac{df}{f},$$

which proves the theorem (for the case $n = 1$). $\square$

Remark 2.2. For general n, if we take suitable connections we see that the difference form is given by

$$c_n(\nabla_0, \nabla_1) = -f^*\beta_n,$$

where $f = (f_1, \dots, f_n)$ and β_n denotes the Bochner-Martinelli kernel on $\mathbb{C}^n$ (Ch.II, 5). This gives a direct proof of Theorem 2.4 below. Thus we reprove the fact that the Grothendieck residue in the above theorem is equal to the mapping degree of f (cf. [GH] Ch.5, 1. Lemma).

(II) Algebraic expression

Theorem 2.3. *In the above situation, we have*

$$\mathrm{Res}_{c_n}(s, E; p) = \dim \mathcal{O}_n/(f_1, \dots, f_n).$$

This can be proved, for example, by perturbing the sections and using the theory of Cohen-Macaulay rings (e.g., [Su5]).

(III) Topological expression

Let S_ε^{2n-1} denote a small $2n-1$ shere in U with center p. Then we have the mapping

$$\varphi = \frac{f}{\|f\|} : S_\varepsilon^{2n-1} \to S^{2n-1},$$

where S^{2n-1} denotes the unit shere in $\mathbb{C}^n$.

Theorem 2.4. *In the above situation, we have*

$$\mathrm{Res}_{c_n}(s, E; p) = \deg \varphi.$$

This can also be proved by perturbing the sections, see [GH], [Su5].

3. Examples I

(a) Poincaré-Hopf index theorem

Let M be a complex manifold of dimension n. We take as E the holomorphic tangent bundle TM. Then a section of TM is a (complex) vector field v. We define the Poincaré-Hopf index $\mathrm{PH}(v, S_\lambda)$ of v at a connected component S_λ of its zero set S by

$$\mathrm{PH}(v, S_\lambda) = \mathrm{Res}_{c_n}(v, TM; S_\lambda).$$

Then, if M is compact, by Proposition 1.4, we have

$$\sum_\lambda \mathrm{PH}(v, S_\lambda) = \int_M c_n(M),$$

where $c_n(M) = c_n(TM)$ and it is known that the right hand side coincides with the Euler-Poincaré characteristic $\chi(M)$ of M ("Gauss-Bonnet formula"). Thus, by Theorem 2.4, we recover the Poincaré-Hopf theorem in case v is holomorphic and the zeros are isolated.

Exercise 3.1. Find all the holomorphic vector field on the Riemann sphere $\mathbb{CP}^1$ and verify the Poincaré-Hopf formula for each of them.

(b) Multiplicity formula

Let M be a complex manifold of dimension n. We take as E the holomorphic cotangent bundle T^*M. For a holomorphic function f on M, its differential df is a section of T^*M. The zero set S of df coincides with the critical set $C(f)$ of f. We define the multiplicity $m(f, S_\lambda)$ of f at a connected component S_λ of $C(f)$ by

$$m(f, S_\lambda) = \mathrm{Res}_{c_n}(df, T^*M; S_\lambda).$$

Note that, if S_λ consists of a point p, it coinsides with the multiplicity $m(f, p)$ of f at p described in Example 6.3 of Ch.III.

Now we consider the global situation. Let $f : M \to C$ be a holomorphic map of M onto a complex curve (Riemann surface) C. The differential

$$df : TM \to f^*TC$$

of f determines a section of the bundle $T^*M \otimes f^*TC$, which is also denoted by df. The set of zeros of df is the critical set $C(f)$ of f. Suppose $C(f)$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then we have the residue $\mathrm{Res}_{c_n}(df, T^*M \otimes f^*TC; S_\lambda)$ for each λ. If M is compact, by Proposition 1.4, we have

$$\sum_\lambda \mathrm{Res}_{c_n}(df, T^*M \otimes f^*TC; S_\lambda) = \int_M c_n(T^*M \otimes f^*TC).$$

We look at the both sides of the above more closely. In the sequel, we set $D(f) = f(C(f))$, the set of critical values. Then, if M is compact, f defines a C^∞ fiber bundle structure on $M \setminus C(f) \to C \setminus D(f)$.

We refer to [IS] for a precise proof of the following

Lemma 3.1. *If M is compact, and if $D(f)$ consists of isolated points,*

$$\int_M c_n(T^*M \otimes f^*TC) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

where F denotes a general fiber of f.

Suppose that $f(S_\lambda)$ is a point. Taking a coordinate on C around $f(S_\lambda)$, we think of f as a holomorphic function near S_λ. Then we may write

$$\mathrm{Res}_{c_n}(df, T^*M \otimes f^*TC; S_\lambda) = \mathrm{Res}_{c_n}(df, T^*M; S_\lambda) = m(f, S_\lambda),$$

the multiplicity of f at S_λ. Thus we have

Theorem 3.2. *Let $f : M \to C$ be a holomorphic map of a compact complex manifold M of dimension n onto a complex curve C. If the critical values $D(f)$ of f consists of only isolated points, then*

$$\sum_\lambda m(f, S_\lambda) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)),$$

where the sum is taken over the connected components S_λ of $C(f)$.

In particular, we have ([I], see also [Fu] Example 14.1.5) :

Corollary 3.3. *In the above situation, if the critical set $C(f)$ of f consists of only isolated points,*

$$\sum_{p \in C(f)} m(f, p) = (-1)^n(\chi(M) - \chi(F)\,\chi(C)).$$

4. Residues of Chern classes on singular varieties

In this section, we deal with the situation more general than the one we discussed in section 1, in two ways. Namely, we consider Chern classes other than the top ones for vector bundles on possibly singular varieties. We refer to [Su2] and [Su5] for details of the material in this section.

Let V be an analytic variety of pure dimension n in a complex manifold W of dimension $n+k$. We denote by $\mathrm{Sing}(V)$ the singular set of V and set $V' = V \setminus \mathrm{Sing}(V)$.

First, suppose V is compact and let $\tilde{U}$ be a regular neighborhood of V in W. Then, as in Ch.II, 4, the cup product in $H^*(\tilde{U}) \simeq H^*(V)$ and the integration

$$\int_V : H^{2n}(\tilde{U}) \to \mathbb{C}$$

induces the "Poincaré homomorphism"

$$P : H^p(V, \mathbb{C}) \longrightarrow H_{2n-p}(V, \mathbb{C}),$$

which is not an isomorphism in general. Note that in [Br] the above homomorphism P, as well as the Alexander homomorphism defined below, are described in a combinatorial way for (co)homology with integral coefficients. The homomorphism P is given by the cap product with the fundamental class $[V]$.

Now suppose V may not be compact. Let S be a compact set in V. We assume that S has a finite number of connected components, $S \supset \mathrm{Sing}(V)$ and that S admits a regular neighborhood in W. Let $\tilde{U}_1$ be a regular neighborhood of S in W and $\tilde{U}_0$ a tubular neighborhood of $U_0 = V \setminus S$ in W. We consider the covering $\mathcal{U} = \{\tilde{U}_0, \tilde{U}_1\}$ of the union $\tilde{U} = \tilde{U}_0 \cup \tilde{U}_1$, which may be assumed to have the same homotopy type as V. We define the subcomplex $A^*(\mathcal{U}, \tilde{U}_0)$ of $A^*(\mathcal{U})$ as in Ch.II, 5. Then we see that

$$H^p_D(\mathcal{U}, \tilde{U}_0) \simeq H^p(V, V \setminus S; \mathbb{C}).$$

Again, as in Ch.II, 5, the cup product and the integration induces the "Alexander homomorphism"

$$A : H^p(V, V \setminus S; \mathbb{C}) \to H_{2n-p}(S, \mathbb{C}),$$

which is not an isomorphism in general.

Suppose V is compact. Then the following diagram is commutative :

$$\begin{array}{ccc} H^p(V, V \setminus S; \mathbb{C}) & \xrightarrow{\ j^*\ } & H^p(V, \mathbb{C}) \\ \big\downarrow{\scriptstyle A} & & \big\downarrow{\scriptstyle P} \\ H_{2n-p}(S, \mathbb{C}) & \xrightarrow{\ i_*\ } & H_{2n-p}(V, \mathbb{C}), \end{array} \tag{4.1}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow V$ and $(V, \emptyset) \hookrightarrow (V, V \setminus S)$.

For a complex vector bundle E over $\tilde{U}$ of rank r, the i-th Chern class $c_i(E)$ is in $H^{2i}(\tilde{U}) \simeq H^{2i}(V)$. The corresponding class in $H^{2i}(V)$ is denoted by $c_i(E|_V)$. The class $c_i(E)$ is represented by a Čech-de Rham cocycle $c_i(\nabla_*)$ on $\mathcal{U}$ given as (3.1) in Ch.III with ∇_0 and ∇_1 connections for E on $\tilde{U}_0$ and $\tilde{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on U_0, since there is a C^∞ retraction of $\tilde{U}_0$ onto U_0. Suppose we have an ℓ-tuple $\mathrm{s} = (s_1, \dots, s_\ell)$ of C^∞ sections linearly independent everywhere on U_0 and let ∇_0 be s-trivial. Then we have the vanishing $c_i(\nabla_0) = 0$, for $i \geq r - \ell + 1$ (Ch.III, Proposition 3.3), and the above cocycle $c_i(\nabla_*)$ defines a class $c_i(E|_V, \mathrm{s})$ in $H^{2i}_D(\mathcal{U}, \tilde{U}_0) \simeq H^{2i}(V, V \setminus S; \mathbb{C})$. It is sent to $c_i(E|_V)$ by the canonical homomorphism $j^* : H^{2i}(V, V \setminus S; \mathbb{C}) \to H^{2i}(V, \mathbb{C})$.

Let $(S_\lambda)_\lambda$ be the connected components of S. Then, for each λ, $c_i(E|_V, \mathbf{s})$ defines the residue $\mathrm{Res}_{c_i}(\mathbf{s}, E|_V; S_\lambda)$ via the Alexander homomorphism

$$A : H^{2i}(V, V \setminus S; \mathbb{C}) \longrightarrow H_{2n-2i}(S, \mathbb{C}) = \bigoplus_\lambda H_{2n-2i}(S_\lambda, \mathbb{C}).$$

For each λ, we choose a neighborhood $\tilde{U}_\lambda$ of S_λ in $\tilde{U}_1$, so that the $\tilde{U}_\lambda$'s are mutually disjoint. Let $\tilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary in $\tilde{U}_\lambda$ containing S_λ in its interior such that the boundary $\partial\tilde{R}_\lambda$ is transverse to V. We set $R_{0\lambda} = -\partial\tilde{R}_\lambda \cap V$. Then the residue $\mathrm{Res}_{c_i}(\mathbf{s}, E|_V; S_\lambda)$ is represented by a $2(n-i)$-cycle C in S_λ satisfying the identity as (1.2) for every closed $2(n-i)$-form τ on $\tilde{U}_\lambda$. In particular, if $i = n$, the residue is a number given by a formula as (1.3).

From the commutativity of (4.1), we have the "residue formula" :

Proposition 4.2. *In the above situation, if V is compact, we have, for $i \geq r-\ell+1$,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_{c_i}(\mathbf{s}, E|_V; S_\lambda) = c_i(E) \frown [V] \qquad \text{in} \ \ H_{2n-2i}(V, \mathbb{C}),$$

where $i_\lambda : S_\lambda \hookrightarrow V$ denotes the inclusion.

Note that the $\mathrm{Res}_{c_i}(\mathbf{s}, E|_V; S_\lambda)$'s are in fact in the integral homology and the above formula holds in the integral homology.

5. Residues at an isolated singularity

Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$, as before. Suppose now that V has at most an isolated singularity at p and let E be a holomorphic vector bundle of rank r $(\geq n)$ on a small coordinate neighborhood $\tilde{U}$ of p in W. We may assume that E is trivial and let $\mathbf{e} = (e_1, \dots, e_n)$ be a holomorphic frame of E on $\tilde{U}$. Let $\ell = r - n + 1$ and suppose we have an ℓ-tuple of holomorphic sections $\tilde{\mathbf{s}}$ of E on $\tilde{U}$. Suppose that $S(\tilde{\mathbf{s}}) \cap V = \{p\}$. Then we have $\mathrm{Res}_{c_n}(\mathbf{s}, E|_V; p)$ with $\mathbf{s} = \tilde{\mathbf{s}}|_V$. In the following, we give various expressions of this number.

We write $\tilde{s}_i = \sum_{j=1}^r f_{ij}\, e_j$, $i = 1, \dots, \ell$, with f_{ij} holomorphic functions on $\tilde{U}$. Let F be the $\ell \times r$ matrix whose (i,j)-entry is f_{ij}. We set

$$\mathcal{I} = \{\, (i_1, \dots, i_\ell) \mid 1 \leq i_1 < \cdots < i_\ell \leq r \,\}.$$

For an element $I = (i_1, \dots, i_\ell)$ in $\mathcal{I}$, let F_I denote the $\ell \times \ell$ matrix consisting of the columns of F corresponding to I and set $\varphi_I = \det F_I$. If we write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, we have

$$\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_\ell = \sum_{I \in \mathcal{I}} \varphi_I\, e_I.$$

Note that $S(\tilde{\mathbf{s}})$ is the set of common zeros of the φ_I's.

(I) Analytic expression

First we recall:

Grothendieck residues relative to a subvariety

Let $\tilde{U}$ be a neighborhood of 0 in $\mathbb{C}^{n+k}$ and V a subvariety of dimension n in $\tilde{U}$ which contains 0 as at most an isolated singular point. Also, let $f_1, \dots, f_n$ be holomorphic functions on $\tilde{U}$ and $V(f_1, \dots, f_n)$ the variety defined by them. We assume that $V(f_1, \dots, f_n) \cap V = \{0\}$. For a holomorphic n-from ω on $\tilde{U}$, the Grothendieck residue relative to V is defined by (e.g., [Su2] Ch.IV, 8)

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ is the n-cycle in V given by

$$\Gamma = \{ q \in \tilde{U} \cap V \mid |f_i(q)| = \varepsilon_i, \quad i = 1, \dots, n \}$$

for small positive numbers ε_i. It is oriented so that $d\arg(f_1) \wedge \cdots \wedge d\arg(f_n) \geq 0$.

If $k = 0$, it reduces to the usual Grothendieck residue (Ch.III, 6), in which case we omit the suffix V.

If V is a complete intersection defined by $h_1 = \cdots = h_k = 0$ in $\tilde{U}$, we have

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \operatorname{Res}_0 \begin{bmatrix} \omega \wedge dh_1 \wedge \cdots \wedge dh_k \\ f_1, \dots, f_n, h_1, \dots, h_k \end{bmatrix}.$$

To get the analytic expression, we first note that, from the assumption $S(\tilde{\mathbf{s}}) \cap V = \{p\}$, we have ([Su4] Lemma 5.6) :

Lemma 5.1. *We may choose a holomorphic frame $\mathbf{e} = (e_1, \dots, e_r)$ of E so that there exist n elements $I^{(1)}, \dots, I^{(n)}$ in $\mathcal{I}$ with $V(\varphi_{I^{(1)}}, \dots, \varphi_{I^{(n)}}) \cap V = \{p\}$.*

Theorem 5.2. *We have*

$$\operatorname{Res}_{c_n}(\mathbf{s}, E|_V; p) = \operatorname{Res}_p \begin{bmatrix} \sigma_n(\Theta) \\ \varphi_{I^{(1)}}, \dots, \varphi_{I^{(n)}} \end{bmatrix}_V,$$

where $\varphi_{I^{(1)}}, \dots, \varphi_{I^{(n)}}$ are chosen so that they satisfy the conditions in Lemma 5.1 and $\sigma_n(\Theta)$ is a holomorphic n-from given in terms of the matrix F (see [Su4] for the precise expression).

Here are some special cases:

1. The case $\ell = 1$ and $r = n$. Let $\mathbf{e} = (e_1, \dots, e_n)$ be an arbitrary frame of E and write $s = \sum_{i=1}^n f_i e_i$. Then we may set $\varphi_{I^{(i)}} = f_i$, $i = 1, \dots, n$, and we have $\sigma_n(\Theta) = df_1 \wedge \cdots \wedge df_n$.

2. The case $n = 1$ and $\ell = r$. Let $\mathbf{e} = (e_1, \dots, e_r)$ be an arbitrary frame of E and write $s_i = \sum_{j=1}^{r} f_{ij}\, e_j$, $i = 1, \dots, r$. Let $F = (f_{ij})$ and set $\varphi = \det F$. Then we may set $\varphi_{I^{(1)}} = \varphi$ and we have $\sigma_n(\Theta) = d\varphi$.

See [Su4] for more cases where the form $\sigma_n(\Theta)$ is computed explicitly.

(II) Algebraic expression

Let $\mathcal{O}_{\tilde{U},p}$ denote the ring of germs of holomorphic functions on $\tilde{U}$ at p, which is isomorphic to the ring $\mathcal{O}_{n+k}$ of convergent power series in $n + k$ variables. We assume that V is a complete intersection defined by $h_1, \dots, h_k$ near p and let $\mathcal{F}(V)_p$ denote the ideal in $\mathcal{O}_{\tilde{U},p}$ generated by (the germs of) the φ_I's and $h_1, \dots, h_k$.

Theorem 5.3. *We have*

$$\mathrm{Res}_{c_n}(\mathbf{s}, E|_V; p) = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{U},p}/\mathcal{F}(V)_p.$$

(III) Topological expression

We again assume that V is a complete intersection in $\tilde{U}$. Let $W_\ell(\mathbb{C}^r)$ denote the Stiefel manifold of ℓ-frames in $\mathbb{C}^r$. It is known that the space $W_\ell(\mathbb{C}^r)$ is $2(r-\ell)$-connected and $\pi_{2n-1}(W_\ell(\mathbb{C}^r)) \simeq \mathbb{Z}$ (recall $2r - 2\ell + 1 = 2n - 1$). Let L denote the link of (V, p). Note that both of $W_\ell(\mathbb{C}^r)$ and L have a natural generator for the $(2n-1)$-st homology. Thus the degree of the map

$$\varphi = \mathbf{s}|_L : L \longrightarrow W_\ell(\mathbb{C}^r)$$

is well-defined.

Theorem 5.4. *We have*

$$\mathrm{Res}_{c_n}(\mathbf{s}, E|_V; p) = \deg \varphi.$$

6. Examples II

(a) Index of a holomorphic 1-form of Ebeling and Gusein-Zade

Let V be a complete intersection in $\tilde{U}$ with an isolated singularity at p and defined by $(h_1, \dots, h_k)$, as before. Also, let L be the link of (V, p). For a holomorphic 1-form θ on $\tilde{U}$, we consider the $(k+1)$-tuple $\tilde{\mathbf{s}} = (\theta, dh_1, \dots, dh_k)$ of sections of $T^*\tilde{U}$, which is of rank $n + k$. Thus $r - \ell + 1 = n + k - (k+1) + 1 = n$. We assume that $S(\tilde{\mathbf{s}}) \cap V = \{p\}$, which means that the pull-back of θ to $V \setminus \{p\}$ by the inclusion $V \setminus \{p\} \hookrightarrow \tilde{U}$ does not vanish. Let $\mathbf{s} = \tilde{\mathbf{s}}|_V$, which defines a map of

$V \setminus \{p\}$ to $W_\ell(\mathbb{C}^r)$. It should be emphasized that here we take the restrictions of components of $\tilde{\mathbf{s}}$ as sections and not as differential forms.

Following [EG1], [EG2], with different naming and notation, we define the *V-index* $\mathrm{Ind}_V(\theta, p)$ of θ at p by

$$\mathrm{Ind}_V(\theta, p) = \deg \mathbf{s}|_L.$$

Then by Theorem 5.4, it coincides with $\mathrm{Res}_{c_n}(\mathbf{s}, T^*\tilde{U}|_V; p)$ and by Theorems 5.2 and 5.3, it has analytic and algebraic expressions. In fact the algebraic one is already given in [EG1], [EG2].

Remark 6.1. For a vector field, there is a similar index, which is called the GSV-index ([GSV], [SS1]). Namely, in the above situation let v be a holomorphic vector field on $\tilde{U}$. Assume that v is tangent to $V \setminus \{p\}$ and non-vanishing there. Set $\tilde{\mathbf{s}} = (v, \overline{\mathrm{grad}\ h_1}, \dots, \overline{\mathrm{grad}\ h_k})$ and $\mathbf{s} = \tilde{\mathbf{s}}|_V$. Then the GSV-index of v at p is defined by

$$\mathrm{GSV}(v, p) = \deg \mathbf{s}|_L.$$

Since $\mathbf{s}$ involves anti-holomorphic objects, we cannot directly apply our previous results. Note that it coincides with the "virtual index" of v ([LSS], [SS2]) and that there is an algebraic formula for it as a homological index, when $k = 1$ ([Gó]).

(b) Multiplicity of a function on a local complete intersection

We refer to [IS] for details of this subsection. Let V be a subvariety of dimension n in a complex manifold W of dimension $n + k$. We assume that V is a local complete intersection defined by a section s of a holomorphic vector bundle N of rank k over W (see Ch.III, 5).

Recall that the restriction of N to the non-singular part V' coincides with the normal bundle of V' in W. We denote the virtual bundle $(T^*W - N^*)|_V$ by τ_V^* and call it the virtual cotangent bundle of V. Let g be a holomorphic function on W and let f and f' be its restrictions to V and V', respectively. We define the singular set $S(f)$ of f by $S(f) = \mathrm{Sing}(V) \cup C(f')$. As in the case of vector bundles, we may define the localization of the n-th Chern class of τ_V^* by df, which in turn defines the residue $\mathrm{Res}_{c_n}(df, \tau_V^*; S)$ at each compact connected component S of $S(f)$. We define the *virtual multiplicity* $\widetilde{m}(f, S)$ of f at S by

$$\widetilde{m}(f, S) = \mathrm{Res}_{c_n}(df, \tau_V^*; S). \tag{6.2}$$

The *multiplicity* of f at S is then defined by

$$m(f, S) = \widetilde{m}(f, S) - \mu(V, S), \tag{6.3}$$

where, $\mu(V,S)$ denotes the (generalized) Milnor number of V at S as defined in [BLSS] (cf. [A], [P], [PP] in the case $k=1$). Note that if S consists of a point p, it is the usual Milnor number $\mu(V,p)$ of the isolated complete intersection singularity (V,p) (cf. Ch.III, 5).

Note that, if S is in V', we have $\mathrm{Res}_{c_n}(df,\tau_V^*;S)=\mathrm{Res}_{c_n}(df,T^*V';S)$. On the other hand, in this case we have $\mu(V,S)=0$ so that $m(f,S)$ coincides with the one in 3 (b).

Let $g:W\to C$ be a holomorphic map onto a complex curve C and set $f=g|_V$, $f'=g|_{V'}$ and $S(f)=\mathrm{Sing}(V)\cup C(f')$. We assume that $S(f)$ is compact. We further set $V_0=V\setminus S(f)$ and $f_0=g|_{V_0}$. Thus df_0 is a non-vanishing section of the bundle $T^*V_0\otimes f_0^*TC$, which is of rank n. If we look at $c_n(\varepsilon)$, $\varepsilon=\tau_V^*\otimes f^*TC$ and we see that there is a canonical localization $c_n(\varepsilon,df)$ in $H^{2n}(V,V\setminus S;\mathbb{C})$ of $c_n(\varepsilon)$.

Let $(S_\lambda)_\lambda$ be the connected components of S and let $(R_\lambda)_\lambda$ be as in 4. Then $c_n(\varepsilon,df)$ defines, for each λ, the residue $\mathrm{Res}_{c_n}(df,\tau_V^*\otimes f^*TC;S_\lambda)$. If V is compact, by Proposition 4.2, we have

$$\sum_\lambda \mathrm{Res}_{c_n}(df,\tau_V^*\otimes f^*TC;S_\lambda)=\int_V c_n(\tau_V^*\otimes f^*TC).$$

The both sides in the above are reduced as follows. If $f(S(f))$ consists of isolated points, we may write

$$\mathrm{Res}_{c_n}(df,\tau_V^*\otimes f^*TC;S_\lambda)=\widetilde{m}(f,S_\lambda)=m(f,S_\lambda)-\mu(V,S_\lambda)$$

and, if moreover, V is compact, we have

$$\int_V c_n(\tau_V^*\otimes f^*TC)=(-1)^n\left(\chi(V)-\chi(F)\,\chi(C)\right)+\sum_\lambda \mu(V,S_\lambda),$$

where F is a general fiber of f ([IS] Lemma 5.2). Thus, in the above situation, we have ([IS] Theorem 5.5) :

$$\sum_\lambda m(f,S_\lambda)=(-1)^n\left(\chi(V)-\chi(F)\,\chi(C)\right).$$

In particular, if $S(f)$ consists only of isolated points,

$$\sum_{p\in S(f)} m(f,p)=(-1)^n\left(\chi(V)-\chi(F)\,\chi(C)\right), \tag{6.4}$$

which generalizes Corollary 3.3 for a singular variety V.

If S_λ consists of a single point p, the residue $\mathrm{Res}_{c_n}(df, \tau_V^*; p)$ is given as follows. Let $\tilde{U}$ be a small neighborhood of p in W so that the bundle N admits a frame $(\nu_1, \dots, \nu_k)$ on $\tilde{U}$. We write $s = \sum_{i=1}^k h_i \, \nu_i$ with h_i holomorphic functions on $\tilde{U}$. Then V is defined by $(h_1, \dots, h_k)$ in $\tilde{U}$. Consider the $(k+1)$-tuple of sections

$$\tilde{\mathrm{s}} = (dg, dh_1, \dots, dh_k)$$

of $T^*\tilde{U}$. By the assumption, we have $S(\tilde{\mathrm{s}}) \cap V = \{p\}$. Since the rank of $T^*\tilde{U}$ is $n+k$, we have the residue $\mathrm{Res}_{c_n}(\mathrm{s}, T^*\tilde{U}|_V; p)$, $\mathrm{s} = \tilde{\mathrm{s}}|_V$. Then we have ([IS] Theorem 4.6)

$$\widetilde{m}(f,p) = \mathrm{Res}_{c_n}(\mathrm{s}, T^*\tilde{U}|_V; p). \tag{6.5}$$

The virtual multiplicity $\widetilde{m}(f,p)$ was defined as the residue of df on the virtual bundle τ_V^* and this definition led us to a global formula as (6.4). The identity (6.5) shows that it coincides with the residue of $\mathrm{s} = (dg|_V, dh_1|_V, \dots, dh_k|_V)$ on the vector bundle $T^*\tilde{U}|_V$. Thus we have various expressions for $\widetilde{m}(f,p)$ as given in the previous sections; by Theorem 5.2 we have a way to compute $\widetilde{m}(f,p)$ explicitly, by Theorem 5.3 we may express

$$\widetilde{m}(f,p) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(g, h_1, \dots, h_k), h_1, \dots, h_k), \tag{6.6}$$

where $J(g, h_1, \dots, h_k)$ denotes the Jacobian ideal of the map $(g, h_1, \dots, h_k)$, i.e., the ideal generated by the $(k+1) \times (k+1)$ minors of the Jacobian matrix $\frac{\partial(g, h_1, \dots, h_k)}{\partial(z_1, \dots, z_{n+k})}$, and by Theorem 5.4,

$$\widetilde{m}(f,p) = \mathrm{Ind}_V(dg, p). \tag{6.7}$$

From (6.3), (6.6) and the identity (cf. [Gr], [Lê1])

$$\mu(V,p) + \mu(V_g, p) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(g, h_1, \dots, h_k), h_1, \dots, h_k),$$

where V_g denotes the complete intersection defined by $(g, h_1, \dots, h_k)$, assuming $g(p) = 0$, we get

$$m(f,p) = \mu(V_g, p). \tag{6.8}$$

(c) Some others

Let V be a complete intersection defined by $(h_1, \dots, h_k)$ in $\tilde{U}$ and p an isolated singularity of V, as before.

The n-the polar multiplicity $m_n(V, p)$ of Gaffney ([Ga]) is defined by

$$m_n(V, p) = \dim_{\mathbb{C}} \mathcal{O}_{n+k}/(J(\ell, h_1, \dots, h_k), h_1, \dots, h_k),$$

where ℓ is a general linear function. By (6.6) and (6.7), we may write

$$m_n(V, p) = \mathrm{Ind}_V(d\ell, p) = \tilde{m}(\ell|_V, p).$$

Also, in the expression

$$\mathrm{Eu}(V, p) = 1 + (-1)^{n+1}\mu(V_\ell, p)$$

for the Euler obstruction $\mathrm{Eu}(V, p)$ of V at p (cf. [D], [Ka], see also [BLS]), we have by (6.8),

$$\mu(V_\ell, p) = m(\ell|_V, p).$$

Note that these local invariants appear in the comparison of the Schwartz-MacPherson, Mather and Fulton-Johnson classes of a local complete intersection with isolated singularities (cf. [OSY], [Su1]).

References

[A] P. Aluffi, *Chern classes for singular hypersurfaces*, Trans. Amer.Math. Soc. **351** (1999), 3989-4026.

[BB] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geom. **7** (1972), 279–342.

[Bo] R. Bott, *Lectures on characteristic classes and foliations*, Lectures on Algebraic and Differential Topology, Lecture Notes in Mathematics 279, Springer-Verlag, New York, Heidelberg, Berlin, 1972, pp. 1–94.

[BT] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics 82, Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[Br] J.-P. Brasselet, *Définition combinatoire des homomorphismes de Poincaré, Alexander et Thom, pour une pseudo-variété*, Caractéristique d'Euler-Poincaré, Astérisque 82-83, Société Mathématiques de France, 1981, pp. 71-91.

[BLS] J.-P. Brasselet, D.-T. Lê and J. Seade, *Euler obstruction and indices of vector fields*, Topology **39** (2000), 1193-1208.

[BLSS] J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor classes of local complete intersections*, Trans. Amer. Math. Soc. **354** (2002), 1351-1371.

[D] A. Dubson, *Classes caractéristiques des variétés singuliéres*, C.R. Acad. Sci. Paris **287** (1978), 237-240.

[EG1] W. Ebeling and S.M. Gusein-Zade, *On the index of a holomorphic* 1*-form on an isolated complete intersection singularity*, Doklady Math. **64** (2001), 221-224.

[EG2] W. Ebeling and S.M. Gusein-Zade, *Indices of* 1*-forms on an isolated complete intersection singularity*, Moscow Math. J. **3** (2003), 439-455.

[Fi] G. Fischer, *Complex Analytic Geometry,* Lecture Notes in Mathematics 538, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[Ga] T. Gaffney, *Multiplicities and equisingularity of ICIS germs*, Invent. Math. **123** (1996), 209-220.

[Gó] X. Gómez-Mont, *An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity*, J. Algebraic Geometry **7** (1998), 731-752.

[GSV] X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. **291** (1991), 737–751.

[Gr] G.-M. Greuel, *Der Gauß-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. **214** (1975), 235–266.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1978.

[GP] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, 1974.

[GR] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, 1965.

[Ham] H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191** (1971), 235–252.

[HL] H. Hamm and D.-T. Lê, *Un théorème de Zariski du type de Lefschetz*, Ann. scient. Éc. Norm. Sup. **6** (1973), 317–366.

[Har] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics 52, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[Hi] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, Heidelberg, Berlin, 1966.

[I] B. Iversen, *Critical points of an algebraic function*, Invent. math. **12** (1971), 210-224.

[IS] T. Izawa and T. Suwa, *Multiplicity of functions on singular varieties*, Intern. J. Math. **14** (2003), 541-558.

[Ka] M. Kashiwara, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. **49** (1973), 803-804.

[Ko] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Springer-Verlag, New York, Heidelberg, Berlin, 1986.

[Lê1] D.-T. Lê, *Calculation of Milnor number of isolated singularity of complete intersection*, Funct. Anal. Appl. **8** (1974), 127–131.

[Lê2] D.-T. Lê, *Some remarks on relative monodromy*, Real and Complex Singularities, Oslo 1976, ed. P. Holm, Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, 1977, pp. 397–403.

[Leh] D. Lehmann, *Systèmes d'alvéoles et intégration sur le complexe de Čech-de Rham*, Publications de l'IRMA, 23, N^o VI, Université de Lille I, 1991.

[LSS] D. Lehmann, M. Soares and T. Suwa, *On the index of a holomorphic vector field tangent to a singular variety*, Bol. Soc. Bras. Mat. **26** (1995), 183–199.

[LS] D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*, J. Differential Geom. **42** (1995), 165–192.

[Lo] E. Looijenga, *Isolated Singular Points on Complete Intersections,* London Mathematical Society Lecture Note Series 77, Cambridge Univ. Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1984.

[Mac] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. **100** (1974), 423–432.

[Mat] H. Matsumura, *Commutative Algebra*, Benjamin/Cummings, 1980.

[Mi] J. Milnor, *Singular Points of Complex Hypersurfaces,* Annales of Mathematics Studies 61, Princeton University Press, Princeton, 1968.

[MS] J. Milnor and J. Stasheff, *Characteristic Classes,* Annales of Mathematics Studies 76, Princeton University Press, Princeton, 1974.

[OSY] T. Ohmoto, T. Suwa and S. Yokura, *A remark on the Chern classes of local complete intersections*, Proc. Japan Acad. **73** (1997), 93-95.

[Ok] M. Oka, *Non-Degenerate Complete Intersection Singularity,* Actualités Mathématiques, Hermann, Paris, 1997.

[Or] P. Orlik, *The multiplicity of a holomorphic map at an isolated critical point*, Real and Complex Singularities, Oslo 1976, ed. P. Holm, Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, 1977, pp. 405–474.

[P] A. Parusiński, *A generalization of the Milnor number*, Math. Ann. **281** (1988), 247–254.

[PP] A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Algebraic Geom. **10** (2001), 63-79.

[SS1] J. Seade and T. Suwa, *A residue formula for the index of a holomorphic flow*, Math. Ann. **304** (1996), 621–634.

[SS2] J. Seade and T. Suwa, *An adjunction formula for local complete intersections*, Intern. J. Math. **9** (1998), 759-768.

[St] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton, 1951.

[Su1] T. Suwa, *Classes de Chern des intersections complètes locales*, C.R. Acad. Sci. Paris **324** (1996), 67–70.

[Su2] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations,* Actualités Mathématiques, Hermann, 1998.

[Su3] T. Suwa, *Dual class of a subvariety*, Tokyo J. Math. **23** (2000), 51-68.

[Su4] T. Suwa, *Residues of Chern classes*, J. Math. Soc. Japan **55** (2003), 269-287.

[Su5] T. Suwa, *Residues of Chern classes on singular varieties*, Singularités Franco-Japonaises, Marseille 2002, Séminaires et Congrès **10**, Soc. Math. France, 2005, pp. 265-285.

DEPARTMENT OF INFORMATION ENGINEERING, NIIGATA UNIVERSITY, 2-8050 IKARASHI, NIIGATA 950-2181, JAPAN

E-mail address: `suwa@ie.niigata-u.ac.jp`