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## Notes on Singularities II

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# Notes on singularities II

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# Lecture 3

## 1 Thom condition

### 1.1 Definition.

Let $f : X \to Y$ be a complex analytic (resp. semi-analytic) map. When one wants to study the geometry of such maps, one is led first to consider properties which characterize how tame these maps are (see [8] and [2]).

**Definition 1.1** *We shall say that $f$ is stratifiable if there exist complex analytic (resp. semi-analytic) Whitney stratifications $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$ and $\mathcal{T} = \{T_\beta\}_{\beta \in B}$ of $X$ and $Y$ respectively, such that, for any $\alpha \in A$, there is $\beta(\alpha)$ such that $f$ induces a submersive map $f_\alpha : S_\alpha \to T_{\beta(\alpha)}$.*

*We shall also say that $f$ is stratified by the Whitney stratifications $\mathcal{S}$ and $\mathcal{T}$.*

In [2] (theorem 1 §4), it is shown that

**Theorem 1.2** *A proper complex analytic (resp. semi-analytic) map is stratifiable.*

When the map is not proper, it is easy to find a counter-example. For instance, consider the graph of the function $x \sin(1/x)$ in $\mathbb{R}^* \times \mathbb{R}$. The restriction to this graph of the projection onto $\{0\} \times \mathbb{R}$ is not stratifiable.

When a map $f$ is stratified by the Whitney stratifications $\mathcal{S}$ and $\mathcal{T}$, for all $\alpha \in A$, the map $f$ induces a stratified map $g_\alpha : f^{-1}(T_{\beta(\alpha)}) \to T_{\beta(\alpha)}$. When $f$ is proper, $g_\alpha$ is also proper and, by applying Thom's first isotopy lemma, we have

**Proposition 1.3** *The map $g_\alpha$ is a locally trivial topological fibration.*

In order to have a relative version of Thom's first isotopy lemma, another important property of complex analytic (resp. semi-analytic) maps is required:

**Definition 1.4** *Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$ and $\mathcal{T} = \{T_\beta\}_{\beta \in B}$ be complex analytic (resp. semi-analytic) Whitney stratifications for which $f$ is stratified. Let $(S_\alpha, S_{\alpha'})$ be a pair of strata of $\mathcal{S}$. We say that the $(S_\alpha, S_{\alpha'})$ satisfies the Thom condition for the map $f : X \to Y$ if, for all $x \in S_\alpha$ and all sequences of points $x_n \in S_{\alpha'}$ which converges to $x$ and for which the sequence of tangent spaces $T_{x_n}(f^{-1}(f(x)) \cap S_{\alpha'})$ converges to $T$, we have*

$$T \supset T_x(f^{-1}(f(x)) \cap S_\alpha).$$

**Remark 1.5** Notice that, since the map $f$ is stratified by $\mathcal{S}$ and $\mathcal{T}$, the restriction of $f$ to $S_{\alpha'}$ has a constant rank; so, the limit of the sequence of tangent spaces $T_{x_n}(f^{-1}(f(x)) \cap S_{\alpha'})$ is taken in some Grassmann space.

Notice that the Thom condition is not always satisfied. For instance, consider the blowing-up $\pi : E_{\{0\}}(\mathbb{C}^2) \to \mathbb{C}^2$ of the point $\{0\}$ in $\mathbb{C}^2$.

This complex analytic map is stratified with $S_1 = E_{\{0\}}(\mathbb{C}^2) \setminus \pi^{-1}(\{0\})$, $S_2 = \pi^{-1}(\{0\})$ and $T_1 = \mathbb{C}^2 \setminus \{0\}$, $T_2 = \{0\}$. The image of $S_1$ is $T_1$ and the image of $S_2$ is $T_2$. However, $(S_1, S_2)$ does not satisfy the Thom condition for the map $\pi$.

In fact, one can prove that $\pi$ above cannot be stratified by any Whitney stratification satisfying Thom's condition relatively to $\pi$ (exercise).

**Definition 1.6** *We say that the Whitney stratifications $\mathcal{S}$ and $\mathcal{T}$ satisfy the Thom condition for the map $f : X \to Y$ if all pairs of strata $(S_\alpha, S_{\alpha'})$ satisfy the Thom condition for the map $f$.*

*We say that the map $f : X \to Y$ is a Thom morphism if it is stratifiable and there exist Whitney stratifications $\mathcal{S}$ and $\mathcal{T}$ of $X$ and $Y$ which satisfy the Thom condition for the map $f$.*

An important result due to H. Hironaka (see [2] corollary 1 §5) is:

**Theorem 1.7** *A complex analytic function $f : X \to \mathbb{C}$ is locally a Thom morphism, i.e., for all $x \in X$, there is an open neighbourhood $\mathcal{U}_x$ of $x$ in $X$ and an neighbourhood $\mathcal{V}_x$ of $f(x)$ in $\mathbb{C}$, such that $f$ induces a Thom morphism from $\mathcal{U}_x$ into $\mathcal{V}_x$.*

## 1.2 Thom second isotopy lemma

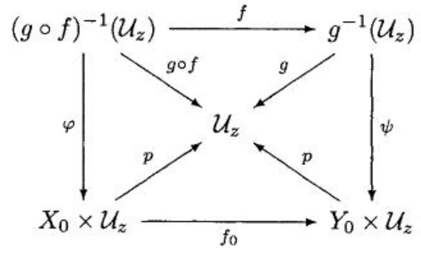
Let $f : X \to Y$ and $f : Y \to Z$ be complex analytic (resp. semi-analytic) maps. We assume that $Z$ is a complex analytic (resp., real analytic) manifold and that $f$ is a stratified map relative to the Whitney stratifications $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$, $\mathcal{T} = \{T_\beta\}_{\beta \in B}$.

**Definition 1.8** *We say that the map $f$ is a Thom morphism above the map $g$ iff*

1. *The maps $f$ and $g$ are proper;*
2. *For each $\beta \in B$, the restriction $g|T_\beta$ is a submersion;*
3. *For each $\alpha \in A$, the map $f$ induces a submersion $f_\alpha : S_\alpha \to T_{\beta(\alpha)}$;*
4. *The map $f$ is a Thom morphism.*

Now we can formulate Thom's second isotopy lemma (see [9]):

**Theorem 1.9** *Assume that the map $f$ is a Thom morphism above $g$. Then, for all $z \in Z$, there is an open neighbourhood $\mathcal{U}_z$ and homeomorphisms $\psi$ and $\varphi$, which preserve strata and are diffeomorphisms when restricted to strata, such that $g^{-1}(\mathcal{U}_z)$ is homeomorphic to $Y_0 \times \mathcal{U}_z$ by $\psi$ and $(g \circ f)^{-1}(\mathcal{U}_z)$ is homeomorphic to $X_0 \times \mathcal{U}_z$ by $\varphi$ and the following diagram is commutative*



*where $p$ is the projection on the second factor and $f_0$ is the morphism induced by $f$.*

## 2 Complex hypersurfaces with isolated singularities

In this section we shall show how one can study singularities in the particular case of complex hypersurfaces with isolated singularities.

Let $f : \mathcal{U} \to \mathbb{C}$ be a non-constant complex analytic function defined on an open connected neighbourhood of 0 in $\mathbb{C}^{n+1}$. We shall assume that $f(0) = 0$. Let $X \subset \mathcal{U} \subset \mathbb{C}^{n+1}$ be the complex hypersurface defined by $f$:

$$X := \{f = 0\}.$$

The first observation is that, by the Nullstellensatz, in the ring of convergent power series $\mathbb{C}\{z_0, \ldots, z_n\}$ at the point 0 of $\mathbb{C}^{n+1}$, the ideal of convergent power series which vanish on $X$ in a neighbourhood of 0 in $\mathbb{C}^{n+1}$ is principal and generated by a power series $g$ obtained from $f$ in the following way: the ring $\mathbb{C}\{z_0, \ldots, z_n\}$ being factorial, the power series of $f$ at 0 equals the product $f_1^{a_1} \ldots f_r^{a_r}$, where the $f_i$ is an irreducible power series and $a_i$ is an integer $\geq 1$, for $i = 1, \ldots, r$. Then $g = f_1 \ldots f_r$. We shall call $g$ the reduced equation of $X$. We shall still denote by $g$ the complex analytic function defined by the series $g$ in an open neighbourhood $\mathcal{V}$ of 0 in $\mathbb{C}^{n+1}$.

As we already saw, a critical point of $g$ is a point of $\mathcal{V}$ where the differential of $g$ vanishes. Equivalently a critical point of $g$ is a point of $\mathcal{V}$ where the partial derivatives of $g$ vanish:

$$\partial g/\partial z_0 = \ldots = \partial g/\partial z_n = 0.$$

By definition (see Definition 2.12) a singular point of the complex hypersurface $X$ is a point which is not smooth. Since a complex hypersurface is pure dimensional (see definition 2.15), a point of $X$ is smooth if and only if it is a critical point of $g$ (see [11] Remark p. 13). So, a singular point of $X$ is a point which satisfies the equations:

$$g = \partial g/\partial z_0 = \ldots = \partial g/\partial z_n = 0.$$

A critical value of $g$ is the value of $g$ at a critical point. By using the curve selection lemma (see lecture 2), we saw that if we choose $\mathcal{V}$ small enough, we may assume that 0 is the only critical value of $g$ (see Theorem 2.45). So, we have:

**Lemma 2.1** *If we choose $\mathcal{V}$ small enough, the set of singular points of $X$ in $\mathcal{V}$ coincides with the set of critical points of $g$ in $\mathcal{V}$.*

In this section, we only want to consider a hypersurface with isolated singularity.

**Definition 2.2** *A point $x \in X$ is an isolated singularity if there is an open neighbourhood $U$ of $x$ in $X$, such that $U \setminus \{x\}$ is non-singular.*

With this definition, in particular, a non-singular point will be considered as an isolated singularity.

Hilbert's Nullstellensatz (see [3] Corollaire 6 p.19-18) gives us:

**Proposition 2.3** *A point $x \in X$ is an isolated singularity if and only if the quotient $\mathbb{C}$-algebra*

$$\mathcal{O}_{\mathbb{C}^{n+1},x}/(g, \partial g/\partial z_0, \ldots, \partial g/\partial z_n)$$

*of the $\mathbb{C}$-algebra $\mathcal{O}_{\mathbb{C}^{n+1},x}$ of convergent power series of $\mathbb{C}^{n+1}$ at $x$ by the ideal generated by the Taylor expansions of $g$ and its partial derivatives at $x$ is a $\mathbb{C}$-vector space of finite complex dimension.*

Since it is equivalent that $x$ is a singular point of $X$ in $\mathcal{V}$ or a point of $X$ which is critical for $g$ in $\mathcal{V}$, we also have

**Proposition 2.4** *A point $x \in X$ is an isolated singularity if and only if the quotient $\mathbb{C}$-algebra*

$$\mathcal{O}_{\mathbb{C}^{n+1},x}/(\partial g/\partial z_0, \ldots, \partial g/\partial z_n)$$

*is a $\mathbb{C}$-vector space of finite complex dimension. This dimension is called the Milnor number of $X$ at $x$, or the Milnor multiplicity of $g$ at $x$.*

In his book [11], J. Milnor studies the topology of complex hypersurfaces with isolated singularities. We shall quickly recall Milnor's results.

In what follows, we shall assume that 0 is an isolated singular point of $X$.

A consequence of the curve selection lemma is that, for $\varepsilon$ small enough, the real sphere $S_\varepsilon(0)$ of $\mathbb{C}^{n+1}$ centered at the singular point 0 intersects transversally $X$, namely (see [11] corollary 2.9, see also definition 2.60):

**Lemma 2.5** *There is an $\varepsilon_0 > 0$ such that, for all $\varepsilon$ such that $\varepsilon_0 > \varepsilon > 0$, the sphere $S_\varepsilon(0)$ intersects transversally $X$ in $\mathbb{C}^{n+1}$ and the diffeomorphism type of the pair $(S_\varepsilon(0), S_\varepsilon(0) \cap X)$ does not depend on $\varepsilon$. The embedded manifold $S_\varepsilon(0) \cap X$ is called the local real link of $X$ at 0.*

The real link of $X$ at an isolated singular point 0 has particular properties.

In the case the point 0 is non-singular, this local real link is a real codimension 2 standard sphere embedded trivially in the $(2n+1)$-sphere $S_\varepsilon(0)$.

In general J. Milnor proved (cf [11] theorem 5.2):

**Proposition 2.6** *Let $\varepsilon_0 > \varepsilon > 0$. The local real link $S_\varepsilon(0) \cap X$ is $(n-2)$-connected.*

When $n = 1$ (plane curve case), the local real link is a disjoint union of embedded circles which define a link in the 3-sphere.

When $n = 2$ (surface case), the local real link is connected. A consequence of a theorem of D. Mumford (see [12]) is that the local real link is simply connected if and only if the point 0 is non-singular in $X$, because a hypersurface of dimension $\geq 2$ with isolated singularity is normal.

When $n \geq 3$, in particular, the local real link is simply-connected. So it is homeomorphic to the standard $(2n-1)$-sphere if and only if it has the homology of a $(2n-1)$-sphere.

The embedding of $S_\varepsilon(0) \cap X$ into $S_\varepsilon(0)$ has the following remarkable property (see [11] theorem 4.8):

**Theorem 2.7** *There is $\varepsilon_1$, such that, for all $\varepsilon$ such that $\varepsilon_1 > \varepsilon > 0$, the quotient $f/|f|$ defines a locally trivial smooth fibration $\varphi_\varepsilon$ of the complement $S_\varepsilon(0) \setminus S_\varepsilon(0) \cap X$ onto the circle $\mathbb{S}^1$. For $\varepsilon_1 > \varepsilon > 0$, all the fibrations $\varphi_\varepsilon$ are diffeomorphic.*

This fibration is called the Milnor fibration of $f$ at 0. We shall call *Milnor spheres of $f$ at* 0 the spheres $\mathbb{S}_\varepsilon(0)$ such that $\varepsilon_1 > \varepsilon > 0$.

In fact, since, for $\varepsilon_0 > \varepsilon > 0$, the sphere $S_\varepsilon(0)$ intersects transversally $X$ in $\mathbb{C}^{n+1}$, there is $\eta_\varepsilon > 0$ such that for $\eta_\varepsilon > |t| > 0$, the hypersurface $X_t := \{f = t\}$ intersects $S_\varepsilon(0)$ transversally $X$ in $\mathbb{C}^{n+1}$.

An immediate consequence of Ehresmann's fibration lemma on manifolds with boundary or Thom's first isotopy lemma implies that

**Proposition 2.8** *Let $\varepsilon_0 > \varepsilon > 0$, $\eta_\varepsilon > \eta > 0$, $\overset{\circ}{B}_\varepsilon(0)$ be the open ball of $\mathbb{C}^{n+1}$ centered at 0 with radius $\varepsilon$ and $\partial D_\eta$ be the boundary of the closed disc $D_\eta$ of $\mathbb{C}$ centered at 0 with radius $\eta$. The complex function $f$ induces a locally trivial smooth fibration $\varphi_{\varepsilon,\eta}$ of $\overset{\circ}{B}_\varepsilon(0) \cap f^{-1}(\partial D_\eta)$ onto $\partial D_\eta$.*

In [11] (theorem 5.11), J. Milnor showed that, for $\min(\varepsilon_0, \varepsilon_1) > \varepsilon > 0$ and $\eta_\varepsilon > \eta > 0$, the space $S_\varepsilon(0) \setminus S_\varepsilon(0) \cap X$ is diffeomorphic to $\mathring{B}_\varepsilon(0) \cap f^{-1}(\partial D_\eta)$ by a diffeomorphism which sends a fiber of $\varphi_\varepsilon$ onto a fiber of $\varphi_{\varepsilon,\eta}$. This implies that the fibrations $\varphi_\varepsilon$ and $\varphi_{\varepsilon,\eta}$ are diffeomorphic. This is why we also call $\varphi_{\varepsilon,\eta}$ a Milnor fibration of $f$ at 0.

In the case $f$ has an isolated critical point at 0, J. Milnor obtained the following remarkable result:

**Theorem 2.9** *The fibers of the Milnor fibration of $f$ at 0 have the homotopy type of a join of $\mu$ $n$-spheres, where $\mu$ is the Milnor multiplicity of $f$ at 0.*

**Proof.** We shall give the proof given in [6]. Let $a_i$ be complex numbers, $0 \leq i \leq n$. We consider the deformation $F$ given by:

$$f_t := f + t \sum_{0 \leq i \leq n} a_i z_i,$$

i.e., $F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ with $F(z,t) = (f_t(z), t)$. In [11] (see Remark of p.113), J. Milnor proved that for almost all $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ in a small neighbourhood of 0 in $\mathbb{C}^{n+1}$, there is $\tau > 0$ and $\varepsilon > 0$ such that the functions $f_t$, for $|t| < \tau$ have only non-degenerate isolated critical points in the open ball $\mathring{B}_\varepsilon(0)$. In $\mathring{B}_\varepsilon(0)$, the function $f_0$ has only the critical point 0 and the functions $f_t$, for $t \neq 0$, have $\mu'$ critical points, where $\mu'$ is the local degree of the finite germ of the map

$$\Phi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$$

defined by $\Phi(z_0, \ldots, z_n) = (\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$. In [13] Palamodov proved that the degree of such finite maps equals the complex dimension of the quotient:

$$\mathcal{O}_{\mathbb{C}^{n+1},x}/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n).$$

So that the degree $\mu'$ equals the Milnor number of $f$ at 0. Now, with $\varepsilon_1$ defined in Theorem 2.7, we may suppose that $\varepsilon_1 > \varepsilon > 0$ and, for $\eta_\varepsilon > \eta > 0$, there is $\tau > 0$ such that, for $\tau > |t|$, the sphere $\mathbb{S}^{2n+1}$ intersects transversally $f_t^{-1}(\partial D_\eta)$.

A version of Ehresmann's fibration lemma for manifolds with boundary and corners or, equivalently, Thom's first isotopy lemma gives that the spaces $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$ are homeomorphic for $\tau > |t|$ (exercise: give a complete proof of this assertion).

Since $B_\varepsilon(0) \cap f_0^{-1}(D_\eta) = B_\varepsilon(0) \cap f^{-1}(D_\eta)$ retracts on $B_\varepsilon(0) \cap f_0^{-1}(0)$ which is a cone by lemma 2.5, it is contractible, so that $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$ is also contractible for $t \neq 0$.

On the other hand, consider the map

$$F_0 : (\mathring{B}_\varepsilon(0) \times D_\tau(0)) \cap F^{-1}(D_\eta \times D_\tau) \to D_\eta \times D_\tau$$

induced by $F$, where $F(z_0, \ldots, z_n, t) = (f_t(z_0, \ldots, z_n), t)$. Since the fiber of $F_0$ over 0 has a singularity only at 0, the restriction of $F_0$ to its critical locus $C(F_0)$ is a finite map, so the image $F_0(C(F_0))$ is a complex curve and, by Ehresmann's fibration lemma, $F_0$ induced a locally trivial fibration over $D_\eta \times D_\tau \setminus F_0(C(F_0))$. This implies that the Milnor fiber of $f$ at 0 is isomorphic to $f_t^{-1}(a,t)$ when $(a,t) \in D_\eta \times D_\tau \setminus F_0(C(F_0))$.

Let us fix $t$ such that $\tau > |t| > 0$. We have a map $\tilde{f}_t : B_\varepsilon(0) \cap f_t^{-1}(D_\eta) \to D_\eta$ induced by $f_t$. Fix $a \in D_\eta$, such that $(a,t) \in D_\eta \times D_\tau \setminus F_0(C(F_0))$. We consider the real smooth function

$$\psi := \frac{(|\tilde{f}_t - a|^2 - r^2)}{(|\tilde{f}_t - a|^2 - r^2) - (|f|^2 - \eta^2)}$$

on $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$. Notice that $\psi^{-1}(0) = B_\varepsilon(0) \cap f_t^{-1}(D_r(a))$ and $\psi^{-1}(1) = B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$.

Let us use Morse Theory (see [10]). We can prove that the critical points of $\psi$ in

$$B_\varepsilon(0) \cap f_t^{-1}(D_\eta) \setminus B_\varepsilon(0) \cap f_t^{-1}(\overset{\circ}{D_r}(a))$$

coincide with the critical points of $f$ inside this domain. Since the complex critical points of $f$ are quadratic non-degenerate, the corresponding critical points of $\psi$ are Morse critical points. One can also check that the index of these Morse points equals $n+1$. Moreover, the restriction of $\psi$ to the boundary of $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$ has no critical point because the fibers of $f$ over $D_\eta$ are transverse to the sphere $\mathbb{S}_\varepsilon(0)$. Morse Theory implies that $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$ is obtained from $B_\varepsilon(0) \cap f_t^{-1}(D_r(a))$ by adding cells of real dimension $n+1$ for each of the critical point of $\psi$ in the difference set

$$B_\varepsilon(0) \cap f_t^{-1}(D_\eta) \setminus B_\varepsilon(0) \cap f_t^{-1}(\overset{\circ}{D_r}(a)).$$

Since we have $\mu$ critical points and that the space $B_\varepsilon(0) \cap f_t^{-1}(D_\eta)$ is homeomorphic to a ball, using Morse theory (see [10]) one can deduce that $B_\varepsilon(0) \cap f_t^{-1}(D_r(a))$ has the homotopy type of a bouquet of $\mu$ $n$-spheres.

The space $B_\varepsilon(0) \cap f_t^{-1}(D_r(a))$ is diffeomorphic to the product $D_r(a) \times f_t^{-1}(a)$, so that the Milnor fiber of $f$ at 0 has the homotopy type of a bouquet of $\mu$ $n$-spheres. □

# 3 Hypersurfaces with non-isolated singularities.

In the case of hypersurfaces with non-isolated singularities, there appears a technical difficulty to obtain results similar to the case of isolated singularities. It comes essentially from the fact that if 0 is a non-isolated critical point of $f$, every sphere centered at 0 will intersect the singular locus of the hypersurface $f=0$.

We saw in the preceding section that, in a small neighbourhood of 0 in $\mathbb{C}^{n+1}$, the set of critical points of $f$ coincide with the singular locus of the hypersurface $f=0$. This comes from theorem 2.45 (see also [11] corollary 2.6). Using the definition of semianalytic sets (complex and real) introduced in lecture 2 (definition 2.32), we have:

**Proposition 3.1** *Let $E$ be a smooth connected $\mathbb{K}$-semianalytic subset of a $\mathbb{K}$-analytic manifold, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $\varphi : E \to \mathbb{K}$ be the restriction to $E$ of a $\mathbb{K}$-analytic map. Let $x \in E$. There is an open neighbourhood $U_x$ of $x$ in $E$ such that the restriction of $\varphi$ to $U_x$ has no critical value in $\mathbb{K} \setminus \{\varphi(x)\}$.*

**Proof.** The proof proceeds nearly as the proof of J. Milnor.

First notice that the set $C(\varphi)$ of critical points of $\varphi$ is a disjoint union

$$C(\varphi) = \coprod_{i \in I} M_i$$

of connected smooth manifolds $M_i$ so that the partition $(M_i)_{i\in I}$ is locally finite. One may consider the filtration $F_0 \subset F_1 \subset \ldots$, where $F_0$ is $C(\varphi)$, $F_1$ the singular locus of $C(\varphi)$, and, for $i \geq 1$, $F_{i+1}$ is the singular locus of $F_i$. By Hilbert's Basis theorem, this is a locally finite descending sequence of analytic spaces and the connected components of $F_i \setminus F_{i-1}$, for $i \geq 0$, give such a partition of $C(\varphi)$.

For each $M_i$, the critical locus of the restriction $\varphi_{|M_i}$ of $\varphi$ to $M_i$ is all of $M_i$, hence, by the curve selection lemma (Lemma 2.36), $\varphi_{|M_i}$ is constant. Now, let $U_x$ be small enough so that the indices $i$ such that $M_i \cap U_x \neq \emptyset$ form a finite set $I_x$, $U_x \cap (\cup_{i\in I_x} M_i)$ is connected and $U_x \cap C(\varphi) = U_x \cap (\cup_{i \in I_x} M_i)$. Since the value of $\varphi$ on $\cup_{i\in I_x} M_i$ is $\varphi(x)$, we have $\varphi^{-1}(t) \cap C(\varphi) \cap U_x = \emptyset$, for $t \neq \varphi(x)$ which means that $t$ is not a critical value of $\varphi|U_x$.□

In particular this proposition shows that, for every $\mathbb{K}$-analytic function $g$ on an open neighbourhood of 0 in $\mathbb{K}^N$, for $\varepsilon > 0$ small enough, the hypersurfaces $\overset{\circ}{\mathbb{B}}_\varepsilon(0) \cap \{g = t\}$ are non-singular for $t \neq g(0)$.

Another way to apply the above Bertini-Sard type result is to consider the restriction of the square of the distance function to a point 0 to the non-singular part $X^0$ of the complex hypersurface $X := \{f = 0\}$. Then, for $\varepsilon > 0$ small enough, the spheres $\mathbb{S}_\varepsilon(0)$ intersect $X^0$ transversally in $\mathbb{C}^{n+1}$.

In the case of an isolated singularity we already saw that small spheres centered at the singularity intersect transversally the hypersurface. This defines the local link of the singularity.

In the case the singularity at 0 is not isolated, all we can say for the moment is that small spheres centered at the singular point 0 intersect transversally only the non-singular part of the hypersurface.

In fact we shall see that in this case of non-isolated singularities, small spheres intersect the singular locus in a rather nice way.

We have:

**Theorem 3.2** *Let $f : U \to \mathbb{C}$ be a complex analytic function defined on an open neighbourhood of 0 in $\mathbb{C}^{n+1}$. Assume that $f(0) = 0$. There is $\epsilon_0 > 0$ such that, for any $\varepsilon$, $\epsilon_0 > \varepsilon > 0$, there is $\eta(\varepsilon)$, such that $f = t$ intersects the sphere $\mathbb{S}_\varepsilon(0)$ transversally for any $\eta(\varepsilon) > |t| > 0$.*

The preceding theorem shows that, even if $\mathbb{S}_\varepsilon(0)$ intersect the singular locus of $f = 0$, it intersects transversally the hypersurfaces $f = t$ for $t \neq 0$ small enough.

In particular it gives immediately the following corollary:

**Corollary 3.3** *Let $\varepsilon$ such that $\epsilon_0 > \varepsilon > 0$. For all $\eta > 0$, $\eta(\varepsilon) > \eta > 0$, the function $f$ induces a locally trivial fibration $\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(0) \cap f^{-1}(\partial D_\eta(0)) \to \partial D_\eta(0)$ on the boundary $\partial D_\eta(0)$ of the disc $D_\eta(0)$ of $\mathbb{C}$ centered at 0 with radius $\eta$.*

The proof of the corollary is another application of Ehresmann's fibration lemma (or Thom's first isotopy lemma (Theorem 2.58)).

Theorem 3.2 is a consequence of Hironaka's result (theorem 1.6). For the convenience of the reader, we shall prove Hironaka's theorem in the case of complex hypersurfaces (see [1] theorem 1.2.1 due to F. Pham):

**Lemma 3.4** *There is an open neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}^{n+1}$ such that there is a Thom stratification of the restriction of $f$ to $\mathcal{U} \cap f^{-1}(\mathring{D}_\eta)$.*

**Proof.** We shall construct a stratification of $\{f = 0\}$ in an open neighbourhood of 0 in $\mathbb{C}^{n+1}$. Adding the complement of the hypersurface, as stratum, to this stratification, will give a stratification of this neighbourhood which will satisfy Thom's condition relative to $f$

Let us introduce another coordinate $u$ and consider the hypersurface $F := f + u^N = 0$ in $\mathbb{C}^{n+1} \times \mathbb{C}$. Note that the gradient of $F$ is given by

$$grad(F(z,u)) = (grad(f(z)), N\overline{u}^{N-1}).$$

In a small neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}^{n+1}$, there is a real number $\theta$, $1 > \theta > 0$, such that we have the Lojasiewicz inequality (see [15]):

$$|f(z)|^\theta \leq \|grad(f(z))\|$$

for any $z \in \mathcal{U}$.

Let $(z, u) \in \{F = 0\}$. We have $|u|^{N-1} = |f|^{\frac{N-1}{N}}$. In addition, for $z$ small enough, say $\|z\| < \varepsilon_1$, we have

$$\|grad(f(z))\| \geq |f|^\theta = |u|^{N\theta}.$$

Therefore, we have for $z$, $\|z\| < \varepsilon_1$

$$N\|grad(f(z))\|^{\frac{(N-1)}{N\theta}} \geq N|u|^{N-1}.$$

which yields

$$\frac{N\|grad(f(z))\|^{\frac{(N-1)}{N\theta}}}{\|grad(f(z))\|} \geq \frac{N|u|^{N-1}}{\|grad(f(z))\|}$$

If one choose $N$ big enough, such that $1 > (N-1)/N > \theta$, we have

$$\frac{(N-1)}{N\theta} > 1,$$

so that for any sequence $(z_n, u_n)$ of points of $F = 0$ which tends to a singular point of $F = 0$ where $\|z\| < \varepsilon_1$, we have that the limit of directions of the gradient vectors of $F$ at the points of this sequence is contained in $\mathbb{C}^{n+1} \times \{0\}$.

Now, one can easily deduce from this geometrical result that a Whitney stratification of $F = 0$ adapted to the subspace $\{f = 0\}$ induces a Whitney stratification of $f$ which satisfies the Thom condition for $f = 0$. In fact, one can notice that one only needs $(a)$-Whitney property to obtain a Thom stratification. □

In fact, Parusiński proved in [14] that, for a complex hypersurface $\{f = 0\}$, a Whitney stratification already has Thom's property for $f$.

From Lemma 3.4, one can prove Theorem 3.2. Let $\varepsilon > 0$ small enough be such that $\mathbb{S}_\varepsilon(0)$ intersect transversally all the strata of a Thom stratification of $\mathcal{U}$ relative to $f$ and adapted to $\{0\}$. For $\eta > 0$ small enough, $\{f = t\}$ is transverse to $\mathbb{S}_\varepsilon(0)$, for $0 < |t| \leq \eta$, otherwise we could construct a sequence of points $x_n \in f^{-1}(t_n) \cap \mathbb{S}_\varepsilon(0)$ where $f^{-1}(t_n)$ does not intersect transversally $\mathbb{S}_\varepsilon(0)$ and $\lim_n t_n = 0$, which contradicts the fact that $\mathbb{S}_\varepsilon(0)$ intersects transversally the strata of the Thom stratification of $\mathcal{U}$.

In [11], J. Milnor also proved the existence of a local fibration even in the case of hypersurface with non-isolated singularities. In fact theorem 2.7 is true in this case:

**Theorem 3.5** *There is $\varepsilon_1 > 0$, such that, for all $\varepsilon$ such that $\varepsilon_1 > \varepsilon > 0$, the quotient $f/|f|$ defines a locally trivial smooth fibration $\varphi_\varepsilon$ of the complement $S_\varepsilon(0) \setminus S_\varepsilon(0) \cap X$ onto the circle $\mathbb{S}^1$. For $\varepsilon_1 > \varepsilon > 0$, all the fibrations $\varphi_\varepsilon$ are diffeomorphic.*

Once it is known that theorem 3.2 is also true, one can easily deduce that the fibrations of 2.7 and of 3.2 are isomorphic by using the result of J. Milnor in [11] (Theorem 5.11) which shows that their fibers are diffeomorphic. But notice that we need first to prove 3.2 to draw this conclusion.

Using the existence of Whitney stratifications, one can also extend the conic structure theorem to non-isolated singularities. We have (see Theorem 2.55):

**Theorem 3.6** *Let $(X, x)$ a germ of an analytic space embedded in $(\mathbb{C}^N, x)$. Let $X \subset U$ be a representative of this germ embedded in an open neighbourhood of $x$ in $\mathbb{C}^N$. There is $\varepsilon(0) > 0$, such that, for all $\varepsilon$ such that $\varepsilon(0) > \varepsilon > 0$, the pair $(\mathbb{B}_\varepsilon(x), \mathbb{B}_\varepsilon(x) \cap X)$ is homeomorphic to the cone over $(\mathbb{S}_\varepsilon(x), \mathbb{S}_\varepsilon(x) \cap X)$.*

**Proof.** Let $(X_i)_{i \in I}$ be a Whitney stratification of $X$. Let $V$ be a sufficiently small neighbourhood of $x$ in $\mathbb{C}^N$. There is a finite number of indices $i$, say $i \in I_x$, such that $X_i \cap V \neq \emptyset$. The Sard-Bertini type lemma shown above implies that, there is $\varepsilon(0) > 0$, such that, for all $\varepsilon$ such that $\varepsilon(0) > \varepsilon > 0$, the sphere $\mathbb{S}_\varepsilon(x)$ is tranverse to $X_i$, for $i \in I_x$.

The square of the distance to $x$ in $\mathbb{C}^N$ defines a smooth map onto $\mathbb{R}$. The restriction of this map to $V$ induces a proper map $h$ of $\mathring{\mathbb{B}}_\varepsilon(x) \setminus \{x\}$ onto $(0, \varepsilon(0)^2)$. The restriction of $h$ to $(\mathring{\mathbb{B}}_\varepsilon(x) \setminus \{x\}) \cap X$ satisfies the hypotheses of Thom's first isotopy lemma. This shows that the pair $(\mathring{\mathbb{B}}_\varepsilon(x) \setminus \{x\}, (\mathring{\mathbb{B}}_\varepsilon(x) \setminus \{x\}) \cap X)$ is homeomorphic to the product of the pair $(\mathbb{S}_\varepsilon(x), \mathbb{S}_\varepsilon(x) \cap X)$ by the interval $(0, \varepsilon]$. This implies our lemma. □

Recall that, for $\varepsilon$ small enough, $\mathbb{S}_\varepsilon(x) \cap X$ is the local real link of $X$ at $x$. In the general case, it is not necessarily a manifold.

# 4 Complex analytic functions on complex analytic spaces.

In this paragraph, we shall consider a complex analytic function $f : X \to \mathbb{C}$ defined on an complex analytic space $X$. We shall assume that $0 \in X$ and $f(0) = 0$. By using the existence of Whitney stratifications adapted to $f^{-1}(0)$ and which satisfies Thom's condition relative to $f$, one can prove similar results as for functions in $\mathbb{C}^{n+1}$.

For instance, by using Thom's first isotopy lemma, we have a local fibration theorem (cf. [7]):

**Theorem 4.1** *Assume that the complex analytic space is closed in an open neighbourhood of $0 \in X$ in $\mathbb{C}^N$. There is $\varepsilon_1 > 0$, such that, for all $\varepsilon$ such that $\varepsilon_1 > \varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that, for all $\eta$ such that $\eta(\varepsilon) > \eta > 0$, we have a locally trivial topological fibration*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(0) \cap X \cap f^{-1}(\partial D_\eta(0)) \to \partial D_\eta(0).$$

*Furthermore all of these topological fibrations are isomorphic as topological fibrations.*

One can also prove the following local fibration:

**Theorem 4.2** *There is $\varepsilon_1$, such that, for all $\varepsilon$ such that $\varepsilon_1 > \varepsilon > 0$, the quotient $f/|f|$ defines a locally trivial smooth fibration $\varphi_\varepsilon$ of the complement $S_\varepsilon(0) \cap X \setminus (S_\varepsilon(0) \cap X \cap f^{-1}(0))$ onto the circle $\mathbb{S}^1$. For $\varepsilon_1 > \varepsilon > 0$, all the fibrations $\varphi_\varepsilon$ are fiber isomorphic to each other and are isomorphic to the fibration $\varphi_{\varepsilon,\eta}$ defined above.*

We shall call the Milnor fibration of $f$ at 0 the fibration defined in 4.1 and its general fiber is called the Milnor fiber of $f$ at 0.

Using Goresky-MacPherson stratified Morse Theory, one can obtain results on the general fiber of these fibrations. However these results involve thé notion of complex link of a Whitney stratum. We shall define this notion in the following lecture.

First, we recall the notion of isolated singularity relative to a Whitney stratification (see Definition 2.4.4.):

**Definition 4.3** *Let $f : X \to \mathbb{C}$ a complex analytic function defined on an complex analytic space $X$. Let $\mathcal{S} = (X_i)_{i \in I}$ be a Whitney stratification of $X$. We say that $f$ has an isolated singularity at the point $0 \in X$ relative to $\mathcal{S}$ if the restriction to each strata which contain 0 in their closure has rank one except maybe at 0.*

In [4], we prove that:

**Theorem 4.4** *The complex links of the strata of a Whitney stratification of a local complete intersection have the homotopy type of joins of spheres of real dimension equal to the complex dimension of the complex link.*

As a consequence, one can prove (see [5]):

**Theorem 4.5** *Let $f : X \to \mathbb{C}$ be a complex analytic function defined on a local complete intersection $X$. Assume that $f$ has an isolated singularity at a point $0 \in X$ relative to a Whitney stratification of $X$. Then, the Milnor fiber of $f$ at 0 has the homotopy type of a join of spheres of dimension $n = \dim X - 1$.*

However, in general complex links of Whitney strata might not be bouquet of spheres of the dimension of the complex link, so that on a singular space the homotopy of the fiber might not be so simple (e.g. see [16]).

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