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Notes on Singularities

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1 Lecture 1

Manifolds and Local, Ambient, Topological-type

We assume that the reader is familiar with the notion of a manifold, but we want to clarify the different types of manifolds that one can discuss.

Throughout these notes, a *topological manifold* is a non-empty, Hausdorff, second-countable, topological space M such that, for all $x \in M$, there exists an open neighborhood $\mathcal{U}_x$ of x in M, a natural number n_x, an open subset $\mathcal{V}_x$ of $\mathbb{R}^{n_x}$, and a homeomorphism $h_x : \mathcal{U}_x \to \mathcal{V}_x$. Such an h_x (or, sometimes, $\mathcal{U}_x$ itself) is called a *coordinate chart (or patch) around x*. The number n_x is uniquely determined and is called the *dimension of M at x*. If n_x is independent of the point $x \in M$, we say that M is *pure-dimensional*; if we call the common value n, then we say that *M is (purely) n-dimensional* or that *M is an n-manifold*, and write $\dim M = n$. If M is a connected manifold, then M is pure-dimensional.

Suppose that M is a topological manifold. If we have two coordinate charts $h_x : \mathcal{U}_x \to \mathcal{V}_x$ and $h_y : \mathcal{U}_y \to \mathcal{V}_y$ such that $\mathcal{U}_x \cap \mathcal{U}_y \neq \emptyset$, then we have the *transition function* $t_{xy} : h_x(\mathcal{U}_x \cap \mathcal{U}_y) \to h_y(\mathcal{U}_x \cap \mathcal{U}_y)$ given by $t_{xy}(v) := h_y(h_x^{-1}(v))$. Note that $t_{yx} = t_{xy}^{-1}$. If it is possible to select all of the coordinate charts so that all of the transition functions are continuously differentiable (or, smooth), then M, together with the collection (an *atlas*) of charts, is called a *smooth manifold*.

One has various notions of smooth manifolds, which correspond to exactly what one means by "continuously differentiable". If the transition functions are all C^r (continuously r times differentiable), then we have a C^r manifold. A C^1 manifold would be the weakest notion that one could use for a "smooth manifold". The case of topological manifolds is the C^0 case. We also have C^∞ (infinitely differentiable) manifolds, and C^ω (real analytic) manifolds. (Recall, that *analytic*, over the real or complex numbers, means a function which can be represented by a convergent power series in a neighborhood of each point.) We refer to all of these cases by simply writing C^r, and allowing r to have the values $0, 1, 2, 3, \dots, \infty, \omega$. Note that in all of these cases, since $t_{yx} = t_{xy}^{-1}$, all of the transition functions are invertible and the inverses are required to be equally as "smooth", i.e., the transition functions are homeomorphisms (in the C^0 case), C^r diffeomorphisms, C^∞ diffeomorphisms, or real analytic isomorphisms; we refer to all of these as *C^r isomorphisms*.

One obtains the notion of a *complex (analytic) manifold* by replacing the open sets $\mathcal{V}_x \subseteq \mathbb{R}^{n_x}$ in the discussion above by open subsets $\mathcal{V}_x \subseteq \mathbb{C}^{n_x}$, and using an atlas for which the transition functions are complex analytic, .i.e., holomorphic. The number n_x here is usually referred to as simply the *dimension of M at x*, since it is usually obvious that we mean the dimension over the complex numbers. However, occasionally, it is necessary to explicitly use the terminology *complex dimension* for this n_x to distinguish it from the *real dimension* of the complex manifold, which would be $2n_x$.

Once one has the notions of C^r and complex analytic manifolds, it is not difficult to define the analogous *morphisms* between two manifolds of the same type; that is, to define C^r functions between two C^r manifolds and complex analytic functions between two complex manifolds. The reader is referred to [Spi70] for a complete treatment. When we say that we have a C^r or complex analytic function between two manifolds, we mean that the two manifolds are also of the corresponding type (i.e., in the appropriate category).

Suppose that N is a smooth n-dimensional manifold. A subset $M \subseteq N$ is a C^r *submanifold of* N if and only if, for all $x \in M$, there exists an open neighborhood $\mathcal{U}$ of x in N, an open neighborhood $\mathcal{V}$ of $\mathbf{0}$ in $\mathbb{R}^n$, a natural number m_x, and a C^r isomorphism $f : \mathcal{U} \to \mathcal{V}$ such that $f(x) = \mathbf{0}$, and $f(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^{m_x} \times \{\mathbf{0}\})$. A C^r submanifold inherits a C^r atlas from the manifold in which it sits, and so C^r submanifolds are C^r manifolds with a prescribed atlas. When discussing submanifolds, the space in which the submanifold sits is frequently referred to as the *ambient space*. Note that the m_x here is the dimension of the manifold M at x. If M is a pure-dimensional submanifold of the pure-dimensional manifold N, then *the codimension of* M *in* N *is* $\dim N - \dim M$.

We shall also be interested in complex submanifolds of a complex n-dimensional manifold N. To obtain the definition of a complex submanifold, one simply takes the definition above, replaces $\mathbb{R}$ with $\mathbb{C}$, and replaces "C^r isomorphism" with complex analytic isomorphism.

Note that when one is dealing with purely topological questions, there is no need to distinguish between the real and complex cases, since $\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$.

In general, we are interested in the topology of how various subsets X are embedded in $\mathbb{R}^n$. If $x \in X \subseteq \mathbb{R}^n$, then the *local, ambient, topological type* of X in $\mathbb{R}^n$ at x is determined by the homeomorphism-type of triples of the form $(\mathcal{W}, \mathcal{W} \cap X, x)$, where $\mathcal{W}$ is an open neighborhood of x in $\mathbb{R}^n$. This means that if $x \in X \subseteq \mathbb{R}^n$ and $y \in Y \subseteq \mathbb{R}^n$, then we would say that the local, ambient, topological-type of X at x is the same as the local, ambient, topological-type of Y at y if and only if there exist open neighborhoods $\mathcal{W}_x$ and $\mathcal{W}_y$ in $\mathbb{R}^n$ around x and y, respectively, and a homeomorphism $g : \mathcal{W}_x \to \mathcal{W}_y$ such that $g(\mathcal{W}_x \cap X) = \mathcal{W}_y \cap Y$ and $g(x) = y$.

For a topological submanifold of $\mathbb{R}^n$, there is no interesting local, ambient topology – the local, ambient topological-type at each point x is prescribed to be the same as that of $\mathbb{R}^{m_x} \times \{\mathbf{0}\}$ inside $\mathbb{R}^n$ at the origin; we refer to such topological-types as *Euclidean* or *trivial*.

The Geometry of the Implicit Function Theorem

We all encountered 1-manifolds in high school. We studied lines, parabolas, circles, ellipses, and hyperbolas. We encountered 2-manifolds in multivariable Calculus, where we saw spheres, ellipsoids, paraboloids, and hyperboloids. Actually, all of these examples are examples of smooth submanifolds of Euclidean space.

One may wonder why equations with such forms as $y = x^2$, $x^2 + y^2 = 4$, $xy = 1$, and $3x^2 + 2y^2 + 5z^2 = 1$ should define smooth (actually, real analytic) submanifolds of $\mathbb{R}^2$ or $\mathbb{R}^3$. In fact this follows from the *Implicit Function Theorem* (see, for instance, [Rud53], Theorem 9.28 and [Gri78], p. 19), which we state in a geometric form.

Theorem 1.1 (Implicit Function Theorem: geometric form) *Let $r \geq 1$ and let f be a C^r function from an $(m + c)$-dimensional manifold N to a c-dimensional manifold P. Suppose that the rank of the derivative, $d_x f$, of f at a point $x \in N$ is equal to c, i.e., $d_x f$ is surjective.*

Then, there exist open subsets $\mathcal{U} \subseteq N$, $\mathcal{V} \subseteq P$, $\mathcal{V}' \subseteq \mathbb{R}^c$, and $\mathcal{W}' \subseteq \mathbb{R}^m$ such that $x \in \mathcal{U}$, $f(\mathcal{U}) = \mathcal{V}$, and such that there exist C^r isomorphisms $\phi : \mathcal{W}' \times \mathcal{V}' \to \mathcal{U}$ and $\psi : \mathcal{V} \to \mathcal{V}'$ such that $\psi \circ f_{|_\mathcal{U}} \circ \phi : \mathcal{W}' \times \mathcal{V}' \to \mathcal{V}'$ is equal to the projection onto $\mathcal{V}'$.

In addition, the complex analytic version of the above statement is also true.

The following corollary is immediate.

Corollary 1.2 *Let $r \geq 1$ and let f be a C^r function from an $(m+c)$-dimensional manifold N to a c-dimensional manifold P. Suppose that $d_x f$ is surjective.*

Then, there exists an open neighborhood $\mathcal{U}$ of x in N such that:

1. *$f_{|\mathcal{U}} : \mathcal{U} \to P$ is an open map;*
2. *for all $y \in \mathcal{U}$, $d_y f$ is surjective; and*
3. *for every C^r submanifold Q of codimension c' in (the open set) $f(\mathcal{U})$, $\mathcal{U} \cap f^{-1}(Q)$ is a C^r submanifold of $\mathcal{U}$ of codimension c'; in particular, $\mathcal{U} \cap f^{-1}(f(x))$ is a C^r submanifold of $\mathcal{U}$ of codimension c.*

In addition, the complex analytic versions of the above statements are also true.

Example 1.3 Let us see how the Implicit Function Theorem or, rather, its corollary implies that the set, M, of points in $\mathbb{R}^2$ which satisfy $x^2 + y^2 = 4$ form a 1-dimensional C^ω submanifold of $\mathbb{R}^2$. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by $f(x, y) = x^2 + y^2 - 4$. Then, f is a real analytic function, and the rank of $d_{(a,b)} f$ will be equal to 1 precisely if one of the partial derivatives of f at (a, b) is non-zero.

As the partial derivatives of f are $2x$ and $2y$, we see that $d_{(a,b)} f$ has rank equal to 1 everywhere except at $(0, 0)$. However, $(0, 0)$ is certainly not in M. Thus, Corollary 1.2 tells us that, for every point $(a, b) \in M$, there exists an open neighborhood $\mathcal{U}'$ of (a, b) in $\mathbb{R}^2$ such that $\mathcal{U} \cap f^{-1}(f(a, b)) = \mathcal{U} \cap M$ is a C^ω 1-submanifold of $\mathbb{R}^2$. Therefore, M is a C^ω 1-submanifold of $\mathbb{R}^2$.

For the remainder of this lecture, suppose that we have $f : N \to P$, a map between pure-dimensional manifolds, that f is a C^r or complex analytic function, where $r \geq 1$, and that $\dim N \geq \dim P$.

Corollary 1.2 tells us that if $x \in N$ is such that $d_x f$ is surjective, then the local, ambient, topological-type of $f^{-1}(f(x))$ at x is Euclidean. Therefore, we can obtain non-Euclidean – that is, **interesting** – local topological-types in $f^{-1}(f(x))$ only at points where the derivative is **not** surjective.

Hence, we make some definitions.

Definition 1.4 *A point $x \in N$ such that $d_x f$ is surjective is called a* **regular point of** *f. A point $x \in N$ at which $d_x f$ is* **not** *surjective is called a* **critical point of** *f. The set of critical points of f is denoted Σf.*

The **set of critical values of** *f is $f(\Sigma f)$. The* **set of regular values of** *f is $P - f(\Sigma f)$. If $\Sigma f = \emptyset$, then f is called a* **submersion**.

Note that Item 2 of Corollary 1.2 implies that Σf is a closed subset of N.

Using the terminology of Definition 1.4, and arguing as in Example 1.3, we immediately obtain the following corollary to the Implicit Function Theorem.

Corollary 1.5 *Suppose $v \in f(N)$ is a regular value of f. Then, $f^{-1}(v)$ is a C^r (or complex analytic) submanifold of N of codimension equal to* $\dim P$.

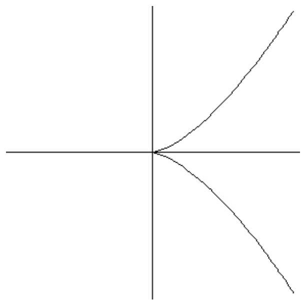
Example 1.6 If $x \in \Sigma f$, then the local topology of $f^{-1}(f(x))$ can certainly be non-trivial. Consider the example from high school: $f : \mathbb{R}^2 \to R$ given by $f(x, y) = xy$, where we are interested in the local topology of $f^{-1}(f(\mathbf{0})) = f^{-1}(0)$ at $\mathbf{0}$. The set $f^{-1}(0)$ consists of the union of the x- and $y-$ axes and, hence, is not even a topological manifold – much less, a topological submanifold of $\mathbb{R}^2$ – in a neighborhood of the origin.

As we are interested in points where spaces fail to be manifolds or submanifolds, as was the case with the origin above, we make a definition.

Definition 1.7 *Suppose that X is a topological space, and $x \in X$. Then, we say that x is a* **singular point of** X *or is a* **singularity of** X *if and only if there is no neighborhood of x which is a topological manifold.*

Suppose that X is a subspace of a C^r, or complex, manifold N, and $x \in X$. Then, we say that x is a C^r **singular point of** X **in** N *or is a C^r* **singularity of** X **in** N *if and only if there is no neighborhood, $\mathcal{U}$, of x in N such that $\mathcal{U} \cap X$ is a C^r, or complex, submanifold of $\mathcal{U}$.*

Example 1.8 Consider the set of points X in $\mathbb{R}^2$ which satisfy $y^2 = x^3$. This is called a *cusp*.



The origin is a C^1 singular point of X in $\mathbb{R}^2$, for there is no smooth way to flatten out the sharp point at the origin.

On the other hand, the origin is **not** a topological (i.e., C^0) singular point of X in $\mathbb{R}^2$. We leave the verification of this as an exercise for the reader.

Note that "singular point" is a type of point associated to a topological space, while a "critical point" is a type of point associated to a function. Of course, the Implicit Function Theorem and its corollaries tell us that these two notions are closely related. We shall return to examples of critical points and singular points in our other lectures.

Henceforth, for simplicity, we will restrict our attention to three cases: the C^∞ case, which we will refer to as the *smooth case*, and the real and complex analytic cases. Thus, we will always assume, from now on, that $f : N \to P$ is at least smooth.

The Theorem of Ehresmann and Integrating Vector Fields

The theorem of Ehresmann [Ehr50] is a theorem which describes a nice topological property of proper submersions.

Recall that a continuous function $g : X \to Y$ between topological spaces is called *proper* provided that for every compact subset C of Y, $g^{-1}(C)$ is compact. It is an easy exercise to show that if $g : X \to Y$ is proper, and Y is Hausdorff and compactly-generated, e.g., a manifold, then g is a closed map. On the other hand, Item 1 of the Corollary 1.2 tells us that submersions are open maps. Therefore, we have:

Proposition 1.9 *If $f : N \to P$ is a proper submersion, and P is connected, then f is a surjection.*

Ehresmann's Theorem refers to smooth, locally trivial, fibrations. We need to define this concept.

Definition 1.10 *A smooth function $g : M \to Q$ between two manifolds is a* **smooth, trivial fibration** *if and only if there exists a smooth manifold F such that $g : M \to Q$ is diffeomorphic to the projection $\pi : Q \times F \to Q$, i.e., there exists a diffeomorphism $\psi : M \to Q \times F$ such that $g = \pi \circ \psi$.*

Note that, if g is a smooth trivial fibration, then each fiber $g^{-1}(q)$, for $q \in Q$, is diffeomorphic to F. The manifold F (or, actually, its diffeomorphism-type) is referred to as the **fiber** *of the trivial fibration.*

A smooth function $g : M \to Q$ between two manifolds is a **smooth, locally trivial fibration** *if and only if, for all $q \in Q$, there exists an open neighborhood $\mathcal{V}$ of q in Q such that $g_{|_{g^{-1}(\mathcal{V})}} : g^{-1}(\mathcal{V}) \to \mathcal{V}$ is a trivial fibration.*

It is easy to show that if $g : M \to Q$ is a smooth, locally trivial fibration, and Q is connected, then the diffeomorphism-type of each of the fibers $g^{-1}(q)$ is independent of q; this common diffeomorphism-type is referred to as the **fiber** *of the locally trivial fibration.*

It is important to note that the "locally" in the term "locally trivial fibration" refers to local in the base space Q, not local in the space M. Later, in Theorem 2.58, we shall need the notion of a (topological) locally trivial fibration; one obtains this notion exactly as above, replacing "smooth" by "continuous" and "diffeomorphism" by "homeomorphism".

Before we state the theorem below, we should point out to the reader that two smooth manifolds are smoothly homotopy-equivalent if and only if they are (topologically) homotopy-equivalent. See [Bot82], p. 36. In particular, there is no difference between a contractible smooth manifold and a smoothly contractible smooth manifold.

The following theorem is well-known; see, for instance, [Ste51], 11.6.

Theorem 1.11 *If $f : N \to P$ is a smooth, locally trivial fibration and P is contractible, then f is a smooth trivial fibration.*

Now we state the theorem of Ehresmann, [Ehr50].

Theorem 1.12 *If $f : N \to P$ is a proper submersion, then it is a smooth, locally trivial fibration.*

One might hope that if f is assumed to be real or complex analytic, then a generalization of Ehresmann's Theorem would imply that the local trivializations could be made real or complex analytic; this is not the case, even though the Implicit Function Theorem tells us that, locally **in** N, we obtain such analytic trivializations. The problem is that one cannot analytically "patch together" the local trivializations in N to obtain an analytic trivialization over open subsets of P. We wish to say more about this, and discuss some important aspects of the proof of Ehresmann's Theorem.

The idea in the proof of Theorem 1.12 is fairly simple. Suppose $p \in P$. As f is assumed to be proper, $f^{-1}(p)$ is compact. By the Implicit Function Theorem, at every $x \in f^{-1}(p)$, there is a local trivialization of f. As $f^{-1}(p)$ is compact, a union of a finite number of these trivializations in N will contain $f^{-1}(p)$. Each of these local trivializations yields a local vector field on N. One then takes a C^∞ partition of unity (see [Spi70], p. 69-70), subordinate to the collection of trivializations, and use this partition of unity to smoothly patch together the local vector fields to obtain a smooth vector field in a neighborhood of $f^{-1}(p)$. One then integrates this vector field, i.e., follows the flow of $f^{-1}(p)$ as points on it "move", along integral curves, with velocities given by the vector field (see [Spi70], p. 203-204 or [Mil63], p. 9-11). As f is proper, this flow yields a smooth family of diffeomorphisms. For details of this proof, see 8.12 of [Bro73].

Integrating along vector fields is a fundamental differential technique for obtaining diffeomorphisms and trivializations. We shall return to this topic in Lecture 2. Note that the existence of C^∞ partitions of unity is used in a strong way above. The non-existence of analytic partitions of unity is what prevents us from proving a real or complex analytic version of Ehresmann's Theorem.

Basic Morse Theory

Ehresmann's Theorem is a theorem about smooth functions with no critical points. Morse Theory is the study of what happens at the most basic type of critical point of a smooth map. The classic, beautiful references for Morse Theory are [Mil63] and [Mil65]. We also recommend the excellent, new introductory treatment in [Mat97].

For the remainder of this lecture, $f : N \to \mathbb{R}$ will be a smooth function from a smooth manifold of dimension n into $\mathbb{R}$. For all $a \in \mathbb{R}$, let $N_a := f^{-1}((-\infty, a])$. Note that if a is a regular value of f, then N_a is a *smooth manifold with boundary* $\partial N_a = f^{-1}(a)$ (see, for instance, [Spi70]).

The following is Theorem 3.1 of [Mil63].

Theorem 1.13 *Suppose that $a, b \in \mathbb{R}$ and $a < b$. Suppose that $f^{-1}([a, b])$ is compact and contains no critical points of f. Then, N_a is a deformation retract of N_b via a smooth isotopy. In particular, N_a is diffeomorphic to N_b.*

As $[a, b]$ is contractible, Theorem 1.13 essentially follows from a combination of Theorem 1.11 and Theorem 1.12. In [Mil63], Milnor uses the existence of a Riemannain metric on M, and then integrates the corresponding normalized gradient vector field on M. As with partitions of unity, this is a C^∞ technique which does not work analytically; Riemannian metrics need only vary in a C^∞ manner as one varies the point in N.

Let $p \in N$, and let $(x_1, ..., x_n)$ be a smooth, local coordinate system for N in an open neighborhood of p.

Definition 1.14 *The point p is a* **non-degenerate critical point of** f *provided that p is a critical point of f, and that the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$ is non-singular.*

The **index of** f **at a non-degenerate critical point** p *is the number of negative eigenvalues of $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$, counted with multiplicity.*

It is left to the reader to check that p being a non-degenerate critical point of f is independent of the choice of local coordinates on N. The reader may also try, as an exercise, to prove that the index of a non-degenerate critical is independent of the coordinate choice; this is also true, but not quite so easy (see [Mil63], p. 4-5). Note that since the Hessian matrix is a real symmetric matrix, it is diagonalizable and, hence, the algebraic and geometric multiplicities of eigenvalues are the same.

The following is Lemma 2.2 of [Mil63], which tells us the basic structure of f near a non-degenerate critical point.

Lemma 1.15 (The Morse Lemma) *Let p be a non-degenerate critical point of f. Then, there is a local coordinate system $(y_1, \ldots, y_n)$ in an open neighborhood $\mathcal{U}$ of p, with $y_i(p) = 0$, for all i, and such that, for all $x \in \mathcal{U}$,*

$$f(x) = f(p) - (y_1(x))^2 - (y_2(x))^2 - \ldots - (y_\lambda(x))^2 + (y_{\lambda+1}(x))^2 + \ldots + (y_n(x))^2,$$

where λ is the index of f at p.

In particular, the point p is an isolated critical point of f, i.e., there is an open neighborhood of p (namely, $\mathcal{U}$) in which p is the only critical point of f.

The fundamental result of Morse Theory is a description of how N_b is obtained from N_a, where $a < b$, and where $f^{-1}([a, b])$ is compact and contains a single critical point of f, and that critical point is contained in $f^{-1}((a, b))$ and is non-degenerate. In [Mil63], Theorem 3.2, , Milnor gives this result up to homotopy. However, we wish to give the stronger "handle" result, as given in [Mil65], Theorems 3.13 and 3.14, and in [Mat97], Theorem 3.2. First, we need a definition. Let B^k denote a closed ball of dimension k.

Definition 1.16 *A smooth n-dimensional manifold M' with boundary is* **obtained from a smooth n-dimensional manifold M with boundary by smoothly attaching a λ-handle** *provided that there is an embedding $i : \partial B^\lambda \times B^{n-\lambda} \to \partial M$ such that M' is diffeomorphic to the space obtained by attaching the space $B^\lambda \times B^{n-\lambda}$ to M via i and then "smoothing the corners"* (see [Mat97], p. 78).

Theorem 1.17 *Suppose that $a < b$, $f^{-1}([a, b])$ is compact and contains a single critical point of f, and that critical point is contained in $f^{-1}((a, b))$ and is non-degenerate of index λ. Then, N_b is obtained from N_a by smoothly attaching a λ-handle.*

In particular, N_b has the homotopy-type of N_a with a λ-cell attached, and so $H_i(N_b, N_a;\ \mathbb{Z}) = 0$ if $i \neq \lambda$, and $H_\lambda(N_b, N_a;\ \mathbb{Z}) \cong \mathbb{Z}$.

Thus, functions $f : N \to \mathbb{R}$ that have only non-degenerate critical points are of great interest, and so we make a definition.

Definition 1.18 *The smooth function $f : N \to \mathbb{R}$ is a* **Morse function** *if and only if all of the critical points of f are non-degenerate.*

Remark 1.19 The reader should be careful when encountering the term "Morse function" in various references. We have given the weakest possible definition. Other authors sometimes mean that f is proper, or that N must be compact, or that the critical values at distinct critical points must be distinct.

If a and b are not critical values of f and $M := f^{-1}([a, b])$ is compact, then the change in diffeomorphism-type between N_a and N_b is determined by the restriction of f to the smooth manifold with boundary M. We make the following definitions (see [Mil65]).

Definition 1.20 *The triple $(M; V_0, V_1)$ is a* **smooth manifold triad** *if and only if M is a compact, smooth, pure-dimensional manifold, and the boundary ∂M is the disjoint union of two open and closed submanifolds V_0 and V_1.*

A **Morse function on the smooth manifold triad** *$(M; V_0, V_1)$ is a smooth function $g : M \to [a, b] \subseteq \mathbb{R}$ such that $g^{-1}(a) = V_0$, $g^{-1}(b) = V_1$, and all of the critical points of g lie in $M - \partial M$ and are non-degenerate.*

Note that a Morse function on a smooth manifold triad has at most a finite number of critical points, since the manifold must be compact and Morse critical points are isolated. Also, note that a smooth manifold triad includes as a special case a compact manifold without boundary, i.e., $V_0 = V_1 = \emptyset$; in this case, a Morse function $g : M \to [a, b]$ would have a below the minimum value of g and b above the maximum.

Definitions 1.18 and 1.20 would not be terribly useful if there were very few Morse functions. However, there are a number of theorems which tell us that Morse functions are very plentiful. We remind the reader that "almost all" means except for a set of measure zero.

Theorem 1.21 ([Mil65], p. 11) *If g is a C^2 function from an open subset $\mathcal{U}$ of $\mathbb{R}^n$ to $\mathbb{R}$, then, for almost all linear functions $L : \mathbb{R}^n \to \mathbb{R}$, the function $g + L : \mathcal{U} \to \mathbb{R}$ has no degenerate critical points.*

Theorem 1.22 ([Mil65], p. 14-17) *Let $(M; V_0, V_1)$ be a smooth manifold triad, and suppose $a < b$. Then, in the C^2 topology, the Morse functions form an open, dense subset of the space of all smooth functions $g : (M, V_0, V_1) \to ([a, b], a, b)$. In particular, there exists a Morse function on the triad.*

Theorem 1.23 ([Mil63], Theorem 6.6) *Let M be a smooth submanifold of $\mathbb{R}^n$, which is a closed subset of $\mathbb{R}^n$. For all $p \in \mathbb{R}^n$, let $L_p : M \to \mathbb{R}$ be given by $L_p(x) := ||x - p||^2$. Then, for almost all $p \in \mathbb{R}^n$, L_p is a proper Morse function such that M_a is compact for all a.*

Corollary 1.24 ([Mil63], p. 36) *Every smooth manifold M possesses a Morse function $g : M \to \mathbb{R}$ such that M_a is compact for all $a \in \mathbb{R}$. Given such a function g, M has the homotopy-type of a CW-complex with one cell of dimension λ for each critical point of g of index λ.*

While we stated the above as a corollary to Theorem 1.23, it also strongly uses two other results: Theorem 3.5 of [Mil63] and Whitney's Embedding Theorem, which tells us that any smooth manifold can be smoothly embedded as a closed subset of some Euclidean space.

To apply Theorem 1.17 at each critical point of a Morse function g on the smooth manifold triad $(M; V_0, V_1)$, we need to know that the critical values of g are distinct.

Theorem 1.25 ([Mil65], Lemma 2.8) *Let g be a Morse function on the smooth manifold triad $(M; V_0, V_1)$ with critical points $p_1, \dots, p_k$ of indices $\lambda_1, \dots, \lambda_k$, respectively. Then, there exists a Morse function h on $(M; V_0, V_1)$ with critical points $p_1, \dots, p_k$ of indices $\lambda_1, \dots, \lambda_k$, respectively, such that all of the critical values $h(p_i)$ are distinct for distinct p_i. Moreover, h can be chosen arbitrarily close to g in the C^2 topology.*

By combining Theorems 1.17, 1.22, and 1.25, we immediately obtain:

Corollary 1.26 *Every smooth compact n-manifold has a finite handlebody structure, i.e., is formed by successively attaching a finite number of handles of various indices.*

In the above corollary, one begins at the global minimum of a Morse function with distinct critical values, and "attaches" a 0-handle (a closed n-ball) to the empty set. The final attaching occurs at the global maximum, where one attaches an n-handle, i.e., attaches an n-ball along its bounding $(n-1)$-sphere.

We now wish to mention a few complex analytic results which are of importance.

Theorem 1.27 ([Mil63], p. 39-41) *Suppose that M is an m-dimensional complex analytic submanifold of $\mathbb{C}^n$. For all $p \in \mathbb{C}^n$, let $L_p : M \to \mathbb{R}$ be given by $L_p(x) := ||x - p||^2$. If $x \in M$ is a non-degenerate critical point of L_p, then the index of L_p at x is less than or equal to m.*

Corollary 1.24 immediately implies:

Corollary 1.28 ([Mil63], Theorem 7.2) *If M is an m-dimensional complex analytic submanifold of $\mathbb{C}^n$, which is a closed subset of $\mathbb{C}^n$, then M has the homotopy-type of an m-dimensional CW-complex. In particular, $H_i(M;\ \mathbb{Z}) = 0$ for $i > m$.*

Note that this result should **not** be considered obvious; m is the **complex** dimension of M. Over the real numbers, M is $2m$-dimensional, and so m is frequently referred to as the **middle dimension**. Thus, the above corollary says that the homology of a complex analytic submanifold of C^n, which is closed in $\mathbb{C}^n$, has trivial homology above the middle dimension.

The reader might hope that the corollary above would allow one to obtain nice results about compact complex manifolds; this is **not** the case. The maximum modulus principle, applied to the coordinate functions on $\mathbb{C}^n$, implies that the only compact, connected, complex submanifold of $\mathbb{C}^n$ is a point.

Suppose now that M is a complex m-manifold, and that $c : M \to \mathbb{C}$ is a complex analytic function. Let $p \in M$, and let $(z_1, ..., z_m)$ be a complex analytic coordinate system for M in an open neighborhood of p.

Analogous to our definition in the smooth case, we have:

Definition 1.29 *The point p is a* **complex non-degenerate critical point of** *c provided that p is a critical point of c, and that the Hessian matrix $\left(\frac{\partial^2 c}{\partial z_i \partial z_j}(p)\right)_{i,j}$ is non-singular.*

There is a complex analytic version of the Morse Lemma, Lemma 1.15:

Lemma 1.30 *Let p be a complex non-degenerate critical point of c. Then, there is a local complex analytic coordinate system $(y_1, \dots, y_m)$ in an open neighborhood $\mathcal{U}$ of p, with $y_i(p) = 0$, for all i, and such that, for all $x \in \mathcal{U}$,*

$$c(x) = c(p) + (y_1(x))^2 + (y_2(x))^2 + \dots + (y_n(x))^2.$$

In particular, the point p is an isolated critical point of c.

The first statement of the following theorem is proved in exactly the same manner as Theorem 1.21; one uses the open mapping principle for complex analytic functions to obtain the second statement.

Theorem 1.31 *If c is a complex analytic function from an open subset $\mathcal{U}$ of $\mathbb{C}^m$ to $\mathbb{C}$, then, for almost all complex linear functions $L : \mathbb{C}^m \to \mathbb{C}$, the function $c + L : \mathcal{U} \to \mathbb{C}$ has no complex degenerate critical points.*

In addition, for all $x \in \mathcal{U}$, there exists an open, dense subset $\mathcal{W}$ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, \mathbb{C}) \cong \mathbb{C}^m$ such that, for all $L \in \mathcal{W}$, there exists an open neighborhood $\mathcal{U}' \subseteq \mathcal{U}$ of x such that $c + L$ has no complex degenerate critical points in $\mathcal{U}'$.

Finally, we leave the following theorem as an exercise for the reader. We denote the real and imaginary parts of c by $\mathrm{Re}\, c$ and $\mathrm{Im}\, c$, respectively.

Theorem 1.32 *Let p be a complex non-degenerate critical point of c. Then, the real functions $\mathrm{Re}\, c : M \to \mathbb{R}$ and $\mathrm{Im}\, c : M \to \mathbb{R}$ each have a (real, smooth) non-degenerate critical point at p of index precisely equal to m, the complex dimension of M. In addition, if $c(p) \neq 0$, then the real function $|c|^2 : M \to \mathbb{R}$ also has a non-degenerate critical point of index m at p.*

2 Lecture 2

Real and Complex Analytic Sets

In Lecture 1, Definition 1.7, we defined a singular point as a point where a topological space is not locally homeomorphic to an open subset of Euclidean space. While this is, in fact, a reasonable definition of a singular point, it is unreasonable to expect to be able to obtain nice results about the local topology of arbitrary topological spaces at such singular points. We need to restrict our attention to topological spaces which are more manageable, and occur "naturally".

Thus, we shall restrict our attention to *algebraic and analytic sets (varieties)* **over a field $\mathfrak{K}$, which we assume to be $\mathbb{R}$ or $\mathbb{C}$**; we will define these notions below. When we write *analytic* below, we mean real analytic if $\mathfrak{K} = \mathbb{R}$, and complex analytic if $\mathfrak{K} = \mathbb{C}$. There are two topologies on $\mathfrak{K}^n$ which we will consider; the classical topology on $\mathbb{R}^n$ or $\mathbb{C}^n$, which is what we used in the first lecture, and the *Zariski topology*, which the reader may be familiar with, but which we will define below. When we use the terms "open" and "closed" without qualification, we mean open or closed in the classical topology. When we want to refer to the Zariski topology, we shall do so explicitly.

In the algebraic setting, we shall consider global algebraic subsets of $\mathfrak{K}^n$, i.e., affine algebraic sets. In the analytic setting, instead of restricting ourselves to connected open subsets of $\mathfrak{K}^n$, it is useful to allow the ambient space to be more general. **Thus, throughout this lecture, unless we specifically state otherwise, M denotes a connected n-dimensional analytic manifold**.

There are many, many basic references for algebraic geometry, and we shall not need very much of the theory. The references for analytic geometry, even in the complex case, are not so widely known. While we shall discuss most of the results that we need, we recommend, for this series of lectures: [Mil68], §2, for the real and complex algebraic cases; [Boc98], Chapters 2 and 3, for the real algebraic case; [Har77], I.1, and [Mum76], Chapter 1, §1, for the complex algebraic case; Chapter 1 of [Kra92] for the real analytic case; for the complex analytic case, [Mum76], Chapter 4A and [Ło91], Chapters II and IV; for a combined treatment of the complex algebraic and analytic cases, [Gri74], Chapter 0, §1 and §2; and, finally, for both the real and complex analytic cases, [Car57] and [Nar66].

We let $\mathcal{O}_{\mathfrak{K}^n} := \mathfrak{K}[x_1, \dots, x_n]$ denote the polynomial ring in n variables over $\mathfrak{K}$. We let $\mathcal{O}_M^{\text{anal}}$ denote the ring of analytic functions from M to $\mathfrak{K}$; these are functions which are locally, on coordinate patches, given by convergent power series in n variables over the field $\mathfrak{K}$; over $\mathbb{C}$, these are simply the holomorphic functions on M. It is important to note that an analytic function is **not** required to have one fixed power series representation that can be used at all points. We write $(x_1, \dots, x_n)$ for local analytic coordinates on M.

Recall the notions of the *germ of a topological space, X,* and *germ of a function on a topological space, $h : X \to Y$* at a point $p \in X$; intuitively, the *germ* at p means the space X or the function h in an arbitrarily small neighborhood of p. Rigorously, the germ of X (resp., h) at p is the equivalence class under the equivalence relation: two spaces (resp., functions) X and X' (resp., h and h') have the same germ at $p \in X \cap X'$ if and only if there is an open neighborhood $\mathcal{U}$ of p in X and $\mathcal{U}'$ of p in X' such that the topological spaces (resp., functions) $\mathcal{U}$ and $\mathcal{U}'$ (resp., $h_{|\mathcal{U}}$ and $h_{|\mathcal{U}'}$) are equal. We denote the germ of X (resp., h) at p by X_p (resp., $[h]_p$).

Now, if $p = (p_1, \dots, p_n) \in M$, we let $\mathcal{O}_{M,p}^{\text{anal}}$ denote the ring of germs of functions which are analytic on some open neighborhood of p in M; these are the power series that converge in some neighborhood of p. Note that $\mathcal{O}_{M,p}^{\text{anal}}$ is a local ring, whose maximal ideal, in local coordinates $(x_1, \dots, x_n)$, is

$$\mathfrak{m}_{M,p} := < x_1 - p_1, \dots, x_n - p_n > \subseteq \mathbb{C}\{x_1 - p_1, \dots, x_n - p_n\} = \mathcal{O}_{M,p}^{\text{anal}}.$$

The following result is fundamental.

Theorem 2.1 (The Principle of Analytic Continuation) *Suppose that $p \in M$, $f \in \mathcal{O}_M^{\text{anal}}$, and $[f]_p = 0$ in $\mathcal{O}_{M,p}^{\text{anal}}$. Then, $f = 0$ in $\mathcal{O}_M^{\text{anal}}$.*

Proof. Let $Z := \{x \in M \mid [f]_x = 0\}$. Then, Z is non-empty (since $p \in Z$) and is clearly open in M. However, as power series representations of functions are unique on the domain of convergence, Z is also closed in M. As M is connected and non-empty, $Z = M$, i.e., $f = 0$ in $\mathcal{O}_M^{\text{anal}}$. □

We have the following basic algebraic result (see [Nar66]).

Theorem 2.2 *The rings $\mathcal{O}_{\mathfrak{K}^n}$ and $\mathcal{O}_{M,p}^{\text{anal}}$ are Noetherian, unique factorization domains (UFDs). The ring $\mathcal{O}_M^{\text{anal}}$ is an integral domain, but need not be Noetherian or a UFD.*

Definition 2.3 *Let $A \subseteq \mathcal{O}_{\mathfrak{K}^n}$ (resp., $A \subseteq \mathcal{O}_M^{\text{anal}}$). Then, we define the* **vanishing locus (or, zero locus) of** A**,** *$V(A)$, to be the set of points in $\mathfrak{K}^n$ (resp., the set of points in M) $V(A) := \{x \in \mathfrak{K}^n$ (resp., M) $\mid f(x) = 0$ for all $f \in A\}$,*

If $A \subseteq \mathcal{O}_{M,p}^{\text{anal}}$, then one makes an analogous definition of $V_p(A)$ as a germ of a set of points in M at p. (We leave this formulation as an exercise for the reader.)

Let $E \subseteq \mathfrak{K}^n$ (resp., $E \subseteq M$). Then, we define the **ideal of polynomials (resp., analytic functions) which vanish on** E *to be $\mathcal{I}(E) := \{f \in \mathcal{O}_{\mathfrak{K}^n}$(resp., $\mathcal{O}_M^{\text{anal}}$) $\mid f(e) = 0$ for all $e \in E\}$.*

If E_p is the germ of a set of points in M at p, then one makes an analogous definition of $\mathcal{I}(E_p)$ as an ideal in $\mathcal{O}_{M,p}^{\text{anal}}$. (We leave this formulation as an exercise for the reader.)

If $A = \{f_1, \ldots, f_j\}$, we write $V(f_1, \ldots, f_j)$ in place of $V(A)$.

The following proposition is a straightforward exercise. Recall that the radical, $\sqrt{J}$, of an ideal J in a ring R is equal to $\{f \in R \mid$ there exists $k \in \mathbb{N}$ such that $f^k \in J\}$. Also, recall that for $A \subseteq R$, $\langle A \rangle$ is the ideal generated by A, i.e., the intersection of all ideals in R which contain A.

Proposition 2.4 *Let $E \subseteq F \subseteq \mathfrak{K}^n$, and $A \subseteq B \subseteq \mathcal{O}_{\mathfrak{K}^n}$. For all α in some indexing set S, let J_α be an ideal in $\mathcal{O}_{\mathfrak{K}^n}$. For all β in some indexing set T, let G_β be a subset of $\mathfrak{K}^n$.*

1. *$\mathcal{I}(E)$ is, in fact, an ideal in the ring $\mathcal{O}_{\mathfrak{K}^n}$, and $\sqrt{\mathcal{I}(E)} = \mathcal{I}(E)$;*
2. *$\mathcal{I}(F) \subseteq \mathcal{I}(E)$;*
3. *$V(B) \subseteq V(A)$;*
4. *$E \subseteq V(\mathcal{I}(E))$;*
5. *$A \subseteq \mathcal{I}(V(A))$;*
6. *$V(A) = V(\mathcal{I}(V(A)))$;*
7. *$\mathcal{I}(E) = \mathcal{I}(V(\mathcal{I}(E)))$;*
8. *$V(A) = V(\langle A \rangle)$ and $\sqrt{\langle A \rangle} \subseteq \mathcal{I}(V(A))$;*
9. *for $\alpha, \beta \in S$, $V(J_\alpha) \cup V(J_\beta) = V(J_\alpha \cap J_\beta) = V(J_\alpha \cdot J_\beta)$;*

10. $V\Big(\sum_{\alpha\in S} J_\alpha\Big) = \bigcap_{\alpha\in S} V(J_\alpha)$*;*

11. for all $\alpha \in S$, $V(\sqrt{J_\alpha}) = V(J_\alpha)$*;*

12. $\mathcal{I}\Big(\bigcup_{\beta\in T} G_\beta\Big) = \bigcap_{\beta\in T} \mathcal{I}(G_\beta)$*.*

Moreover, the analogous statements for $\mathcal{O}^{\text{anal}}_{M,p}$ *and* $\mathcal{O}^{\text{anal}}_{M}$ *are also true.*

Definition 2.5 *An* **algebraic subset of** $\mathfrak{K}^n$ *or an* **affine algebraic set** *is a set of the form* $V(A)$*, where* $A \subseteq \mathcal{O}_{\mathfrak{K}^n}$*. As* $V(A) = V(\langle A\rangle)$ *and* $\mathcal{O}_{\mathfrak{K}^n}$ *is Noetherian, this is equivalent to saying that an algebraic subset of* $\mathfrak{K}^n$ *is defined by the vanishing of a finite number of polynomials, i.e., is of the form* $V(f_1, \ldots, f_j)$*.*

A subset $X \subseteq M$ *is an* **analytic subset of** M *if and only if* X *is closed in* M *and, for all* $x \in X$*, there exists an open neighborhood* $\mathcal{W}$ *of* x *in* M *and a finite collection* $f_1, \ldots, f_j \in \mathcal{O}^{\text{anal}}_{\mathcal{W}}$ *such that* $V(f_1, \ldots, f_j) = \mathcal{W} \cap X$*.*

A subset E *of* M *is* **locally analytic** *if and only if, for all* $p \in E$*, there exists an open neighborhood* $\mathcal{W}$ *of* p *in* M *such that* $\mathcal{W} \cap E$ *is an analytic subset of* $\mathcal{W}$*.*

Remark 2.6 Item 6 of Proposition 2.4 tells us that if X is an algebraic set (resp., X_p is the germ of an analytic subset), then $V(\mathcal{I}(X)) = X$ (resp., $V_p(\mathcal{I}(X_p)) = X_p$).

However, an analytic subset X of M need not be defined by the vanishing of a collection of analytic functions which are defined **everywhere** on M. In this case, while it is still true that $X \subseteq V(\mathcal{I}(X))$, the containment may well be proper. A particularly striking example is given in [Car57], §11, where a real analytic subset $S \subseteq \mathbb{R}^3$ is defined by the vanishing of a C^∞ function, and it is shown that $\mathcal{I}(S) = \langle 0\rangle$.

We should also remark that some authors do not require an analytic subset to be closed. Their notion of an analytic subset of M is what we have called a locally analytic subset. One must be careful when reading the definition of "analytic subset" in a given source. Another source of possible confusion is that it is common to require an analytic subset X to be closed without explicitly stating that X is closed; some authors take our definition, omit the phrase "X is closed in M", but change "for all $x \in X$" to "for all $x \in M$".

The following theorem follows from Item 10 of Proposition 2.4 in the algebraic case. In the analytic case, it is substantially more difficult; see [Whi72], Theorem 9C and [Nar66], Cor. 2, p. 100.

Theorem 2.7 *Intersections of affine algebraic sets are affine algebraic sets. Intersections of analytic subsets of* M *are analytic subsets of* M*; in particular, if* $A \subseteq \mathcal{O}^{\text{anal}}_{M}$*, then* $V(A)$ *is an analytic subset of* M*.*

In light of this theorem, and the fact that $V(1) = \emptyset$ and $V(0)$ is the entire ambient space, $\mathfrak{K}^n$ or M, we may make the following definition.

Definition 2.8 *The* **algebraic Zariski topology** *on* $\mathfrak{K}^n$ *is the topology in which the closed sets are precisely the algebraic subsets.*

The **analytic Zariski topology** *on* M *is the topology in which the closed sets are precisely the analytic subsets.*

We denote the topological closure of $E \subseteq M$ *by* $\overline{E}$ *and the* **analytic closure***, i.e., the closure in the analytic Zariski topology, by* $\overline{E}^{\text{anal}}$*.*

Note that the Zariski topologies are very coarse; the open subsets are very big and special. In particular, a Zariski-open (resp., closed) set is open (resp., closed) in the classical topology. We remind the reader that when we use the terms "open" and "closed", or other topological notions, without qualification, we mean in the classical topology.

Definition 2.9 *A non-empty algebraic subset in* $\mathfrak{K}^n$ *(resp., analytic subset of* M*) is* **irreducible** *if and only if it cannot be written as the union of two proper algebraic (resp., analytic) subsets.*

An analytic subset $X \subseteq M$ *is* **irreducible at a point** $p \in X$ *(or* **the germ** X_p **is irreducible***) if and only if the* X_p *cannot be written as the union of two proper germs of analytic spaces at* p*.*

Note that it is immediate from the definition that an irreducible analytic subset of M must be connected. Moreover, the following proposition is an easy exercise:

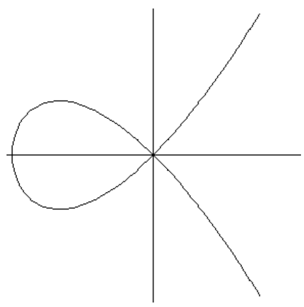
Proposition 2.10

1. *If* X *is an irreducible algebraic set in* $\mathfrak{K}^n$*, then* $\mathcal{I}(X)$ *is a prime ideal in* $\mathcal{O}_{\mathfrak{K}^n}$*. Moreover, the analogous statements for irreducible analytic sets in* M *and irreducible germs are also true.*
2. *If* $A \subseteq \mathcal{O}_{\mathfrak{K}^n}$*,* $X := V(A)$*, and* $\mathcal{I}(X)$ *is a prime ideal in* $\mathcal{O}_{\mathfrak{K}^n}$*, then* X *is irreducible. Moreover, the analogous statement in* $\mathcal{O}^{\text{anal}}_{M,p}$ *is also true.*

In particular, an algebraic set (resp., an analytic germ) is irreducible if and only if the ideal of polynomials vanishing on the set (resp., the ideal of germs of analytic functions vanishing on the germ) is prime.

Note that we are **not** claiming that Item 2 of Proposition 2.10 holds for analytic subsets $X \subseteq M$.

Example 2.11 Consider $X := V(y^2 - x^3 - x^2) \subseteq \mathbb{R}^2$.



This algebraic/analytic set is both algebraically and analytically irreducible in $\mathbb{R}^2$. However, X is not analytically irreducible at $\mathbf{0}$. In the graph, one sees that in a small neighborhood of the origin, X consists of two smooth curves which transversely intersect each other. One detects this analytically by factoring $y^2 - x^3 - x^2$ as $(y + x\sqrt{1+x})(y - x\sqrt{1+x})$. Here, $\sqrt{1+x}$ denotes one of the two convergent power series at $\mathbf{0}$ whose square is $1 + x$, and the equality $y^2 - x^3 - x^2 = (y + x\sqrt{1+x})(y - x\sqrt{1+x})$ holds only on neighborhoods of the origin where the power series for $\sqrt{1+x}$ converges, i.e, inside an open ball $\overset{\circ}{B}$ of radius 1. Thus, by Item 9 of Proposition 2.4, inside $\overset{\circ}{B}$,

$$X = V(y + x\sqrt{1+x}) \ \cup \ V(y - x\sqrt{1+x}),$$

and so the germ of X at $\mathbf{0}$ has two irreducible "components" (see below).

Note that the Inverse Function Theorem ([Rud53], 9.24; [Kra92], 1.8.1; [Whi72], Appendix II, Lemma 2.A) implies that the map $(x, y) \mapsto (x\sqrt{1+x}, y)$ is a local, analytic change of coordinates at the origin. This means that, up to an analytic isomorphism in a neighborhood of the origin, the analytic set $V(y^2 - x^3 - x^2)$ is the same as $V(y + x) \cup V(y - x)$, i.e., two intersecting lines.

Our final comment on this example is that we could have written all of the above with x and y as complex variables and X being a complex analytic subset of $\mathbb{C}^2$. Of course, $\mathbb{C}^2$ is real 4-dimensional, and so we have no hope of drawing an accurate picture over $\mathbb{C}$. It is common in low-dimensional complex analytic geometry to draw the picture of the corresponding situation over $\mathbb{R}$ and hope that the picture over the real numbers provides one with some intuition for what happens over $\mathbb{C}$.

There are, unfortunately, quite a few definitions of *smooth*, *regular*, and *singular* points in the literature. The issues involved in the various definitions are whether one uses globally-defined polynomials or local analytic functions, and whether one allows regular points of more than one dimension. Throughout our lectures, we will deal with the local analytic situation, and below we adopt notation and terminology for this situation. In the remark following the definition, we discuss the algebraic situation.

Definition 2.12 *Let X be an analytic subset of M. A point $p \in X$ is called* **smooth***, of dimension d, if and only if there exists an open neighborhood $\mathcal{W}$ of p in M such that $\mathcal{W} \cap X$ is an analytic submanifold of $\mathcal{W}$ of dimension d (over the field $\mathfrak{K}$). Thus, $p \in X$ is smooth of dimension d if and only if there exists an open neighborhood $\mathcal{W}$ of p in M and $f_{d+1}, \dots f_n \in \overset{\text{anal}}{\mathcal{O}_{\mathcal{W}}}$ such that $\mathcal{W} \cap X = V(f_{d+1}, \dots, f_n)$ and, for all $x \in \mathcal{W}$, $d_x f_{d+1}, \dots, d_x f_n$ are linearly independent.*

We denote the set of smooth points of X by $\overset{\circ}{X}$, and the set of smooth points of dimension d by $\overset{\circ}{X}{}^{(d)}$. A smooth point of X of highest dimension is a **regular point***; we denote the set of regular points of X by X_{reg}.*

A point $p \in X$ which is not smooth is called a **singular point** *(or, a* **singularity***). We denote the set of singular points, the* **singular locus***, of X by ΣX. The set of* **exceptional points** *of X is $X - X_{\text{reg}}$, and is denoted by X_{exc}.*

Remark 2.13 In the above definition, we have followed the standard analytic definition, as given in [Car57], [Nar66], [Ło91], and [Whi72] – though, we have used the term *smooth* where some authors write *regular*. Our uses of the terms *regular* and *exceptional* are not standard in many references. The reader should be careful to look up the definitions of any such terms in any given source.

The reader should also note that the definitions of smooth/regular/simple/non-singular and singular points given in [Mil68] are different; Milnor is dealing with the algebraic case and, in addition, gives a definition which is satisfactory only when all of the regular points have the same dimension; see [Mil68], p.10 and be sure to read the footnote. In the real or complex **algebraic** setting, it is standard to use globally-defined polynomials in the definitions of regular and singular points, as Milnor does in [Mil68]; see, in addition, [Boc98], §3.3 and [Mum76], §1A and §1.B.

Using global polynomial functions, one can still define *smooth points of dimension d* (as in [Boc98]), and avoid the "problem" mentioned in the footnote of [Mil68], p. 10. The set of algebraic smooth points $\overset{\circ}{X}{}^{\text{alg}}$ is the union of the smooth points of all dimensions, and the complement of this set is one notion of the algebraic singular locus of X, which we will denote by $\Sigma_{\text{alg}} X$. The algebraic regular points are the algebraic smooth points of highest dimension; we denote the set of algebraic regular points by $X^{\text{alg}}_{\text{reg}}$. If X is algebraically irreducible, then $X^{\text{alg}}_{\text{reg}} = \overset{\circ}{X}{}^{\text{alg}}$. The set of **algebraic exceptional points** of X is $X - X^{\text{alg}}_{\text{reg}}$, and is denoted by $X^{\text{alg}}_{\text{exc}}$.

Clearly, $\Sigma X \subseteq \Sigma_{\text{alg}} X \subseteq X^{\text{alg}}_{\text{exc}}$. An important question, of course, is whether or not $\Sigma X = \Sigma_{\text{alg}} X$.

In fact, in the complex algebraic case, Milnor's remark and proof in [Mil68], p. 13-14, shows that $\Sigma X = \Sigma_{\mathrm{alg}} X$. However, in the **real** algebraic case, it is **false**, in general, that $\Sigma X = \Sigma_{\mathrm{alg}} X$; see Example 3.3.12a of [Boc98]. However, the algebraic situation over the real numbers is not **so** badly behaved; over the reals or complexes, $X^{\mathrm{alg}}_{\mathrm{exc}}$ is an algebraic set; see [Boc98], Proposition 3.3.14, which is essentially the easy argument given by Milnor in [Mil68], p. 11.

The real analytic situation is much more problematic; see Example 2.31.

Theorem 2.14 *Let X be an analytic subset of M. Suppose that $0 \leq d \leq n$.*

1. *$\overset{\circ}{X}{}^{(d)}$ is a d-dimensional analytic submanifold of M, and is an open subset of X;*
2. *the $\overset{\circ}{X}{}^{(d)}$ are disjoint for different d, $\overset{\circ}{X} = \overset{\circ}{X}{}^{(0)} \cup \dots \overset{\circ}{X}{}^{(n)}$, $\overset{\circ}{X}$ is an analytic submanifold of M, and $\overset{\circ}{X}$ is open in X;*
3. *if p is a smooth point of X, then the germ X_p is irreducible;*
4. *$\overset{\circ}{X}$ is dense in X; and*
5. *ΣX is a closed, nowhere dense subset of X.*

Proof. Items 1 and 2 follow at once from the definitions.

At a smooth point p of dimension d, the Implicit Function Theorem tells us that there is an analytic isomorphism in a neighborhood of p which takes the ideal $\langle [f_{d+1}]_p, \dots, [f_n]_p \rangle$ in $\overset{\mathrm{anal}}{\mathcal{O}}_{M,p}$ to the ideal $\langle [x_{d+1}]_p, \dots, [x_n]_p \rangle$ in $\overset{\mathrm{anal}}{\mathcal{O}}_{\mathcal{U},p}$, where $\mathcal{U}$ is an open subset of $\mathfrak{K}^n$ and $(x_1, \dots, x_n)$ are coordinates on $\mathcal{U}$. As $\langle [x_{d+1}]_p, \dots, [x_n]_p \rangle$ is obviously a prime ideal in $\overset{\mathrm{anal}}{\mathcal{O}}_{\mathcal{U},p}$, $\langle [f_{d+1}]_p, \dots, [f_n]_p \rangle$ is prime in $\overset{\mathrm{anal}}{\mathcal{O}}_{M,p}$. Item 2 of Proposition 2.10 now implies that X_p is irreducible.

Item 5 follows immediately from Items 2 and 4.

Item 4 is difficult; see [Nar66], p. 41, Theorem 1. $\square$

Definition 2.15 *The* **dimension** *(over $\mathfrak{K}$), $\dim X$, of an analytic set $X \subseteq M$ is the largest d such that $\overset{\circ}{X}{}^{(d)}$ is non-empty. The* **dimension** *of X* **at a point** *$p \in X$, $\dim_p X$, is the largest d such that p is in the closure of $\overset{\circ}{X}{}^{(d)}$. We say that X is* **pure-dimensional** *if and only if the dimension of X at each point $p \in X$ is independent of p.*

We make the analogous definitions for local analytic sets.

We have defined dimension geometrically above. However, there are algebraic characterizations. See [Nar66], p. 32-40. Below, we use the Krull dimension; this follows quickly from [Nar66], since the Krull dimension of an integral extension of a ring R is the same as that of R (for Noetherian rings). For the global dimension statement for an algebraic set, there are many references; see, for instance, [Mum76], Chapter 1, in the complex case, and [Boc98], §3.3 in the real case.

Theorem 2.16 *The dimension of an analytic subset $X \subseteq M$ at a point $p \in X$ is equal to the (Krull) dimension of the quotient ring $\overset{\mathrm{anal}}{\mathcal{O}}_{M,p}/\mathcal{I}(X_p)$ and, thus, $\dim X$ is the maximum of these dimensions as p varies over all of the points of X.*

If X is an algebraic subset of $\mathfrak{K}^n$, then $\dim X$ is equal to the dimension of $\mathcal{O}_{\mathfrak{K}^n}/\mathcal{I}(X)$.

We have the following corollary, over $\mathbb{R}$ or $\mathbb{C}$, which is a very easy exercise, but is still extremely useful for proving local irreducibility:

Corollary 2.17 *Suppose that α is an ideal in $\overset{\text{anal}}{\mathcal{O}}_{M,p}$ such that $\sqrt{\alpha}$ is a prime ideal $\mathfrak{p}$, and such that*

$$\dim\left(\frac{\overset{\text{anal}}{\mathcal{O}}_{M,p}}{\alpha}\right) = \dim V_p(\alpha).$$

Then, $\mathcal{I}(V_p(\alpha)) = \mathfrak{p}$ and, hence, $V_p(\alpha)$ is irreducible.

In particular, if f is an irreducible element (in particular, not a unit or zero) of $\overset{\text{anal}}{\mathcal{O}}_{M,p}$ such that $\dim V_p(f) = n-1$, then $\langle f\rangle = \mathcal{I}(V_p(f))$ and $V_p(f)$ is irreducible.

Remark 2.18 It is tempting to read more into Theorem 2.16 and Corollary 2.17 than they actually say. Suppose that X_p is irreducible of dimension d, and $f \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$ is such that $f(p) = 0$ and $X_p \not\subseteq V_p(f)$. Then, one might think that Krull's Hauptidealsatz would easily imply that $\dim(X_p \cap V_p(f)) = d-1$. While this is true if $\mathfrak{K} = \mathbb{C}$, it is **not** true if $\mathfrak{K} = \mathbb{R}$.

What the Hauptidealsatz actually does tell us is that $\dim \dfrac{\overset{\text{anal}}{\mathcal{O}}_{M,p}}{\mathcal{I}(X_p) + \langle f\rangle} = d-1$. As

$$\dim \frac{\overset{\text{anal}}{\mathcal{O}}_{M,p}}{\mathcal{I}(X_p) + \langle f\rangle} = \dim \frac{\overset{\text{anal}}{\mathcal{O}}_{M,p}}{\sqrt{\mathcal{I}(X_p) + \langle f\rangle}}.$$

and, by Item 1 of Proposition 2.4, $\mathcal{I}(X_p \cap V_p(f))$ is equal to its own radical, the question is: does $\mathcal{I}(X_p \cap V_p(f))$ equal $\sqrt{\mathcal{I}(X_p) + \langle f\rangle}$? The answer is "yes" if $\mathfrak{K} = \mathbb{C}$, by Hilbert's Nullstellensatz (see Theorem 2.19 below), and "no", in general, if $\mathfrak{K} = \mathbb{R}$.

It is easy to see/explain this failure over $\mathbb{R}$; if $f_1, \dots, f_k$ are real analytic functions and $f := f_1^2 + \dots + f_k^2$, then $V(f_1, \dots, f_k) = V(f)$. Thus, being locally defined by the vanishing of a single real analytic function should not have any nice dimensionality properties. For instance, if $p = (p_1, \dots, p_n)$, $\dim X_p \geq 2$, and we choose $f = (x_1 - p_1)^2 + \dots + (x_n - p_n)^2 \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$, then $\dim(X_p \cap V_p(f)) = 0$, while $\dim \dfrac{\overset{\text{anal}}{\mathcal{O}}_{M,p}}{\sqrt{\mathcal{I}(X_p) + \langle f\rangle}} = (\dim X_p) - 1 \geq 1$.

In the complex algebraic setting, there are a large number of references for Hilbert's Nullstellensatz; see, for instance, [Mum76], Theorem 1.5. In the local complex analytic setting, see [Ło91], III.4.1.

Theorem 2.19 (Hilbert's Nullstellensatz) *Let $\mathfrak{K} = \mathbb{C}$. Suppose that α is an ideal in $\mathcal{O}_{\mathfrak{K}^n}$ (resp., in $\overset{\text{anal}}{\mathcal{O}}_{M,p}$). Then, $\sqrt{\alpha} = \mathcal{I}(V(\alpha))$ (resp., $= \mathcal{I}(V_p(\alpha))$).*

Our discussion in Remark 2.18 immediately yields:

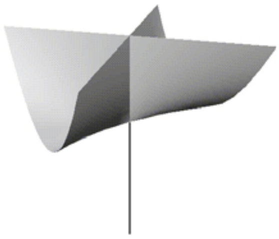
Proposition 2.20 *Let $\mathfrak{K} = \mathbb{C}$, $f \in \overset{\text{anal}}{\mathcal{O}}_{M}$, and $p \in V(f)$. Suppose that $[f]_p \neq 0$. Then, $\dim V_p(f) = n-1$.*

In particular, if $f \not\equiv 0$, then $V(f)$ is purely $(n-1)$-dimensional (this vacuously allows for $V(f) = \emptyset$).

In analogy with the manifold terminology, we say, in the setting above, that $V(f)$ has *codimension* 1 everywhere.

It should come as no surprise that Proposition 2.20 fails over the real numbers, even if one assumes that $V(f)$ is irreducible of dimension $n-1$.

Example 2.21 A famous example of a real algebraic set which is "troublesome" is the **Whitney umbrella**, $X := V(y^2 - zx^2) \subseteq \mathbb{R}^3$.



Note the 2-dimensional portion above the xy-plane, and that when $z < 0$, the only points of X are on the z-axis. One might suspect that X is not irreducible, as X is certainly the union of the z-axis (the *handle* of the umbrella) and the 2-dimensional "umbrella" portion. However, $X \cap \{(x, y, z) \mid z \geq 0\}$ is **not** an analytic set (use Corollary 2.17 at the origin), and X is, in fact, irreducible.

Thus, the Whitney umbrella is an irreducible analytic set which is **not** pure-dimensional.

Proposition 2.22 *Suppose that $f \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$ has an irreducible decomposition $uf_1^{\alpha_1} f_2^{\alpha_2} \dots f_l^{\alpha_l}$, where u is a unit, and the f_i are non-associated, irreducible elements of $\overset{\text{anal}}{\mathcal{O}}_{M,p}$. Then, $V_p(f)$ is an $(n-1)$-dimensional analytic submanifold inside the germ of M at p if and only if there exists i_0 such that $d_p f_{i_0} \neq 0$ and such that, for all $i \neq i_0$, $V(f_i) \subseteq V(f_{i_0})$ and $\dim V_p(f_i) < \dim V_p(f_{i_0})$.*

Therefore, if $\mathfrak{K} = \mathbb{C}$ and $f \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$, then $V_p(f)$ is an $(n-1)$-dimensional analytic submanifold inside the germ of M at p if and only if there exists an irreducible $g \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$ and $\alpha \in \mathbb{N}$ such that $f = g^\alpha$ and $d_p g \neq 0$.

Proof. Suppose that there exists i_0 such that $d_p f_{i_0} \neq 0$ and such that, for all $i \neq i_0$, $V(f_i) \subseteq V(f_{i_0})$ and $\dim V_p(f_i) < \dim V_p(f_{i_0})$. Then, Item 9 of Proposition 2.4 tells us that $V_p(f) = V_p(f_{i_0})$, and the Implicit Function Theorem (Corollary 1.2) tells us that $V_p(f_{i_0})$ is an $(n-1)$-dimensional analytic submanifold of the germ of M at p.

Now suppose that $V_p(f)$ is an $(n-1)$-dimensional analytic submanifold inside the germ of M at p. Then, $\mathcal{I}(V_p(f))$ equals a prime ideal $\mathfrak{p}$ and, since $V_p(f) = \cup_i V_p(f_i)$, we also have that $\mathcal{I}(V_p(f)) = \cap_i \mathcal{I}(V_p(f_i))$. Therefore, there exists i_0 such that $\mathfrak{p} = \mathcal{I}(V_p(f_{i_0}))$; it follows that $V_p(f) = V_p(f_{i_0})$ and $\dim V_p(f_{i_0}) = n-1$. Thus, for all i, $V(f_i) \subseteq V(f_{i_0})$.

Now, note that we are in the situation of the last statement of Corollary 2.17, and so we conclude that $\mathcal{I}(V_p(f)) = \mathcal{I}(V_p(f_{i_0})) = \langle f_{i_0} \rangle$. As $V_p(f)$ is an $(n-1)$-dimensional analytic submanifold, there exists some $g \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$ such that $V_p(g) = V_p(f)$ and $d_p g \neq 0$. Since $g \in \mathcal{I}(V_p(f)) = \langle f_{i_0} \rangle$, there exists $q \in \overset{\text{anal}}{\mathcal{O}}_{M,p}$ such that $g = q \cdot f_{i_0}$. The product rule yields $d_p g = q(p) d_p f_{i_0} + f_{i_0}(p) d_p q = q(p) d_p f_{i_0}$, and so $d_p f_{i_0}$ must be unequal to zero.

Suppose that for some $i \neq i_0$, $\dim V_p(f_i) = n-1$. Then, applying Corollary 2.17 again, we conclude that $\mathcal{I}(V_p(f_i)) = \langle f_i \rangle$. Thus, we obtain a containment of prime ideals $\langle f_{i_0} \rangle = \mathcal{I}(V_p(f)) \subseteq \mathcal{I}(V_p(f_i)) = \langle f_i \rangle$, where $\dim \mathcal{O}^{\text{anal}}_{M,p}/\langle f_{i_0} \rangle = \dim \mathcal{O}^{\text{anal}}_{M,p}/\langle f_i \rangle$. Hence, $\langle f_{i_0} \rangle = \langle f_i \rangle$, which contradicts that f_i is not associated to f_{i_0}.

This finishes the proof, except for the final statement, which now follows at once from Proposition 2.20 (and the fact that one can extract arbitrary roots of units over $\mathbb{C}$). □

Example 2.23 Over $\mathbb{R}$, one does not have the nice result that one has over $\mathbb{C}$ in Proposition 2.22. Consider the germ of the real function $f = -(x^2+y^2)^2x^2$ at the origin in $\mathbb{R}^2$. Then, $V(f) = V(x)$ is certainly a smooth 1-manifold at the origin, and yet one cannot eliminate the x^2+y^2 factors from f nor "absorb" the unit -1 into the powers of irreducible elements.

For lack of a convenient reference, we will now prove:

Theorem 2.24 *Suppose that X is an analytic subset of M, and E is a connected, pure-dimensional, locally analytic subset of M such that, for all $x \in E$, E_x is irreducible. Finally, suppose that $p \in E$ is such that $E_p \subseteq X_p$. Then, $E \subseteq X$.*

Proof. Let $F := \{x \in E \mid E_x \subseteq X_x\}$. We will show that F is both open and closed in E. As E is connected and $p \in F$, it will follow that $F = E$, and so $E \subseteq X$. Let d denote the dimension of E at each of its points.

By definition of the containment of germs, the set F is open in E. It remains for us to show that the complement of F in E is open in E.

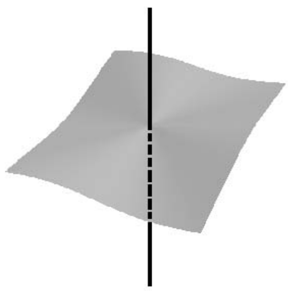
Suppose that $q \in E$ and $E_q \not\subseteq X_q$. As $E_q \not\subseteq X_q$, $\mathcal{I}(X_q) \not\subseteq \mathcal{I}(E_q)$. Therefore, there exists an open neighborhood $\mathcal{W} \subseteq M$ of q and $f \in \mathcal{O}^{\text{anal}}_{\mathcal{W}}$ such that $f_{|\mathcal{W}\cap X} \equiv 0$ and $f \notin \mathcal{I}(E_q)$. It follows that $E_q \cap V(f)$ is a proper analytic germ inside the irreducible germ E_q. By Proposition 7, p. 41, of [Nar66], $\dim_q(E \cap V(f)) < d$. Let $\mathcal{W}' \subseteq \mathcal{W}$ be an open neighborhood of q in M such that $\dim(\mathcal{W}' \cap E \cap V(f)) < d$. We claim that, for all $x \in \mathcal{W}' \cap E$, $E_x \not\subseteq X_x$, i.e., that the complement of F in E is open.

Suppose, to the contrary, that $x \in \mathcal{W}' \cap E$ and $E_x \subseteq X_x$. Then, as $f_{|\mathcal{W}\cap X} \equiv 0$, $x \in V(f)$, and so $\dim E_x \cap V(f) < d$. However, $E_x \subseteq X_x \subseteq V(f)$ and, hence, $\dim E_x \cap V(f) = \dim E_x = d$. This contradiction concludes the proof. □

The following corollary is an easy exercise.

Corollary 2.25 *Suppose that E is a connected, pure-dimensional, locally analytic subset of M such that, for all $x \in E$, E_x is irreducible. Then, $\overline{E}^{\text{anal}}$ is irreducible in M.*

Example 2.26 Theorem 2.24 is false without the assumption that E is pure-dimensional. Consider the set $E := V(z(x^2+y^2) - x^3) \subseteq \mathbb{R}^3$. This is an example of H. Cartan from p. 93 of [Car57]; see also [Nar66], Example 1, p. 106. We refer to this example as the **Cartan top**.



The Implicit Function Theorem tells us that outside $Z := \{(0,0,z) \mid z \in \mathbb{R}\}$ (i.e., the z-axis), E is an an analytic 2-manifold. Therefore, $E-Z$ is irreducible at each point. In addition, at any point p on $Z-\{\mathbf{0}\}$, E_p is certainly irreducible since it agrees with Z_p. Finally, use Corollary 2.17, or refer to [Car57] and [Nar66], for the irreducibility of E at $\mathbf{0}$.

Thus, E is connected and locally irreducible. Let $p := (0,0,-1)$. Then, $E_p \subseteq Z_p$, but certainly $E \not\subseteq Z$.

The following result is very useful.

Theorem 2.27 *Suppose that X is a connected, pure-dimensional, locally irreducible, analytic subset of M. Suppose that $\mathcal{W}$ is a non-empty, open subset of X. Then, the analytic closure of $\mathcal{W}$ in M is equal to X.*

Proof. Let Ω be the interior of $\overline{\mathcal{W}}^{\text{anal}}$ in X. Then, Ω is a non-empty open subset of X. We shall show that Ω is closed in X. As X is connected, that will conclude the proof. Let d denote the common dimension of X at each of its points.

Let $x \in \overline{\Omega} \subseteq \overline{\mathcal{W}}^{\text{anal}}$; we will show that $x \in \Omega$. For every open neighborhood $\mathcal{V}$ of x in X, $\Omega \cap \mathcal{V} \neq \emptyset$ and, thus, $\dim \left(\overline{\mathcal{W}}^{\text{anal}}\right)_x = d$. As $\left(\overline{\mathcal{W}}^{\text{anal}}\right)_x \subseteq X_x$, and X_x is irreducible is dimension d, we may use Proposition 7, p. 41, of [Nar66] to conclude that $\left(\overline{\mathcal{W}}^{\text{anal}}\right)_x = X_x$. Therefore, there is an open neighborhood $\mathcal{V}$ of x in X such that $\mathcal{V} \cap \overline{\mathcal{W}}^{\text{anal}} = \mathcal{V}$, i.e., such that $\mathcal{V} \subseteq \overline{\mathcal{W}}^{\text{anal}}$. Thus, by definition of the interior, $x \in \Omega$. $\square$

Corollary 2.28 *Suppose that Y is an affine linear subspace of $\mathfrak{K}^n$, and that $\mathcal{W}$ is a non-empty open subset of Y. Then, the analytic closure of $\mathcal{W}$ in $\mathfrak{K}^n$ is Y.*

We shall now give a number of fundamental results which hold over the complex numbers, but not over the real numbers. In Item 1 below, we use the convention that $\dim \emptyset = -\infty$.

Theorem 2.29 *Let X be a non-empty analytic subset of M. Let V be an irreducible analytic subset of M. Then, the following statements hold for $\mathfrak{K} = \mathbb{C}$ and are false, in general, for $\mathfrak{K} = \mathbb{R}$.*

1. *ΣX is an analytic subset of M of dimension strictly less than $\dim X$.*
2. *V is pure-dimensional.*
3. *If $p \in V$ and $V_p \subseteq X_p$, then $V \subseteq X$.*
4. *X is irreducible if and only if it is the topological closure of a non-empty, connected analytic submanifold.*
5. *If $V \not\subseteq X$, then $V - X$ and $\mathring{V} - X$ are connected and dense in V.*
6. *For every connected component C of $\mathring{X}$, the closure $\overline{C}$ is an irreducible analytic subset of M, and distinct connected components C yield distinct $\overline{C}$. The collection $\mathcal{K} := \{\overline{C} \mid C \text{ a connected component of } \mathring{X}\}$ is countable and locally finite. None of the elements of $\mathcal{K}$ is contained in the union of the others, and the union of all of the elements of $\mathcal{K}$ is X. If $V \subseteq X$, then there exists $Y \in \mathcal{K}$ such that $V \subseteq Y$.*
7. *Suppose that $X = \bigcup_i X_i$, where the X_i are irreducible analytic subsets of M, $\{X_i\}_i$ is locally finite (or countable), and $X_{i_1} \not\subseteq X_{i_2}$ if $i_1 \neq i_2$. Then, $\{X_i\}_i$ equals the set $\mathcal{K}$ from Item 6.*
8. *If X is connected and irreducible at each point, then X is irreducible.*

9. If $X \subseteq V$*, and* $\dim X = \dim V$*, then* $X = V$*.*

Proof. We give references for each of these results over $\mathbb{C}$, except we will prove Item 8, and leave Item 9 as an exercise. We give examples below which show that the general statements are false if $\mathfrak{K} = \mathbb{R}$.

While these results can be found in many places, all of our references are to [Ło91]. Item 1 is Theorem 1, p. 211. Item 2 is immediate from Item 4. Item 3 is Proposition 4, p. 217. Item 4 is Corollary 3 on p. 216. Item 5 is Corollary 1 to Proposition 3 on p. 216. Items 6 and 7 are the corollary and Theorem 4 on p. 217.

Finally, we use Items 3 and 6 to prove Item 8. Suppose that X is connected and irreducible at each point. Write $X = \bigcup_C \overline{C}$, where the C's are the connected components of $\overset{\circ}{X}$. We need to show that there is only one such C. Suppose not. Fix a C_0. As X is connected, $\overline{C_0}$ cannot be disjoint from $\bigcup_{C \neq C_0} \overline{C}$ (locally finite unions of closed sets are closed). Let $p \in \overline{C_0} \cap \bigcup_{C \neq C_0} \overline{C}$. As $X = \bigcup_C \overline{C}$, which is a locally finite union, and we are assuming that X_p is irreducible, there must be a C_i such that $X_p = (\overline{C_i})_p$. However, this implies that for every other distinct C_j such that $p \in \overline{C_j}$, $(\overline{C_j})_p \subseteq X_p = (\overline{C_i})_p$. By Item 3, this implies $\overline{C_j} \subseteq \overline{C_i}$; a contradiction of Item 6. □

Definition 2.30 *When* $\mathfrak{K} = \mathbb{C}$*, we refer to the* $\overline{C}$ *of Item 6, or the* X_i *of Item 7, in Theorem 2.29 as the* **irreducible components of** X*.*

We will now give examples which show that all of the items of Theorem 2.29 fail over $\mathbb{R}$.

Example 2.31

• Looking at Definition 2.12, it is tempting to believe that an easy argument – such as that mentioned in [Mil68], p. 11 – in terms of determinants of the minors of the matrix of partial derivatives would allow one to conclude in the real or complex analytic case that ΣX is analytic. However, the result is simply false if $\mathfrak{K} = \mathbb{R}$; see Example 3 on pages 106-107 of [Nar66]. In this example, an analytic subset $A \subseteq \mathbb{R}^3$ is described such that $\dim(\overline{\Sigma A}^{\text{anal}}) = \dim A$.

• The Whitney umbrella of Example 2.21 or the Cartan top of Example 2.26 are examples of irreducible real analytic sets which are not pure-dimensional.

• Let V equal the Whitney umbrella or the Cartan top, and let X denote the z-axis in $\mathbb{R}^3$. Let $p = (0, 0, -1)$. Then V is irreducible, $V_p \subseteq X_p$, but $V \not\subseteq X$. Thus, Item 3 of Theorem 2.29 fails over $\mathbb{R}$. Also, even though V is irreducible, it is not the closure of a **connected** analytic submanifold; therefore, Item 4 of Theorem 2.29 fails over $\mathbb{R}$.

Note, however, that the other implication in Item 4 remains valid even when $\mathfrak{K} = \mathbb{R}$. That is, if an analytic subset X of M is the topological closure of a non-empty, connected analytic submanifold, then X is irreducible. This follows immediately from Corollary 2.25.

• As $\mathbb{R} - \{\mathbf{0}\}$ is not connected, certainly the connectedness conclusion of Item 5 is terribly false over $\mathbb{R}$. In addition, removing the z-axis from V, where V is the Whitney umbrella or Cartan top, leaves one with a non-dense subset of V, which means that the other conclusion of Item 5 is not generally true over $\mathbb{R}$.

• No reasonable concept of an analytic irreducible component of a space X, as appears in Items 6 and 7, exists over the real numbers. Certainly the topological closure of the 2-dimensional connected component of the smooth points of the Whitney umbrella or the Cartan top are not analytic sets. However, the situation is significantly worse than this.

Example 4 on p. 107 of [Nar66] is an example of an analytic subset S of $\mathbb{R}^3$ such that if $S = B \cup C$, where B and C are analytic subsets and B is irreducible, then $C = S$.

• Suppose that Y is the Cartan top, $V(z(x^2+y^2)-x^3) \subseteq \mathbb{R}^3$. Let Z be a shifted copy of the Cartan top, $Z := V((z+1)(x^2+y^2)-x^3)$. Then, $Y \cap Z$ is the z-axis. Let $X := Y \cup Z$. Then, X is connected and locally irreducible, but not irreducible. Therefore, Item 8 does not hold over $\mathbb{R}$.

• Example 5 on p. 108 of [Nar66] is an example of an irreducible 2-dimensional analytic subset S of $\mathbb{R}^3$ containing proper 2-dimensional analytic subsets. Thus, Item 9 fails over $\mathbb{R}$.

Real and Complex Semianalytic Sets

When dealing with analytic subsets of an analytic manifold M, we frequently use constructions which require intersecting the analytic set with a closed ball, or use some other non-analytic intersection. Therefore, it is useful to expand our point-of-view somewhat and discuss *semianalytic* subsets. See [Ło65] and [Bie88].

Definition 2.32 *Suppose that $\mathfrak{K} = \mathbb{R}$. Let $\mathcal{W}$ be an open subset of M. A* **basic semianalytic subset of** $\mathcal{W}$ *is a subset of the form $V(f_1, \dots f_k) \cap \{x \in \mathcal{W} \mid g_\lambda(x) > 0, \lambda = 1, \dots, l\}$, where $f_1, \dots, f_k, g_1, \dots, g_l \in \mathcal{O}_{\mathcal{W}}^{\text{anal}}$.*

A subset X in M is **semianalytic** *if and only if, for all $p \in M$, there exists a open neighborhood $\mathcal{W}$ of p such that $\mathcal{W} \cap X$ is a finite union of basic semianalytic subsets of $\mathcal{W}$.*

Note that, by negating functions, the definition of a basic semianalytic subset can also contain $g_\lambda(x) < 0$ and so, after taking unions, we can also obtain $g_\lambda(x) \neq 0$, $g_\lambda(x) \geq 0$, and $g_\lambda(x) \leq 0$.

Now, suppose that $\mathfrak{K} = \mathbb{C}$. Let $\mathcal{W}$ be an open subset of M. A **basic semianalytic subset of** $\mathcal{W}$ *is a subset of the form $V(f_1, \dots f_k) \cap \{x \in \mathcal{W} \mid g_\lambda(x) \neq 0, \lambda = 1, \dots, l\}$, where $f_1, \dots, f_k, g_1, \dots, g_l \in \mathcal{O}_{\mathcal{W}}^{\text{anal}}$.*

A subset X in M is **semianalytic** *if and only if, for all $p \in M$, there exists a open neighborhood $\mathcal{W}$ of p such that $\mathcal{W} \cap X$ is a finite union of basic semianalytic subsets of $\mathcal{W}$.*

If X is a semianalytic subset of M, and Y is a semianalytic subset of an analytic manifold N, then a function $f : X \to Y$ is **semianalytic** *if and only if the graph of f is a semianalytic subset of $M \times N$.*

We summarize a number of properties of semianalytic sets below. The first item is an easy exercise and the others can be found in [Bie88], §2.

Theorem 2.33 *Let $\mathfrak{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let X be a semianalytic subset of M*

1. *The collection of semianalytic subsets of M is closed under finite unions, finite intersections, and taking complements.*
2. *Every connected component of X is semianalytic.*
3. *The family of connected components of X is locally finite.*
4. *X is locally connected.*
5. *The closure and interior of X is semianalytic. In fact, if $\mathfrak{K} = \mathbb{C}$, $\overline{X}$ is an analytic subset of M.*

Remark 2.34 Later, we shall see that Item 4 above can be substantially strengthened; semianalytic subsets are, in fact, locally contractible.

The following is Corollary 2.9 of [Bie88].

Proposition 2.35 *Let $\mathfrak{K} = \mathbb{R}$. Let X be a semianalytic subset of M.*

1. *Let V be a semianalytic subset of M such that $V \subseteq X$, and V is open in X. Then, for all $x \in X$, there exists an open neighborhood $\mathcal{W}$ of x in M such that $\mathcal{W} \cap V$ is a finite union of sets of the form*
$$\mathcal{W} \cap \{x \in X \mid f_1(x) > 0, \dots, f_k(x) > 0\},$$
where $f_1, \dots, f_k \in \overset{\text{anal}}{\mathcal{O}_{\mathcal{W}}}$.
2. *If X is closed in M, then, for all $x \in X$, there exists an open neighborhood $\mathcal{W}$ of x in M such that $\mathcal{W} \cap X$ is a finite union of sets of the form*
$$\{x \in \mathcal{W} \mid f_1(x) \geq 0, \dots, f_k(x) \geq 0\},$$
where $f_1, \dots, f_k \in \overset{\text{anal}}{\mathcal{O}_{\mathcal{W}}}$.

The following lemma is an extremely useful tool; see [Mil68], §3 and [Loo84], §2.1. The complex analytic statement uses Lemma 3.3 of [Mil68]. Below, $\overset{\circ}{\mathbb{D}}_\delta$ denotes an open disk of radius $\delta > 0$ centered at the origin in $\mathbb{C}$.

Lemma 2.36 (Curve Selection Lemma) *Let $\mathfrak{K} = \mathbb{R}$, and let $p \in M$. Let Z be a semianalytic subset of M such that $p \in \overline{Z}$. Then, there exists a real analytic curve $\gamma : [0, \delta) \to M$ with $\gamma(0) = p$ and $\gamma(t) \in Z$ for $t \in (0, \delta)$.*

Let $\mathfrak{K} = \mathbb{C}$, and let $p \in M$. Let Z be a semianalytic subset of M such that $p \in \overline{Z}$. Then, there exists a complex analytic curve $\gamma : \overset{\circ}{\mathbb{D}}_\delta \to M$ with $\gamma(0) = p$ and $\gamma(t) \in Z$ for $t \in \overset{\circ}{\mathbb{D}}_\delta - \{\mathbf{0}\}$.

Theorem 2.37 *Let $f \in \overset{\text{anal}}{\mathcal{O}_M}$. Let $p \in M$. Then, there exists an open neighborhood $\mathcal{W}$ of p in M such that the critical locus $\mathcal{W} \cap \Sigma f \subseteq f^{-1}(f(p))$, i.e., locally in M, the critical values of f are isolated.*

Proof. Suppose to the contrary that, for every open neighborhood $\mathcal{W}$ of p in M, $\mathcal{W} \cap \big(\Sigma f - f^{-1}(f(p))\big) \neq \emptyset$, i.e., that $p \in \overline{\Sigma f - f^{-1}(f(p))}$. Let γ denote either a real or complex curve as guaranteed by the Curve Selection Lemma; so, $\gamma(0) = p$ and $\gamma(t) \in \Sigma f - f^{-1}(f(p))$ for small $t \neq 0$. Then, the derivative $(f \circ \gamma)'(t)$ is an analytic function which is zero for all small $t \neq 0$, and so must also be zero at $t = 0$. Therefore, $(f \circ \gamma)(t)$ must be constant. As $f(\gamma(0)) = f(p)$, it follows that $f(\gamma(t)) = f(p)$ for all small t, i.e., $\gamma(t) \in f^{-1}(f(p))$ for all small t. This is a contradiction. □

Remark 2.38 In the algebraic setting, there is a much stronger result. For an algebraic set $X \subseteq \mathfrak{K}^n$, recall the definition of $\overset{\circ}{X}{}^{\text{alg}}$ from Remark 2.13. The following is Corollary 2.8 of [Mil68]:

Let X be an algebraic subset of $\mathfrak{K}^n$, and let $f \in \mathcal{O}_{\mathfrak{K}^n}$. Then, the restriction of the polynomial map f to the set $\overset{\circ}{X}{}^{\text{alg}}$ has a finite number of critical values.

Definition 2.39 *Let X be a semianalytic subset of M.*

Then, we define **smooth points**, **regular points**, **singular points**, *and* **exceptional points** *exactly as we did for analytic subsets in Definition 2.12, and we use the same notations.*

As we shall see below (Corollary 2.48), the smooth points of a semianalytic set are dense, and so we define the **dimension of** X, *and the* **dimension of** X **at a point** *exactly as we did for analytic subsets in Definition 2.15.*

Remark 2.40 As with analytic sets, it follows at once from the definitions that, for $0 \leq d \leq n$, $\overset{\circ}{X}{}^{(d)}$ is open in X and, hence, that $\overset{\circ}{X}$ and X_{reg} are open in X. Thus, ΣX and X_{exc} are closed in X.

The following result is from [Ło65], and [Bie88], Theorem 7.2 and Remark 7.3..

Theorem 2.41 *Let X be a non-empty semianalytic subset of M. Suppose that $0 \leq d \leq n$. Then, $\overset{\circ}{X}{}^{(d)}$ is semianalytic and, thus, so are $\overset{\circ}{X}$, ΣX, X_{reg}, and X_{exc}. In addition,* $\dim X_{\text{exc}} < \dim X$.

Partitions and Stratifications

For the remainder of this lecture, it will be necessary to consider both the smooth and semianalytic case. **Thus, below, when we are in the smooth case, X will denote a subset of a smooth, connected ambient manifold M and, when we are in the analytic/semianalytic case, X will denote a semianalytic subset of the connected analytic manifold M.**

If N is a smooth (resp., analytic) manifold, then we define $f : X \to N$ to be *smooth* (resp., *analytic*) if and only if it extends to a smooth (resp., analytic) function on an open subset of M.

One approach to describing a singular subset $X \subseteq M$ is to "subdivide" or "partition" the set into "smooth pieces", and then try to describe how these pieces are put together to form the analytic subset. As a first attempt at such a partitioning, one would proceed inductively: write $X = \overset{\circ}{X} \cup \Sigma X$, and then write $\Sigma X = (\overset{\circ}{\Sigma X}) \cup \Sigma(\Sigma X)$, and so on. If X is complex analytic, then Item 1 of Theorem 2.29 tells us that ΣX is complex analytic, and so this "inductive" process works well. However, Example 2.31 tells us that, for a real analytic X, ΣX need not be real analytic. On the other hand, Theorem 2.41 says that if X is semianalytic, then ΣX is also semianalytic. Thus, we make the following definition (see [Bie88]).

Definition 2.42 *A collection $\mathcal{S} := \{S_\alpha\}_\alpha$ of non-empty subsets of M is a* **smooth (resp., semianalytic) partition of** X *if and only if*

1. *X is the disjoint union of the S_α;*
2. *Each S_α is a smooth (resp., analytic) submanifold of M and is a connected (semianalytic) subset of M;*
3. *$\mathcal{S}$ is locally finite.*

Given a smooth (resp., semianalytic) partition $\mathcal{S}$ of X, we refer to the elements of $\mathcal{S}$ as **strata of** X.

A **smooth (resp., semianalytic) stratification** *$\mathcal{S}$ of X is a smooth (resp., semianalytic) partition of X which satisfies (the Condition of the Frontier): if $S_\alpha, S_\beta \in \mathcal{S}$, $S_\alpha \neq S_\beta$, and $S_\alpha \cap \overline{S_\beta} \neq \emptyset$, then $S_\alpha \subseteq \overline{S_\beta}$ and* $\dim S_\alpha < \dim S_\beta$.

A smooth (resp., semianalytic) partition/stratification $\mathcal{S}'$ of X is a **refinement** *of a smooth (resp., semianalytic) partition/stratification $\mathcal{S}$ of X if and only if the strata in $\mathcal{S}'$ are unions of the strata in $\mathcal{S}$.*

If X is a complex analytic subset of M, and $\mathcal{S}$ is a semianalyic stratification of X, then, for all $S \in \mathcal{S}$, in addition to S being a complex analytic submanifold of M, $\overline{S}$ is an irreducible complex analytic subset of M, and $\overline{S} - S$ is a complex analytic subset of M. Therefore, we refer to semianalytic stratifications of complex analytic subsets X of M simply as **(complex) analytic stratifications of** X.

Note that, in the Condition of the Frontier, it is irrelevant whether the closures are in X or in M; however, using closures in X, the Condition of the Frontier says precisely that the closure of a stratum S_α is equal to the union of S_α and smaller-dimensional strata.

Also, note that, in our definition, we have required the strata to be connected; not all authors require this.

Example 2.43 Consider the analytic set $X := V(xy) \subseteq \mathbb{R}^2$ consisting of the x- and y-axes. Then,

$$\mathcal{S} := \big\{V(x),\ \{(x,0) \mid x > 0\},\ \{(x,0) \mid x < 0\}\big\}$$

is a semianalytic partition, but is **not** a stratification. One must refine the partition by including the origin, in order to make the Condition of the Frontier hold. Thus,

$$\mathcal{S}' := \big\{\{(0,y) \mid y > 0\},\ \{(0,y) \mid y < 0\},\ \{(x,0) \mid x > 0\},\ \{(x,0) \mid x < 0\},\ \{\mathbf{0}\}\big\}$$

is the "best", i.e., most coarse, stratification of X.

Definition 2.44 *Let $\mathcal{S}$ be a partition of X, let N be a smooth (resp., analytic) manifold, and let $f : X \to N$ be a smooth (resp., analytic) function. Then, the* **stratified critical locus of** *f (with respect to $\mathcal{S}$) is $\Sigma_{\mathcal{S}} f := \bigcup_{S\in\mathcal{S}} \Sigma(f_{|S})$. Naturally, we define a* **stratified critical value of** *f (with respect to $\mathcal{S}$) to be be an element of $f(\Sigma_{\mathcal{S}} f)$.*

We leave the proof of the following generalization of Theorem 2.37 as an exercise.

Theorem 2.45 *Let $\mathcal{S}$ be a semianalytic partition of X, and let $f : X \to \mathfrak{K}$ be an analytic function. Let $p \in X$. Then, there exists an open neighborhood $\mathcal{W}$ of p in X such that $\mathcal{W} \cap \Sigma_{\mathcal{S}} f \subseteq f^{-1}(f(p))$, i.e., locally in X, the stratified critical values of f are isolated.*

If, in addition, f is a proper function, then the set of stratified critical values of f is discrete.

The following theorem is in [Ło65] and is Corollary 2.11 of [Bie88].

Theorem 2.46 *Let $\{X_i\}_i$ be a locally finite collection of semianalytic subsets of M. Then, there exists a semianalytic stratification $\mathcal{S} := \{S_\alpha\}_\alpha$ of M which is compatible with $\{X_i\}_i$, i.e., each X_i is a union of elements of $\mathcal{S}$.*

Remark 2.47 Let X be a semianalytic subset of M. Then, one easily concludes from Theorem 2.46 that there is a semianalytic stratification $\mathcal{S}$ of M such that $M - X$ and the connected components of $\overset{\circ}{X}$ are strata, and X is a union of strata. Therefore, $\mathcal{S}' := \{S \in \mathcal{S} \mid S \subseteq X\}$ is a semianalytic stratification of X.

As an example of how one can use Theorem 2.46, we leave it as an exercise for the reader to prove:

Corollary 2.48 *Let X be a semianalytic subset of M. Let $\mathcal{S}$ be a semianalytic stratification of X. Let $p \in X$, and let S be a stratum in $\mathcal{S}$ such that S has maximal dimension among all strata contained p in their closures. Then, $S \subseteq \overset{\circ}{X}$.*

In particular, $\overset{\circ}{X}$ is dense in X.

A stratification is useful since it decomposes a space into pieces which are smooth manifolds. However, to understand the original space better, one would like to know how a given stratum behaves as it approaches points in another stratum. The classic conditions to desire/require are the *Whitney conditions*: Whitney's condition a) and condition b). See [Whi65b], [Whi65a], [Mat70], and [Gor88].

Definition 2.49 *Let $\mathcal{S}$ be a smooth (resp., semianalytic) partition of X. Let $S_\alpha, S_\beta \in \mathcal{S}$, and let $p \in S_\alpha$. Fix a local coordinate system for M at p.*

A **Whitney a) sequence in** S_β **at** p *is a sequence of points $p_i \in S_\beta$ such that $p_i \to p$ and the tangent spaces $T_{p_i}S_\beta$ converge to some $\mathfrak{T}$ in the appropriate Grassmanian.*

A **Whitney b) pair of sequences in** (S_β, S_α) **at** p *is a pair of sequences of points $p_i \in S_\beta$ and $q_i \in S_\alpha$ such that $\{p_i\}$ is a Whitney a) sequence in S_β at p, $q_i \to p$ and the lines (using our fixed coordinate system) $\overline{p_iq_i}$ converge to some line $\mathfrak{l}$.*

The pair (S_β, S_α) **satisfies Whitney's condition a) at** p *if and only if, for all Whitney a) sequences $\{p_i\}$ in S_β at p, $T_pS_\alpha \subseteq \lim_{i\to\infty} T_{p_i}S_\beta$.*

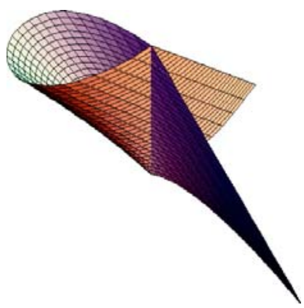
The pair (S_β, S_α) **satisfies Whitney's condition b) at** p *if and only if, for all pairs of Whitney b) sequences $\{p_i\}$, $\{q_i\}$ in (S_β, S_α) at p, $\lim_{i\to\infty} \overline{p_iq_i} \subseteq \lim_{i\to\infty} T_{p_i}S_\beta$.*

The Whitney conditions are independent of the choice of local coordinates on M, and are vacuously satisfied if $p \notin \overline{S_\beta}$.

The smooth (resp., semianalytic) partition $\mathcal{S}$ is a **smooth (resp., semianalytic) Whitney a) partition of X** *if and only if, for all $S_\beta, S_\alpha \in \mathcal{S}$, the pair (S_β, S_α) satisfies Whitney's condition a) at each point of S_α. If $\mathcal{S}$ is, in fact, a stratification, we naturally refer to $\mathcal{S}$ as a* **Whitney a) stratification**.

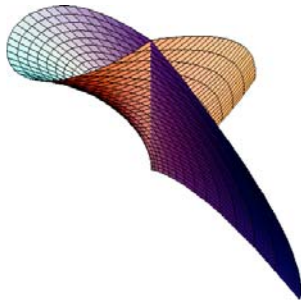
The smooth (resp., semianalytic) partition $\mathcal{S}$ is a **smooth (resp., semianalytic) Whitney b) partition of X** *if and only if, for all $S_\beta, S_\alpha \in \mathcal{S}$, the pair (S_β, S_α) satisfies Whitney's condition b) at each point of S_α. By [Mat70], Proposition 2.4 and Corollary 10.5, a Whitney b) partition also satisfies Whitney's condition a) and the condition of the frontier; hence, we generally refer to a Whitney b) partition as a* **Whitney stratification**.

Example 2.50 Consider a family of nodes degenerating to a cusp, given by $V(f) \subseteq \mathfrak{K}^3$, where $\mathfrak{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and $f = y^2 - x^3 - tx^2$.



Then, $\Sigma V(f) = V(x, y)$, and $\{V(f) - V(x, y), V(x, y)\}$ is a semianalytic stratification of $V(f)$. However, this stratification is **not** a Whitney a) stratification; the tangent planes to $V(f) - V(x, y)$ along the line where $y = 0$ and $x = -t$ approach a plane which does not contain $T_{\mathbf{0}}V(x, y)$.

Now consider another family of nodes degenerating to a cusp, given by $V(g) \subseteq \mathfrak{K}^3$, where $g = y^2 - x^3 - t^2x^2$.



Then, $\Sigma V(g) = V(x, y)$, and $\{V(g) - V(x, y), V(x, y)\}$ is a semianalytic Whitney a) stratification of $V(g)$. However, this stratification is **not** a Whitney stratification. We leave it as an exercise for the reader to show that Whitney's condition b) fails at the origin.

The following proposition is an easy exercise.

Proposition 2.51 *Let N be a smooth manifold and let $f : X \to N$ be a smooth function on X. Suppose that $\mathcal{S}$ is a Whitney a) partition of X. Then, $\Sigma_{\mathcal{S}} f$ is closed in X.*

The following strengthened version of Theorem 2.46 follows from [Ło65].

Theorem 2.52 *Let $\{X_i\}_i$ be a locally finite collection of semianalytic subsets of M. Then, there exists a semianalytic Whitney stratification $\mathcal{S} := \{S_\alpha\}_\alpha$ of M which is compatible with $\{X_i\}_i$, i.e., each X_i is a union of elements of $\mathcal{S}$.*

Remark 2.53 In the general smooth case, one is not guaranteed that Whitney stratifications exist. We could look at *subanalytic* subsets of M, for which *subanalytic Whitney stratifications* always exist (see [Har75] and [Hir73]), but we do not wish to discuss this generalization here.

Henceforth, **we fix a smooth (resp., semianalytic) Whitney stratification $\mathcal{S}$ of** X. In the smooth case, the existence of a Whitney stratification is a new condition which we place on the space X.

The following proposition is a good exercise.

Proposition 2.54 *Let N be a smooth (resp., an analytic) submanifold of M which transversely intersects a stratum $S_0 \in \mathcal{S}$ at a point $p \in S_0$.*

Then, there exists an open neighborhood $\mathcal{U}$ of p in M such that, for all $S \in \mathcal{S}$, $\mathcal{U} \cap N$ transversely intersects $\mathcal{U} \cap S$ in $\mathcal{U}$; in addition, in every such neighborhood $\mathcal{U}$, $\{\mathcal{U} \cap S \cap N \mid S \in \mathcal{S}\}$ is a smooth (resp., semianalytic) Whitney stratification of $\mathcal{U} \cap X \cap N$.

We naturally refer to the stratification $\{\mathcal{U} \cap S \cap N \mid S \in \mathcal{S}\}$ above as the *stratification induced by the transverse intersection.*

We now wish to consider definitions that involve "closed balls in M". Locally, we could simply use closed balls with respect to any coordinate choice. However, we wish to compare the structure of Whitney stratified spaces at different points in a given stratum, and thus will need to take closed balls at various points.

Hence, **we now assume that** M **is endowed with a Riemannian metric**. By taking infimums of lengths of piecewise smooth paths between pairs of points, this Riemannian metric induces a global (topological) metric $r : M \times M \to \mathbb{R}$. The closed ball of radius $\epsilon > 0$ centered at a point $p \in M$ is, of course, $B_\epsilon(p) := \{x \in M \mid r(x,p) \leq \epsilon\}$. We let $S_\epsilon(p)$ denote the sphere $\partial B_\epsilon(p)$.

We shall also need to use the notion of the abstract *cone* on a set. If Y is a topological space, then, by cone(Y), we shall mean the pair of spaces $(Y \times [0,1]/Y \times \{0\}, Y \times \{0\})$, where $Y \times \{0\}$ is the equivalence class which is normally referred to as *the cone point*.

The following result follows from [Ło65], [Mat70], and §1.4 of [Gor88].

Theorem 2.55 *Let $p \in M$. Then, for all sufficiently small $\epsilon > 0$, $S_\epsilon(p)$ transversely intersects all of the strata of $\mathcal{S}$, and there is a homeomorphism $h_\epsilon : \mathrm{cone}(S_\epsilon(p) \cap X) \to (B_\epsilon(p) \cap X, \{p\})$ which "preserves the strata", i.e., for all $S \in \mathcal{S}$, $h_\epsilon((S_\epsilon(p) \cap S) \times (0,1]) \subseteq S$, and which is a diffeomorphism when restricted to each $(S_\epsilon(p) \cap S) \times (0,1)$ for $S \in \mathcal{S}$. In particular, X is locally contractible.*

The stratified homeomorphism-type of $S_\epsilon(p) \cap X$ is independent of the choice of the Riemannian metric on M and of the choice of small ϵ, i.e., if $S_\epsilon(p)$ and $S'_{\epsilon'}(p)$ are spheres centered at p with respect to possibly different Riemannian metrics and of different radii, then, for both ϵ and ϵ' sufficiently small, there is a homeomorphism $k : S_\epsilon(p) \cap X \to S'_{\epsilon'}(p) \cap X$ such that, for all $S \in \mathcal{S}$, $k(S_\epsilon(p) \cap S) = S'_{\epsilon'}(p) \cap S$ and $k_{|_{S_\epsilon(p) \cap S}}$ is a diffeomorphism.

Definition 2.56 *The stratified homeomorphism-type of $S_\epsilon(p) \cap X$ in Theorem 2.55 is called the* **real link of** X **at** p.

Note that Theorem 2.55 implies that $B_\epsilon(p) \cap X$, itself, has a canonical Whitney stratification for small $\epsilon > 0$.

We now wish to give a substantial generalization of the Theorem of Ehresmann (Theorem 1.12). First, we need another definition. Let N be a smooth manifold.

Definition 2.57 *Let $f : X \to N$ be a smooth function on X. Then, f is a* **stratified submersion** *(with respect to our fixed stratification $\mathcal{S}$) if and only if, for all $S \in \mathcal{S}$, $f_{|_S}$ is a submersion.*

The complete proof of the following theorem can be found in [Mat70].

Theorem 2.58 (Thom's first isotopy lemma) *Suppose that $f : X \to N$ is a smooth, proper, stratified submersion. Then, f is a (topological) locally trivial fibration, and the local trivializations can be chosen to respect the strata and to be diffeomorphisms when restricted to strata.*

We wish to be clear about the conclusion of Thom's first isotopy lemma. Suppose that $f : X \to N$ is a smooth, proper, stratified submersion, and let $q \in \operatorname{im} f$. By Proposition 2.54, $\{S \cap f^{-1}(q) \mid S \in \mathcal{S}\}$ is a smooth Whitney stratification of $f^{-1}(q)$. The conclusion of Theorem 2.58 is that there exists an open neighborhood $\mathcal{U}$ of q in N and a homeomorphism $h : f^{-1}(\mathcal{U}) \to \mathcal{U} \times f^{-1}(q)$ such that $f_{|_{f^{-1}(\mathcal{U})}} = \pi \circ h$, where π is the projection from $\mathcal{U} \times f^{-1}(q)$ onto $\mathcal{U}$; moreover, for all $S \in \mathcal{S}$, the restriction of h is a diffeomorphism from $S \cap f^{-1}(\mathcal{U})$ to $\mathcal{U} \times (S \cap f^{-1}(q))$. Below, we will describe this type of situation by saying that $f^{-1}(\mathcal{U})$ *and $\mathcal{U} \times f^{-1}(q)$ have the same stratified topological-type, and the same smoothness structure along the strata.*

As we discussed briefly in Lecture 1, the Theorem of Ehresmann is proved by integrating a smooth vector field. Thom's first isotopy lemma is also proved by integrating a vector field – one which is smooth along each stratum, but is only continuous on all of X.

The following corollary tell us that X is topologically trivial along each Whitney stratum, and that the topological-type is determined by the *normal slice to the stratum*; see [Gor88], §1.4. Recall that our strata are assumed to be connected.

Corollary 2.59 *Let $p \in S_0 \in \mathcal{S}$. Let $\mathcal{U}$ be an open neighborhood of p in M, and let N be a smooth submanifold of $\mathcal{U}$, which transversely intersects all of the strata of $\mathcal{S}$ in $\mathcal{U}$, such that $N \cap S_0 = \{p\}$ (and, hence, $\dim N = \dim M - \dim S$).*

Then, for all sufficiently small $\epsilon > 0$, the stratified topological-type and smoothness structure along the strata of the pair $(N \cap X \cap B_\epsilon(p), N \cap X \cap S_\epsilon(p))$ is independent of the point p in S_0, the choice of the Riemannian metric on M, the transverse submanifold N, and ϵ.

In addition, for a fixed choice of all of the above data, there exists a (non-open) neighborhood T of p in X, an open neighborhood $\mathcal{W}$ of p in S_0, and a stratum-preserving homeomorphism $h : \mathcal{W} \times \big(N \cap X \cap B_\epsilon(p)\big) \to T$ which is a diffeomorphism when restricted to each stratum and is the "identity map" on $\{p\} \times \big(N \cap X \cap B_\epsilon(p)\big)$. In particular, the stratified topological-type of X is trivial along a Whitney stratum.

Definition 2.60 *The $N \cap X \cap B_\epsilon(p)$ above (or, sometimes, simply N itself) is referred to as a* **normal slice to S_0 at** *p, and $N \cap X \cap S_\epsilon(p)$ is called the* **real link of S_0 at** *p (with respect to N at radius ϵ). The stratified topological-type, and smoothness structure along the strata, of the pair $(N \cap X \cap B_\epsilon(p), N \cap X \cap S_\epsilon(p))$ is called the* **normal data of S_0** *(in X).*

Remark 2.61 Suppose that $p \in S_0 \in \mathcal{S}$. Then we may simply declare $\{p\}$ to be a "point-stratum" by refining the Whitney stratification $\mathcal{S}$; one defines a new Whitney stratification

$$\mathcal{S}' := \{S \in \mathcal{S} \mid S \neq S_0\} \cup \{S_0 - \{p\}, \{p\}\}.$$

In this sense, the real-link defined in Definition 2.56 can be considered the real link of a stratum.

Basic Stratified Morse Theory

Thom's first isotopy lemma is a generalization of the Theorem of Ehresmann to the Whitney stratified case. For the remainder of this lecture, we will discuss Goresky and MacPherson's generalization of Morse Theory to the Whitney stratified case, as presented in [Gor88].

It is easy to state an intuitive form of the main theorem of stratified Morse Theory: at a well-behaved isolated critical point p of a smooth function f on a Whitney stratified space, the local Morse data, which measures how the topology changes at p with respect f, is a product of tangential Morse data and Morse data which is normal to the stratum containing p. Making this statement precise is the goal of the remainder of this lecture.

We now assume that X is a semianalytic subset of a real analytic Riemannian manifold M, that $\mathcal{S}$ is a semianalytic Whitney stratification of X, and that $f : X \to \mathbb{R}$ is a proper function which extends to a real analytic function $\tilde{f}$ on an open neighborhood of X in M. We fix an $S_0 \in \mathcal{S}$ and a point $p \in S_0$.

Remark 2.62 We could, in fact, deal with a more general situation; the condition that we need is that p is a *nondepraved* critical point of f; see [Gor88], §2.3. However, we do not wish to go into the details of this definition. By §2.4 and §2.6 of [Gor88], isolated critical points of real analytic functions on real analytic manifolds are nondepraved.

Definition 2.63 *The point p is* **nondegenerate** *(with respect to f and $\mathcal{S}$) if and only if, for all $S \in \mathcal{S}$ such that $S \neq S_0$ and $S_0 \subseteq \overline{S}$, for all sequences $p_i \in S$ such that $p_i \to p$ and $T_{p_i}S$ converges to some $\mathfrak{T}$, it follows that $d_p\tilde{f}(\mathfrak{T}) \not\equiv 0$.*

The point of saying that p is nondegenerate is that it implies that small perturbations of f will not have critical points near p, unless those critical points are on S_0 itself.

We return to our notation from Lecture 1; for $b \in \mathbb{R}$, we let $X_b := f^{-1}((-\infty, b])$. In addition, if $a, b \in \mathbb{R}$, we let $X_{[a,b]} := f^{-1}([a, b])$.

A technical issue arises in what follows: if D is an arbitrary subset of X, we still wish to keep track of the intersections of strata with D, even when this does not produce a stratification of D.

Definition 2.64 *Let $A, B \subseteq X$.*

Then, an **$\mathcal{S}$-function** *$g : A \to B$ is a function such that, for all $S \in \mathcal{S}$, $g(A \cap S) \subseteq B \cap S$. Naturally, an* **$\mathcal{S}$-homeomorphism** *$h : A \to B$ is a homeomorphism such that h and h^{-1} are $\mathcal{S}$-functions, i.e., for all $S \in \mathcal{S}$, $h(A \cap S) = B \cap S$.*

We say that A and B have the same $\mathcal{S}$-topological-type if and only if there exists an $\mathcal{S}$-homeomorphism $h : A \to B$.

We now assume that p is a nondegenerate, isolated critical point of $f_{|_{S_0}}$, and that p is the only stratified critical point with critical value $v := f(p)$. Let $\delta > 0$ be small enough so that the only stratified critical value of f in $[v - \delta, v + \delta]$ is v (which we may assume by Theorem 2.45); **thus, we are assuming that $\Sigma_{\mathcal{S}} f \cap f^{-1}([v - \delta, v + \delta]) = \{p\}$.** Note that this implies that $f^{-1}(v - \delta)$, $f^{-1}(v + \delta)$, $X_{v-\delta}$, and $X_{v+\delta}$ all have induced Whitney stratifications.

The following is a combination of Propositions 3.5.3 and 3.6.2 of [Gor88].

Theorem 2.65 *For all sufficiently small $\epsilon > 0$, for all sufficiently small $\delta' > 0$ (small compared to ϵ), the $\mathcal{S}$-topological-type of the pair $(B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, B_\epsilon(p) \cap X_{v-\delta'})$ is independent of the choice of the Riemannian metric and the choices of ϵ and δ'.*

If N is a smooth (not necessarily analytic) normal slice to S_0 at p, then, for all sufficiently small $\epsilon > 0$, for all sufficiently small $\delta' > 0$, the $\mathcal{S}$-topological-type of the pair

$$(N \cap B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, N \cap B_\epsilon(p) \cap X_{v-\delta'})$$

is also independent of the choice of the Riemannian metric and the choices of ϵ and δ'.

Definition 2.66 *The $\mathcal{S}$-topological-type of the pair $(B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, B_\epsilon(p) \cap X_{v-\delta'})$ above is the* **local Morse data for f at p.**

The $\mathcal{S}$-topological-type of the pair $(N \cap B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, N \cap B_\epsilon(p) \cap X_{v-\delta'})$ above is the **normal Morse data for f at p.**

The local Morse data for $f_{|_{S_0}}$ at p is the **tangential Morse data for f at p.**

Note that, in Definition 2.66, one may assume that $\delta' \leq \delta$; moreover, by re-choosing δ after ϵ, one may assume that $\delta' = \delta$. However, below, it is important that δ does not depend on the choice of ϵ.

The theorem below is Theorem 3.5.4 of [Gor88]. The point of this theorem is that the change in the topological-type as one passes from $X_{v-\delta}$ to $X_{v+\delta}$ is a result of activity in a small neighborhood of the unique stratified critical point in $X_{[v-\delta, v+\delta]}$.

Theorem 2.67 *The stratified space $X_{v+\delta}$ is obtained from $X_{v-\delta}$ by a stratified attaching of the local Morse data for f at p; more precisely, with ϵ and δ' as in Theorem 2.65 and Definition 2.66, there is an $\mathcal{S}$-embedding $j : B_\epsilon(p) \cap X_{v-\delta'} \to X_{v-\delta}$ and an $\mathcal{S}$-homeomorphism (in the obvious sense) from $X_{v+\delta}$ to the identification space $X_{v-\delta} \cup_{B_\epsilon(p)\cap X_{v-\delta'}} \big(B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}\big)$.*

Therefore, to understand the change in the topology as one passes from $X_{v-\delta}$ to $X_{v+\delta}$, it is enough to understand the local Morse data of f at p.

We are now ready to state the main theorem of stratified Morse Theory. We use our notation from Theorem 2.65 and Definition 2.66. Recall that the product, $(P, Q) \times (J, K)$, of two pairs of spaces is defined to be $\big(P \times J, (P \times K) \cup (Q \times J)\big)$.

Theorem 2.68 (The Fundamental Theorem of Stratified Morse Theory) *Let (A, B) equal the local Morse data $(B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, B_\epsilon(p) \cap X_{v-\delta'})$ for f at p. Let (P, Q) equal the tangential Morse data*

$$(B_\epsilon(p) \cap (S_0)_{[v-\delta', v+\delta']}, B_\epsilon(p) \cap (S_0)_{v-\delta'})$$

for f at p. Let (J, K) equal the normal Morse data $(N \cap B_\epsilon(p) \cap X_{[v-\delta', v+\delta']}, N \cap B_\epsilon(p) \cap X_{v-\delta'})$ for f at p.

Then, there is a homeomorphism of pairs $h : (A, B) \to (P, Q) \times (J, K)$ such that, for all $S \in \mathcal{S}$, $h(A \cap S) \subseteq P \times S$, i.e., local Morse data is, up to homeomorphism, the product of the tangential Morse data and the normal Morse data.

The reader needs to understand the point of this theorem. The tangential Morse data is Morse data on an analytic manifold; this is the classical situation that we discussed in Lecture 1. The tangential Morse data at an isolated critical point is analyzed by perturbing f slightly (as in Theorem 1.21, for instance) and thereby "splitting" the isolated critical point into a collection of non-degenerate critical points, and then applying Theorem 1.17.

The normal Morse data is the really new piece of data that needs to be understood in order for one to apply the Fundamental Theorem of Stratified Morse Theory. This data can, in fact, be very complicated. However, a stunning thing happens in the case where X is a complex semianalytic subset of a complex analytic manifold, and is endowed with a complex analytic Whitney stratification: the normal Morse data is independent of the function f. This fact underlies much of the beauty of applications of stratified Morse Theory to complex stratified spaces.

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