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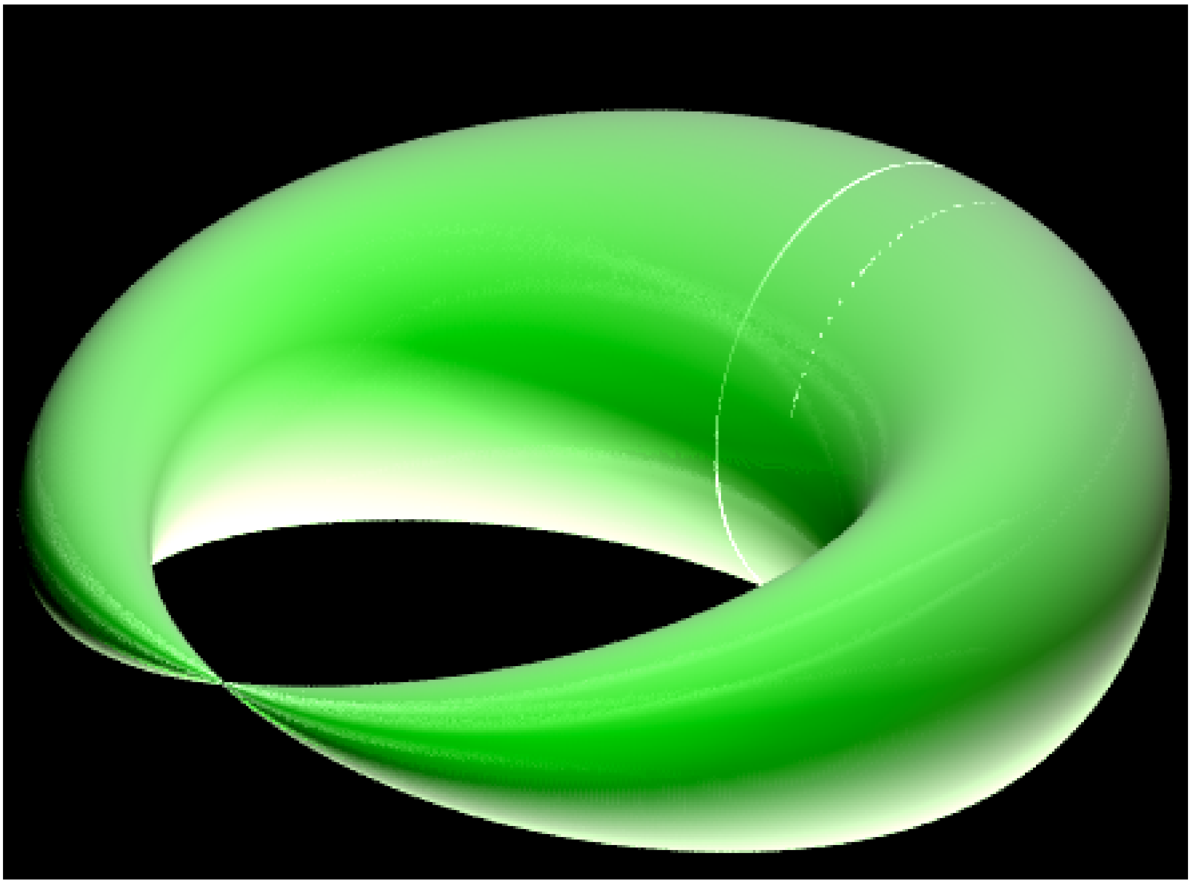

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1 Introduction

The Euler-Poincaré characteristic has been the first characteristic class to be introduced. For a triangulated (possibly singular) compact variety X without boundary, it has been defined, as

$$\chi(X) = \sum (-1)^i n_i \,,$$

where n_i is the number of i-dimensional simplices of the triangulation of X. It is also equal to $\sum(-1)^i b_i$ where b_i is the i-th Betti number, rank of $H_i(X)$. The Poincaré-Hopf theorem says that, if M is a (compact) manifold and v a continuous vector field with a finite number of isolated singularities a_k with indices $I(v, a_k)$, then

$$\chi(M) = \sum I(v, a_k) \,.$$

This means that the Euler-Poincaré characteristic is a measure of the obstruction to the construction of a non-zero vector field tangent to M.

During several years, the attractiveness of the axiomatic properties of Chern classes caused the viewpoint of obstruction theory to be somewhat forgotten. It is interesting to see that this viewpoint came back on the scene with the question of defining characteristic classes for singular varieties.

There are various definitions of characteristic classes for singular varieties. In the real case, there is a combinatorial definition, which simplifies the problem. In the complex case, the situation is more complicated (and certainly more interesting !), due to the fact that there is no combinatorial definition of Chern classes. Thinking of the obstruction theory point of view, one has to find a substitute to the tangent bundle; in fact there are various candidates to substitute the tangent bundle and to each of them corresponds a different definition of Chern class for singular varieties.

If X is a singular complex analytic variety, equipped with a Whitney stratification and embedded in a smooth complex analytic manifold M one can consider the union of tangent bundles to the strata, that is a subspace E of the tangent bundle to M. The space E is not a bundle but it generalizes the notion of tangent bundle in the following sense: A section of E over X is a section v of $TM|_X$ such that in each point $x \in X$, then $v(x)$ belongs to the tangent space of the stratum containing x. Such a section is called a stratified vector field over X. To consider E as the substitute to the tangent bundle of X and to use obstruction theory is the M.H. Schwartz point of view (1965, [Sc1]), in the case of analytic complex varieties. Another possibility is to consider the space of all possible limits of tangent vector spaces $T_{x_i}(X_{\mathrm{reg}})$ where x_i is a sequence of points in the regular part X_{reg} of X converging in $x \in X$. That point of view leads to the notion of Mather classes, which are an ingredient in the MacPherson definition in the case of algebraic complex varieties (1974, [MP]) . The other main ingredient for these classes is the notion of Euler local obstruction. Finally, when there exists a normal bundle N to X in M, for example in the case of local complete intersections, one can consider the virtual bundle $TM|_X \setminus N$ as a substitute to the tangent bundle of X. That point of view is the one of Fulton (1980, [Fu]).

There are many relations between the classes obtained by the previous constructions. First of all, the Schwartz and MacPherson classes coincide, via Alexander duality (1979, [BS]). The relation between Mather classes on the one side and Schwartz-MacPherson classes on the other side follows form the MacPherson's definition itself: His construction uses Mather classes, taking into account the local complexity of the singular locus along Whitney strata. This is the role of the local Euler obstruction.

A natural question raised to compare the Schwartz-MacPherson and the Fulton-Johnson classes. A result of Suwa [Su] shows that in the case of isolated singularities, the difference of these classes is given by the sum of the Milnor numbers in the singular points. It was natural to call Milnor classes the difference arising in the general case. This difference has been described by several authors by different methods (P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura).

In this lecture, one will see that all known characteristic classes for singular varieties contain (under suitable hypothesis) a cycle defined by the same formula issued from obstruction theory:

$$\boxed{\sum_{\substack{\sigma \subset X \\ \dim \sigma = 2(r-1)}} \alpha(\sigma) I(v^{(r)}, \hat{\sigma})\ \sigma}$$

for a suitable constructible function α (depending on the characteristic class) and with notations defined in the following.

2 Euler-Poincaré characteristic and Poincaré-Hopf Theorem

In this section, the varieties we consider are possibly singular varieties.

2.1 Euler-Poincaré characteristic

2.1.1 Combinatorial definition

History of characteristic classes begins with the discovery of the so-called Euler formula, published by Euler in 1758. For a 2-dimensional polyhedron P in $\mathbf{R}^3$, homeomorphic to the sphere S^2, one has

$$n_0 - n_1 + n_2 = 2$$

where n_0 is the number of vertices in P, n_1 is the number of segments and n_2 the number of faces.

For a general 2-dimensional polyhedron P in $\mathbf{R}^3$, the alternative sum

$$\chi(P) = n_0 - n_1 + n_2$$

is called Euler characteristic of P.

Poincaré generalized the result in 1893 for finite polyhedra P of any dimension and showed the real meaning of this "magic" number. Let us denote by n_i the number of i-dimensional simplices of a finite n-dimensional polyhedron P, the Euler-Poincaré characteristic of P is defined by

$$\chi(P) = \sum_i (-1)^i n_i.$$

The important result due to Poincaré is the following:

Theorem 2.1 (Poincaré) *Let (P, h) and (P', h') be two triangulations of the same space X, one has $\chi(P) = \chi(P')$.*

This number, independent of the triangulation of X is called Euler-Poincaré characteristic of X and denoted by $\chi(X)$. The result proves that Euler-Poincaré characteristic is a topological invariant of the space X. For example, the Euler-Poincaré characteristic of the sphere S^n is $\chi(S^n) = 1 + (-1)^n$. Also the Euler-Poincaré characteristic of the real projective space $\mathbf{P}^n$ is $\chi(\mathbf{P}^n) = 0$ if n is odd and $\chi(\mathbf{P}^n) = 1$ if n is even.

2.1.2 Betti numbers

The Betti numbers of P are defined as

$$\beta_i(P) = \mathrm{rk}\ (H_i(P, \mathbf{Z})) = \mathrm{rk}\ (F_i(P)).$$

where $F_i(P)$ is the free subgroup of $H_i(P, \mathbf{Z})$.

Noting that $\beta_i(P) = 0$ if $i > \dim P = n$, one has the Poincaré Theorem:

Theorem 2.2 (Poincaré Theorem) *Let P be a finite polyhedron in $\mathbf{R}^m$, with Betti numbers $\beta_i(P)$, one has*

$$\chi(P) = \sum_{i=0}^{n} (-1)^i \beta_i(P).$$

2.2 Poincaré-Hopf Theorem

2.2.1 The smooth case

The Poincaré-Hopf Theorem has been proved by Poincaré in 1885, in the 2-dimensional case, and by Hopf in 1927 for any dimension. This result is the first apparition of Euler-Poincaré characteristic in differential geometry, out of combinatorial topology.

Let us recall that a n-dimensional topological manifold is a topological space M satisfying the following property:

Property 2.3 For every point a in M, there is a neighborhood U_a of a in M homeomorphic to a ball $B^n \subset \mathbf{R}^n$ via a homeomorphism $\phi : B^n \to U_a$ such that $\phi(0) = a$ and the boundary of U_a, called the link of a, is homeomorphic to the sphere S^{n-1}.

Let M be a compact differentiable manifold of dimension n and let v be a continuous vector field on M with isolated singularities. A singularity a of the vector field v is a point in which the vector field is zero or is not defined. In such a point the index of the vector field v can be defined in the following way: Let us consider a ball $B(a)$ with center a sufficiently small so that a is the only singular point of v in $B(a)$. The vector field v is well defined and without singularity on $S(a) = \partial B(a)$. The sphere $S(a) \cong S^{n-1}$ is oriented, according to the orientation of $\mathbf{R}^n$. Let us define the Gauss map

$$\gamma : S(a) \cong S^{n-1} \longrightarrow S^{n-1}$$

by $\gamma(x) = v(x)/\|v(x)\|$. The degree of the Gauss map is geometrically the number of times the cycle $\gamma(S(a))$ recovers S^{n-1}, i.e. the degree of the map

$$\gamma_* : H_{n-1}(S^{n-1}) \cong \mathbf{Z} \longrightarrow H_{n-1}(S^{n-1}) \cong \mathbf{Z}.$$

We call it the index of v in the point a and we denote it by $I(v, a)$. This index does not depend on the choice of the ball $B(a)$ satisfying the previous conditions.

Theorem 2.4 [Poincaré-Hopf Theorem] *Let M be a compact oriented differentiable manifold, and let v be a continuous vector field on M with finitely isolated singularities a_i. One has*

$$\chi(M) = \sum_i I(v, a_i)$$

As important consequence of the Poincaré-Hopf Theorem, one has the following

Corollary 2.5 *If $\chi(M) \neq 0$, then any continuous vector field tangent to a surface M admits at least a singular point. Reciprocally, every compact manifold such that $\chi(M) = 0$ admits a continuous tangent vector field without singularities.*

2.2.2 The singular case

A singular variety is a variety which contains points for which the property 2.3 is not satisfied.

Examples of singular varieties are the following: The pinched torus : the pinched point a does not admit any neighborhood satisfying the property 2.3. In that case, the link of an "elementary neighborhood" of a is the union of two not connected circles. Another example is provided by the suspension of the torus. The two pinched points a and b of the suspension of the torus are singular points, in that case, the link of a (or b) is a torus, it is not a sphere.

If X is a singular variety, the Poincaré-Hopf Theorem fails to be true, the main reason is that there is no more tangent space in singular points. The definition of the index of a vector field in one of its singular points takes sense on a smooth manifold only. In particular the singular point must have a neighborhood isomorphic to the ball B^m and whose boundary is isomorphic to the sphere S^{m-1}. Let us consider the example of the pinched torus X in $\mathbf{R}^3$. The pinched point a is a singular point of X, in fact it constitutes the singular part of the pinched torus. The only 'natural' way to define an index of a vector field in the point a is to consider a vector field v defined in a ball $B^3(a)$ centered in a, in $\mathbf{R}^3$, with an isolated singularity in a, such that if $x \in X \setminus \{a\}$, then $v(x)$ is tangent to the smooth manifold $X \setminus \{a\}$ and such that v does not have other singularity in $B^3(a)$.

Let us consider two examples of such vector fields:

a) The vector field tangent to the parallels of the torus T determines on the pinched torus X a vector field v going inward the ball $B^3(a)$ along one of the two circles intersection of $\partial B^3(a)$ with X and going outward the ball along the other circle. On the one hand, this vector field, defined on $\partial B^3(a) \cap X$ is the restriction of a vector field w defined on $\partial B^3(a)$ and with index 0 in a. On the other hand, there is no more singularity of v on $X \setminus \{a\}$. In this case, the Poincaré-Hopf Theorem is not satisfied : one has

$$\chi(X) = 1 \neq 0 = I(w, a)$$

b) Let us consider a vector field v going outward the ball $B^3(a)$ along $\partial B^3(a)$ and tangent to X along the restriction $\partial B^3(a) \cap X$. This vector field has index $+1$ in the point a, it can be extended on the pinched torus as a continuous vector field without other singularity. In fact, one can define an extension v such that the angle of $v(x)$ with the tangent space to the meridian containing x goes down with the distance to a until to be 0 for the meridian opposed to a. In that case, the Poincaré-Hopf Theorem is valid:

$$\chi(X) = 1 = I(v, a)$$

The vector field v is the first example of M.H. Schwartz radial vector field, of which we will make a systematic study.

3 Review of obstruction theory

Let us recall the idea of constructing characteristic classes by obstruction theory, following Steenrod [Ste], part III.

The Poincaré-Hopf Theorem says that $\chi(M)$ is a measure of the obstruction for the construction of a vector field tangent to the manifold M. In the same way, the objective of the obstruction theory is to define invariant objects which evaluate the measure of the obstruction to the construction of linearly independent sections of vector bundles. In a more precise way, the objective is to answer to questions like:

Let E be a vector bundle of rank n on a variety X and fix r such that $1 \leq r \leq n$. Is it possible to construct r sections of E, linearly independent everywhere ?

It is obviously possible to define such sections on the 0-skeleton of a triangulation of X. So, the question becomes the following:

Performing the construction of r independent sections by increasing dimension of simplices of a triangulation of X, up to what dimension can we proceed ? When arriving to the obstruction dimension, is it possible to evaluate this obstruction ?

In a first step, we study the case of the (real) tangent bundle to a differentiable smooth manifold or the (complex) tangent bundle to an analytic complex manifold. We will denote by $\mathbf{K}$ the field $\mathbf{R}$ or $\mathbf{C}$, according to the situation.

Let M be a manifold of dimension m, over $\mathbf{K}$, endowed with an euclidean (or hermitian) metric. The tangent bundle to M, denoted by TM, is a $\mathbf{K}$-vector bundle of rank n, whose fiber in a point x of M is the tangent vector space to M in x, denoted by $T_x(M)$ and is isomorphic to $\mathbf{K}^m$. The vector bundle TM is locally trivial, i.e. there is a covering of M by open subsets such that the restriction of TM to U is isomorphic to $U \times \mathbf{K}^m$.

The objective is to evaluate the obstruction to the construction of r sections of TM linearly independent (over $\mathbf{K}$) in each point, i.e. an r-frame.

Let us consider the fiber bundle $T^r(M)$, with basis M, associated to TM and whose fiber in the point x of M is the set of r-frames of $T_x(M)$. This bundle is no more a vector bundle. The "typical" fiber of $T^r(M)$ is the set of all r-frames of $\mathbf{K}^m$, i.e. the Stiefel manifold $V_{m,r}(\mathbf{K})$. To construct r linearly independent sections of TM over a subset A of M is equivalent to construct a section of $T^r(M)$ over A.

Let us consider a triangulation (K) of M sufficiently small so that every simplex σ is contained in an open subset U over which $T^r(M)$ is trivial.

We are interested by the following question:

Let us suppose that there is a section v^r of $T^r(M)$ on the boundary $\partial\sigma$ of the k-dimensional simplex σ. Is it possible to extend this section in the interior of σ ? Is the answer is no, what is the obstruction for such an extension ?

The section v^r on the boundary of σ defines a map

$$\partial\sigma \xrightarrow{v^r} T^r(M)|_U \cong U \times V_{m,r}(\mathbf{K}) \xrightarrow{pr_2} V_{m,r}(\mathbf{K})$$

where pr_2 is the projection on the second factor. We obtain a map

$$S^{k-1} \cong \partial\sigma \xrightarrow{pr_2 \circ v^r} V_{m,r}(\mathbf{K})$$

which induces an element of $\pi_{k-1}(V_{m,r}(\mathbf{K}))$ denoted by $[\gamma(v^r,\sigma)]$. In order to answer to the previous question, we need to know the homotopy groups of $V_{m,r}(\mathbf{K})$.

If one has $[\gamma(v^r,\sigma)] = 0$, then, by classical homotopy theory, the map $S^{k-1} \to V_{m,r}(\mathbf{K})$ can be extended inside the ball B^k. In another words, the map $\partial\sigma \to V_{m,r}(\mathbf{K})$ can be extended inside σ.

$$
\begin{array}{ccc}
\partial\sigma \cong S^{k-1} & \longrightarrow & V_{m,r}(\mathbf{K}) \\
\big\downarrow & ? \nearrow & \\
\sigma \cong B^k & &
\end{array}
$$

In that case, there is no obstruction to the extension of the section v^r inside σ. This happens for example in the case $\pi_{k-1}(V_{m,r}(\mathbf{K})) = 0$.

The homotopy groups $\pi_{k-1}(V_{m,r}(\mathbf{K}))$ have been computed by Stiefel and by Whitney (see [Ste]) in the cases $\mathbf{K} = \mathbf{R}$ and $\mathbf{C}$. One has the following result:

Let $V_{m,r}(\mathbf{R})$ be the Stiefel manifold of r-frames in $\mathbf{R}^m$, one has:

$$
\pi_i(V_{m,r}(\mathbf{R})) = \begin{cases} 0 & \text{for } i < m-r \\ \mathbf{Z} & \text{for } i = m-r \text{ even or } i = m-1 \text{ if } r = 1 \\ \mathbf{Z}_2 & \text{for } i = m-r \text{ odd and } r > 1 \end{cases} \tag{3.1}
$$

For the Stiefel manifold of r-frames in $\mathbf{C}^m$, one has:

$$
\pi_i(V_{m,r}(\mathbf{C})) = \begin{cases} 0 & \text{for } i < 2m-2r+1 \\ \mathbf{Z} & \text{for } i = 2m-2r+1 \end{cases} \tag{3.2}
$$

Let us denote $2p = 2(m-r+1)$. A generator of $\pi_{2p-1}(V_{m,r}(\mathbf{C}))$ can be described in the following way: let us choose a $(r-1)$-frame in $\mathbf{C}^m$. It defines a $(r-1)$-subspace of $\mathbf{C}^m$ whose complementary is a complex space $\mathbf{C}^p$. The unit sphere in $\mathbf{C}^p$ denoted by S^{2p-1} is oriented, with orientation induced by the natural one of $\mathbf{C}^p$. Let us consider, for every point of the sphere, a r-frame consisting of the vector w and the fixed $(r-1)$-frame, one obtains an element of $V_{m,r}(\mathbf{C})$. The induced map from the oriented sphere S^{2p-1} to $V_{m,r}(\mathbf{C})$ defines a generator of $\pi_{2p-1}(V_{m,r}(\mathbf{C}))$.

4 Applications: Chern classes

We recall briefly the principle of the construction of Chern classes by obstruction theory. In this section, M is a complex analytic manifold of (complex) dimension m.

The result of the previous section implies that one can construct an r-frame $v^{(r)}$, i.e. a section of T^rM, by induction on the dimension of cells of the given cell decomposition of M without singularity up to the $(2m-2r+1)$-skeleton and with isolated singularities on the $2p = 2(m-r+1)$-skeleton. For each $2p$-cell $d = d(\sigma)$, the r-frame $v^{(r)}$ is well defined on the boundary $\partial d(\sigma)$ and can be extended in the interior of $d(\sigma)$ by homothety whose center is the center $\hat{\sigma} = d(\sigma) \cap \sigma$ of $d(\sigma)$, i.e. the barycenter of σ. The extension admits an isolated singularity in $d(\sigma)$ at that point. The index of the complex r-frame $v^{(r)}$ at its singular point $\hat{\sigma}$ in d is $I(v^{(r)}, \hat{\sigma}) = [v^{(r)}; \partial d] \in \mathbf{Z}$.

On can define a $2p$-cochain in $C^{2p}(K, \mathbf{Z})$, whose value on each $2p$-cell $d(\sigma)$ is $[v^{(r)}; \partial d]$. The cochain is in fact a cocycle and defines an element in $Z^{2p}(M; \mathbf{Z})$. One has:

Lemma 4.1 *Let us consider two r-fields defined by the previous construction, then the difference of the two corresponding cocycles is a coboundary.*

That justifies the following definition:

Definition 4.2 [Ch] The p-th (cohomology) *Chern class* of M, $c^p(M) \in H^{2p}(M; \mathbf{Z})$ is the cohomology class of the obstruction cocycle.

The Chern classes do not depend of the choices we made.

By Poincaré duality isomorphism, the image of $c^p(M)$ in $H_{2(r-1)}(M)$ is the $(r-1)$-st homology Chern class of M, represented by the cycle

$$\boxed{\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \hat{\sigma})\ \sigma}$$

In particular, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic $\chi(M)$.

5 Singular case : substitutes of the tangent bundle

In the case of a complex analytic singular variety X, there is no more tangent bundle TX. The different notions of Chern classes, in the singular setting, correspond to different notions of substitute to the tangent bundle. There are (at least) three ways to define such a substitute in the case of a singular variety X embedded in a manifold M:

1. let us consider the union $\mathcal{T}$ of tangent spaces to the strata of a stratification of X and consider the sections of TM whose images are in $\mathcal{T}$. This is the method used by M.H. Schwartz. She shows that it is not possible to process obstruction theory, using any vector fields, but that one has to use radial ones (see the previous example and section 6.1).

2. let us consider the set of all possible limits of tangent spaces to sequences of points in the regular part of X, that is the Nash transformation and one has the Nash bundle on it (see section 7.1).

3. let us consider the virtual bundle. That is the method used by Fulton (see section 9). If X is smooth, one has an exact sequence

$$0 \to TX \to TM|_X \to N_X M \to 0$$

where $N_X M$ is the normal bundle of X in M. In the case of a singular variety such that the normal bundle $N_X M$ exists (for instance hypersurfaces or local complete intersections), one can define the virtual bundle (in the Grothendieck group $KU(X)$) as

$$\tau_X = TM|_X - N_X M.$$

In the case of singular spaces, there are no characteristic classes in cohomology. The different notions define characteristic classes in homology.

6 The Schwartz classes

In the following, M will be a complex m-dimensional manifold equipped with an analytic stratification $\{V_i\}$: for every stratum V_i, the closure $\bar{V}_i$ and the boundary $\dot{V}_i = \bar{V}_i \setminus V_i$ are analytic sets, union of strata. We denote by $X \subset M$ a compact complex analytic variety stratified by $\{V_i\}$.

The first definition of Chern class for singular varieties was given in 1965 by M.H. Schwartz in two "Notes aux CRAS" [Sc1]. For this section, see [Sc1], [Sc2], [Sc3].

Definition 6.1 A *stratified vector field* v on X is a (continuous) section of the tangent bundle of M, $T(M)$, such that, for every $x \in X$, one has $v(x) \in T(V_{i(x)})$ where $V_{i(x)}$ is the stratum containing x.

6.1 Triangulations and cellular decompositions

Let $X \subset M$ be a singular n-dimensional complex variety embedded in a complex m-dimensional manifold. Let us consider a Whitney stratification $\{V_i\}$ of M [Wh] such that X is a union of strata and denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum.

The first nice observation of M.H. Schwartz concerns the triangulations:

We denote by (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. Each cell of (D) is transverse to the strata. This implies that if d is a cell of dimension $2p = 2(m - r + 1)$ and V_i is a stratum of dimension $2s$, then $d \cap V_i$ is a cell such that

$$\dim(d \cap V_i) = 2(s - r + 1)$$

That means that if d is a cell whose dimension is the dimension of obstruction to the construction of an r-frame tangent to M, then $d \cap V_i$ is a cell whose dimension is exactly the dimension of obstruction to the construction of an r-frame tangent to the stratum V_i.

The second nice construction of M.H. Schwartz is the construction of radial vector fields. Let us give an idea of that construction: Given the vector field w on a stratum V_i, we can firstly define, using Whitney property (a), a "parallel" extension which is a stratified vector field admitting "disks" of singularities, transverse to V_i in each singular point of w. Then, we can consider the vector field which is the gradient of the distance to the stratum V_i. Using the Whitney property (b), we deduce a "transverse" stratified vector field, which is zero along V_i and growing with the distance to the stratum. The sum of the parallel and of the transverse vector fields is the radial extension v. The radial extension of w is defined in a suitable geodesic tube (see [Sc2] (c) of the proof of Lemme 3.1.2). The radial extension v admits the same singularities than w in the geodesic tube.

One can perform the same construction for frames instead of vector fields, on a suitable skeleton of the dual cell decomposition and such that singularities remain singularities of the last vector of the frame. In fact, one has:

Proposition 6.2 *[Sc1], [Sc3] One can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

(i) $v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.

(ii) Let $a \in V_i \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_i. If $s > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_i \cap (D)^{2p}$ considered as an r-frame tangent to V_i. If $s = r-1$, then $I(v^{(r)}, a) = +1$.

(iii) Inside a $2p$-cell d which meets several strata, the only singularities of $v^{(r)}$ are inside the lowest dimensional one (in fact located in the barycenter of d).

(iv) The r-frame $v^{(r)}$ is pointing outwards a (particular) regular neighborhood U of X in M. It has no singularity on ∂U.

The procedure of the construction of radial frames is made by induction on the dimension of the strata, using the properties of Whitney stratifications for proving the existence of frames pointing outward regular neighborhoods and satisfying property (ii). For strata V_i^s of dimension $s = r-1$, the obstruction dimension to the construction of a r-frame tangent to the strata is 0. For each $(r-1)$-simplex σ of the triangulation of M contained in V_i^s, one can construct an r-frame tangent to the $2p$-cell $d(\sigma)$ dual of the simplex, with a singularity in the center $\hat{\sigma} = d(\sigma) \cap \sigma$ and pointing outwards along the boundary of the cell. This provides an r-frame pointing outwards the $2p$-skeleton of a tubular neighborhood of V_i^{2s}. Then one proceeds on the previous way, by induction on the dimension of strata. One obtains an r-frame on the $2p$-skeleton of a tubular neighborhood of X satisfying the properties of Proposition 6.2.

6.2 Obstruction cocycles and classes

Let us denote by $\mathcal{T}$ the tubular neighborhood of X in M consisting of the (D)-cells which meet X. Let us denote by d^* is the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\sum_{\substack{d(\sigma)\in\mathcal{T} \\ \dim d(\sigma)=2p}} I(v^{(r)}, \hat{\sigma})\; d^*(\sigma)$$

This cochain actually is a cocycle whose class $c^p(X)$ lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction along the rays of $\mathcal{T}$ and the second by excision (by $M \setminus \mathcal{T}$).

Definition 6.3 [Sc1],[Sc3] The p-th Schwartz class is the class

$$c^p(X) \in H^{2p}(M, M \setminus X).$$

The Schwartz classes do not depend on the different choices we made: stratification, triangulation, r-frame satisfying the previous conditions.

7 Euler local obstruction

Euler local obstruction has been defined by MacPherson [MP]. One will use the equivalent definition by Brasselet-Schwartz [BS].

7.1 Nash transformation

Let M be an analytic manifold, of complex dimension m. Let X be an subanalytic complex variety, $X \subset M$, of complex dimension n. Let us denote by $\Sigma = X_{\text{sing}}$ the singular part of X and by $X_{\text{reg}} = X \setminus \Sigma$ the regular part.

The Grassmanian of complex n-planes in $\mathbf{C}^m$ is denoted by $G(n,m)$. Let us consider the Grassmann bundle of n (complex) planes in $T(M)$, denoted by G. The fibre G_x over $x \in M$ is the set of n-planes in $T_x(M)$, it is isomorphic to $G(n,m)$. An element of G is denoted by (x, P) where $x \in M$ and $P \in G_x$.

On the regular part of X, one can define the Gauss map $\sigma : X_{\text{reg}} \longrightarrow G$ by

$$\sigma(x) = (x, T_x(X_{\text{reg}})).$$

Definition 7.1 The Nash transformation $\widetilde{X}$ is defined as the closure of the image of σ in G. It is equipped with a natural analytic projection $\nu : \widetilde{X} \longrightarrow X$.

$$\begin{array}{ccc} & & G \\ & \nearrow\sigma & \downarrow \\ X_{\text{reg}} & \hookrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{X} = \overline{\text{Im}\sigma} & \hookrightarrow & G \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array} \tag{7.3}$$

In general, $\widetilde{X}$ is not smooth. Nevertheless, it is an analytic variety and the restriction $\nu : \widetilde{X} \to X$ of the bundle projection $G \to M$ is analytic.

The fiber E_P of the tautological bundle E over G, in a point $(x, P) \in G$, is the set of the vectors v of the n-plane P.

$$E_P = \{v(x) \in T_xM \,:\, v(x) \in P, \quad x = \nu(P)\}$$

Let us define $\widetilde{E} = E|_{\tilde{X}}$, then $\widetilde{E}|_{\tilde{X}_{\rm reg}} = T(X_{\rm reg})$ where $\widetilde{X}_{\rm reg} = \nu^{-1}(X_{\rm reg}) \cong X_{\rm reg}$ and

$$\widetilde{E} = E \times_G \widetilde{X} = \{(v(x), \tilde{x}) \in E \times \widetilde{X} \ : v(x) \in \tilde{x}\}$$

$\tilde{x} \in \widetilde{X}$ is a n-complex plane in $T_x(M)$ and $x = \nu(\tilde{x})$.

One has a diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

An element of $\widetilde{E}$ is written (x, P, v) with $x \in X$, P is a n-plane in $T_x(M)$ and v is a vector in P.

We continue to denote by (V_i) a complex analytic stratification of X satisfying the *Whitney conditions*. The following lemma is fundamental for the understanding of the geometrical definition of the local Euler obstruction.

Lemma 7.2 ([BS], Proposition 9.1) *A stratified vector field v on a part $A \subset X$ admits a canonical lifting $\tilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{E}$.*

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\nu_*} & TM|_X \\ \tilde{v}\uparrow\downarrow & & v\uparrow\downarrow \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array} \qquad \nu_*(v(x), \tilde{x}) = v(\nu(\tilde{x})) = v(x).$$

Let us recall that a *radial* vector field v in a neighborhood of the point $\{0\} \in X$ is a stratified vector field so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outwards the ball B_ε along the boundary $S_\varepsilon = \partial B_\varepsilon$. By the Bertini-Sard theorem, S_ε is transverse to the strata V_i if ε is small enough, so the definition takes sense.

Theorem 7.3 Theorem-Definition [BS] *Let v be a radial vector field over $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of v over $\nu^{-1}(X \cap S_\varepsilon)$. The local Euler obstruction* $\mathrm{Eu}_0(X)$ *is the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\tilde{E}$ over $\nu^{-1}(X \cap B_\varepsilon)$, evaluated on the orientation class $\mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$:*

$$Eu_0(X) = Obs(\tilde{v}, \widetilde{E}, \nu^{-1}(X \cap B_\varepsilon)).$$

Theorem 7.4 ([BS], Théorème 11.1) (Proportionality Theorem for frames). *Let v^r be a radial r-frame with isolated singularities on the $2p$-cells $d = d(\sigma)$ with index $I(v^r, \hat{\sigma})$ in the barycenter $\{\hat{\sigma}\} = d(\sigma) \cap \sigma$. Then the obstruction to the extension of $\tilde{v}^r$ as a section of $\widetilde{E}^r$ on $\nu^{-1}(d \cap X)$ is*

$$Obs(\tilde{v}^r, \widetilde{E}^r, \nu^{-1}(d \cap X)) = Eu_{\hat{\sigma}}(X) \cdot I(v^r, \hat{\sigma}).$$

8 MacPherson and Mather classes

The MacPherson and Mather classes have been defined by R.MacPherson [MP]. Firstly let us recall some basic definitions.

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\alpha : X \to \mathbf{Z}$ is a function such that $\alpha^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathbf{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, α is a constructible function if and only if there is a triangulation (K) of X such that α is constant on the interior of each simplex of (K). Such a triangulation of X is called α-adapted.

The correspondence $\mathbf{F} : X \to \mathbf{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbf{F}(Y) \to \mathbf{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathbf{F}(X)$. The following result was conjectured by Deligne and Grothendieck in 1969.

Theorem 8.1 *[MP] Let $\mathbf{F}$ be the covariant functor of constructible functions and let $H_*(\ ;\mathbf{Z})$ be the usual covariant $\mathbf{Z}$-homology functor. Then there exists a unique natural transformation*

$$c_* : \mathbf{F} \to H_*(\ ;\mathbf{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

The theorem means that for X algebraic variety, one has a functor $c_* : \mathbf{F}(X) \to H_*(X;\mathbf{Z})$ satisfying the following properties:

1. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$ for α and β in $\mathbf{F}(X)$,
2. $c_*(f_*\alpha) = f_*(c_*(\alpha))$ for $f : X \to Y$ algebraic map and $\alpha \in \mathbf{F}(Y)$,
3. $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

8.1 Mather classes

The first approach to the proof of the Deligne-Grothendieck's conjecture is given by the construction of Mather classes. Let $X \subset M$ a possibly singular algebraic complex variety embedded in a smooth one. Let us define the Nash transformation $\widetilde{X}$ of X, as in section 7.1 and the Nash bundle $\widetilde{E}$ on $\widetilde{X}$.

Definition 8.2 The Mather class of X is defined by:

$$c^M(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}])$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Mather classes do not satisfy the Deligne-Grothendieck's conjecture. One has to perform the following construction, due to MacPherson.

8.2 MacPherson classes

The MacPherson's construction uses both the constructions of Mather classes and local Euler obstruction.

For a Whitney stratification, we have the following lemma:

Lemma 8.3 [MP] *There are integers n_i such that, for every point $x \in X$, we have:*

$$\sum_i n_i \mathrm{Eu}_x(\overline{V_i}) = 1$$

where the sum goes on strata V_i containing x in their closure.

Definition 8.4 [MP] The MacPherson class of X is defined by

$$c_*(X) = c_*(\mathbf{1}_X) = \sum_i n_i \; j_* c_*^M(\overline{V_i})$$

where j denotes the inclusion $\overline{V_i} \hookrightarrow X$ and the sum goes on the strata V_i contained in X.

Theorem 8.5 *[BS] The MacPherson class is the image of the Schwartz class by Alexander duality isomorphism*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

Corollary 8.6 *The Schwartz-MacPherson class $c_{r-1}(X)$ is represented by the cycle:*

$$\boxed{\sum_{\substack{\sigma \subset X \\ \dim \sigma = 2(r-1)}} I(v^{(r)}, \hat{\sigma}) \; \sigma}$$

Theorem 8.7 *[BS] The Chern-Mather class $c^M_{r-1}(X)$ is represented by the cycle:*

$$\boxed{\sum_{\substack{\sigma \subset X \\ \dim \sigma = 2(r-1)}} Eu_{\hat{\sigma}}(X) I(v^{(r)}, \hat{\sigma}) \; \sigma}$$

9 Fulton classes

Fulton classes have been defined in a general setting by Fulton (see [Fu]). One gives a very brief idea of that notion in the case of hypersurface (or more generaly a local complete intersection). Definition and results of this section will be found from [Fu] and [BSS].

Definition 9.1 If X is a *hypersurface*, then the normal bundle of X_{reg} in M extends canonically to X as a vector bundle $N_X M$ and one can define the Fulton classes by

$$c^F(X) = c(TM|_X)c(N_X M)^{-1} \cap [X] = c(\tau_X) \cap [X].$$

Here $\tau_X = TM|_X - N_X M$ denotes the virtual tangent bundle on X, defined in the Grothendieck group of vector bundles on X.

Theorem 9.2 *[BSS] Let us assume that $X \subset M$ is a hypersurface, defined by a section s of a holomorphic line bundle L over M. Assume further that L also admits a section s_0 which is everywhere transverse to the zero-section. For each point $a \in X$, let F_a denote a local Milnor fiber, and let $\chi(F_a)$ be its Euler-Poincaré characteristic. Then the Fulton-Johnson class $c^{FJ}_{r-1}(X)$ of X of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle*

$$\boxed{\sum_{\substack{\sigma\subset X \\ \dim\sigma = 2(r-1)}} \chi(F_{\hat\sigma})\, I(v^{(r)}, \hat\sigma)\ \sigma}$$

The question of understanding the difference between the Schwartz-MacPherson and the Fulton-Johnson classes has been addressed by several authors, see for instance P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura. This led to the concept of *Milnor classes*, defined by

$$\mu_*(X) \;=\; (-1)^{n+1}\left(c_*(X) - c^{FJ}_*(X)\right), \qquad n = \dim X.$$

Let us define by $\mu(X,a) = (-1)^{n+1}(1-\chi(F_a))$ the local Milnor number of X at the point $a \in X$; it coincides with the usual Milnor number when a is an isolated singularity of X. It is non zero only on the singular set Σ of X. We have the following immediate consequence of the previous Theorem:

Corollary 9.3 *Under the assumptions of the previous Theorem, the Milnor class $\mu_{r-1}(X)$ in $H_{2(r-1)}(X)$ is represented by the cycle*

$$\boxed{\sum_{\substack{\sigma\subset X \\ \dim\sigma = 2(r-1)}} \mu(X,\hat\sigma_\alpha)\, I(v^{(r)}, \hat\sigma)\ \sigma}$$

10 Schwartz-MacPherson classes of Thom spaces associated to embeddings

In this section and as a matter of example, we compute the Schwartz-MacPherson classes of the Thom spaces associated to Segre and Veronese embeddings. Results of this section come from [BFK], [BG1] and [BG2].

10.1 The projective cone

Let us consider an n-dimensional projective variety Y in $\mathbf{P}^m = \mathbf{PC}^m$ and let us denote by L the restriction of the hyperplane bundle of $\mathbf{P}^m$ to Y. We denote by E the completed projective space of the total space of L, i.e. $\mathbf{P}(L \oplus 1_Y)$ where 1_Y is the trivial bundle of complex rank 1 on Y. The canonical projection $p\colon E \to Y$ admits two sections, zero and infinite, with images $Y_{(0)}$ and $Y_{(\infty)}$. The projective cone KY is obtained as a quotient of E by contraction of $Y_{(\infty)}$ in a point $\{s\}$. It is the Thom space associated to the bundle L, with basis Y.

Let us consider $p\colon E \to Y$ as a sphere bundle with fiber S^2, subbundle of a bundle $\bar{p}\colon \bar{E} \to Y$ with fiber the ball B^3. We denote by $\theta_{\bar{E}} \in H^3(\bar{E}, E)$ the associated Thom class; One has a Gysin exact sequence

$$\ldots \to H_{j+1}(Y) \to H_{j-2}(Y) \xrightarrow{\gamma} H_j(E) \xrightarrow{p_j} H_j(Y) \to \ldots;$$

in which the gysin map γ is the composition of

$$H_{j-2}(Y) \xrightarrow{(\bar{p}_{j-2})^{-1}} H_{j-2}(\bar{E}) \xrightarrow{(\cap\theta_{\bar{E}})^{-1}} H_{j+1}(\bar{E}, E) \xrightarrow{\partial} H_j(E)$$

and can be explicited in the following way: If ζ is a cycle representing the class $[\zeta]$ of $H_{j-2}(Y)$, then $\gamma([\zeta])$ is the class of the cycle $p^{-1}(\zeta)$ in $H_j(E)$.

Let π the canonical projection $\pi\colon E \to KY$.

Proposition 10.1 *The Chern classes of E and Y are related by the formula*

$$c_*(E) = (1 \ + \ \eta_0 \ + \ \eta_\infty) \cap \gamma(c_*(Y)), \tag{10.4}$$

where $\eta_j := c^1(\mathcal{O}(Y_{(j)})) \in H^2(E)$ for $j = 0, \ \infty$, and $\cap$ denotes the usual cap-product.

PROOF: The vertical tangent bundle T_v of $p\colon E \to Y$ is defined by the exact sequence:

$$0 \to T_v \to TE \to p^*TY \to 0.$$

We have, in $H^*(E)$

$$c^*(E) = c^*(T_v) \cup c^*(p^*(TY)). \tag{10.5}$$

The sheaf of sections of the bundle T_v is the sheaf canonically associated to the divisor $Y_{(0)} + Y_{(\infty)}$, denoted by $\mathcal{O}_E(Y_{(0)} + Y_{(\infty)})$. By Poincaré isomorphism in Y, the divisor $[Y_{(j)}] \in H_{2n}(E)$ is identified to the class $\eta_j \in H^2(E)$. The Chern class of T_v is

$$c^*(T_v) = 1 \ + \ \eta_0 \ + \ \eta_\infty.$$

By definition of the Gysin map γ, one has a commutative diagram

$$\begin{array}{ccc} H^i(Y) & \xrightarrow{\cap[Y]} & H_{2n-i}(Y) \\ \downarrow{\scriptstyle p^i} & & \downarrow{\scriptstyle \gamma} \\ H^i(E) & \xrightarrow{\cap[E]} & H_{2n+2-i}(E) \end{array}$$

and by Poincaré duality

$$c^*(p^*(TY)) \cap [E] = p^*(c^*(TY)) \cap [E] = \gamma(c^*(TY) \cap [Y]) = \gamma(c_*(Y)). \tag{10.6}$$

Using formulae 10.5 and 10.6, one obtains

$$c_*(E) = (1 \ + \ \eta_0 \ + \ \eta_\infty) \ \cap \ \gamma(c_*(Y))$$

□

10.2 Schwartz-MacPherson classes of the projective cone

Definition 10.2 We call homological projective cone and we denote by K the composition $K = \pi_*\gamma : H_{j-2}(Y) \to H_j(KY)$ for $j \geq 2$. We let $K(0) := [s] \in H_0(KY)$ for $0 = H_{-2}(Y)$.

Let us remark that K is an homomorphism, out of $j = 0$.

Theorem 10.3 *Let $Y \subset \mathbf{P}_N$, be a projective variety and $\imath{:}Y \hookrightarrow KY$ the canonical inclusion in the projective cone KY on Y with vertex $\{s\}$. Let us denote also by K : $H_*(Y) \to H_{*+2}(KY)$ the homological projective cone, one has*

$$c_j(KY) = \imath_* c_j(Y) + K c_{j-1}(Y), \tag{10.7}$$

where $Kc_{-1}(Y)$ denotes the class $[s] \in H_0(KY)$

Proof: Let $\mathbf{1}_E$ the constructible function which is the characteristic function of E, one has

$$\pi_*(\mathbf{1}_E)(x) = \begin{cases} \chi(Y), & \text{if } x = s \\ 1, & \text{elsewhere,} \end{cases}$$

i.e.

$$\pi_*(\mathbf{1}_E) = \mathbf{1}_{KY} + (\chi(Y) - 1)\mathbf{1}_{\{s\}}.$$

As one has

$$\pi_* c_*(\mathbf{1}_E) = c_*(\pi_*(\mathbf{1}_E))$$

one obtains

$$\pi_* c_*(E) = c_*(KY) + (\chi(Y) - 1)\,[s]. \tag{10.8}$$

On another hand, from the formula (10.4) one obtains:

$$\pi_* c_*(E) = \pi_* \gamma(c_{*-1}(Y)) + \pi_*(\eta_0 \cap \gamma(c_*(Y))) + \pi_*(\eta_\infty \cap \gamma(c_*(Y))). \tag{10.9}$$

Let $\iota_0 : Y \hookrightarrow E$ and $\iota_\infty : Y \hookrightarrow E$ be the inclusions of Y as zero and infinite sections of E respectively. By definition of γ, one has for every cycle ζ in Y and for $j = 0$ or ∞

$$\eta_j \cap \gamma([\zeta]) = (\iota_j)_*([\zeta])$$

then

$$\pi_*(\eta_j \cap \gamma(c_*(Y))) = \pi_* \iota_{j*} c_*(\mathbf{1}_Y) = \pi_* c_*(\mathbf{1}_{Y_{(j)}}) = c_* \pi_*(\mathbf{1}_{Y_{(j)}}).$$

Let us denote by $\iota = \pi \circ \iota_0 : Y \hookrightarrow KY$ the natural inclusion of Y in KY, one has

$$\pi_*(\mathbf{1}_{Y_{(0)}}) = \mathbf{1}_{\iota(Y)} \quad \text{and} \quad \pi_*(\mathbf{1}_{Y_{(\infty)}}) = \chi(Y)\mathbf{1}_{\{s\}}.$$

One obtains

$$\pi_*(\eta_0 \cap \gamma(c_*(Y))) = c_*(\mathbf{1}_{\iota(Y)}) = \iota_* c_*(Y),$$

and

$$\pi_*(\eta_\infty \cap \gamma(c_*(Y))) = \chi(Y) c_*(\mathbf{1}_{\{s\}}) = \chi(Y)[s],$$

where $[s]$ is the class of the vertex s in $H_0(KY)$. The comparizon of the formulae 10.8 and 10.9 gives:

$$c_*(KY) = \iota_* c_*(Y) + \pi_* \gamma c_{*-1}(Y) + [s],$$

and the Theorem 10.3. $\square$

10.3 Case of the Segre and Veronese embeddings

The previous construction associates canonically a Thom space to the embedding of a smooth variety Y in $\mathbf{P}^m$. Let us consider in particular the following examples: the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$, defined in homogeneous coordinates by

$$(x_0 : x_1) \times (y_0 : y_1) \;\mapsto\; (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1),$$

and the Veronese embedding $\mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ defined by

$$(x_0 : x_1 : x_2) \;\mapsto\; (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

With the previous construction, KY is the Thom space associated to the fiber bundle L, of complex rank 1 and restriction to Y of the hyperplane bundle of $\mathbf{P}^m$. Chern classes and intersection homology of these examples have been computed in [BG1]. In the case of the Segre embedding, let d_1 and d_2 two fixed lines belonging each to a system of generatrices of the quadric $Y = \mathbf{P}^1 \times \mathbf{P}^1$. Let us denote by ω the canonical generator of $H^2(\mathbf{P}^1)$, one has $c^*(\mathbf{P}^1) = 1 + 2\omega$ and

$$c_*(Y) = c_*(\mathbf{P}^1 \times \mathbf{P}^1) = ([Y] + 2[d_1]) * ([Y] + 2[d_2]) = [Y] + 2([d_1] + [d_2]) + 4[a]$$

where a is a point in Y and where $*$ denotes the intersection of cycles or homology classes. One has

$$K(c_*(Y)) = [KY] + 2([Kd_1] + [Kd_2]) + 4[Ka].$$

let us denote by $\sim$ the homology relation of cycles. In KY, one has [BG1], 3:

$$Y \sim Kd_1 + Kd_2, \quad d_1 \sim d_2 \sim Ka, \quad a \sim s,$$

and, with 10.3

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{3([Kd_1] + [Kd_2])}_{H_4(KY)} + \underbrace{8[Ka]}_{H_2(KY)} + \underbrace{5[s]}_{H_0(KY)},$$

which is the result of [BG1].

In the case of the Veronese embedding, let d be a projective line in $Y := \mathbf{P}^2$, one has: $c^*(\mathbf{P}^2) = 1 + 3\omega + 3\omega^2$ where ω is the canonical generator of $H^2(\mathbf{P}_2)$, and is dual, by Poincaré isomorphism of the class $[d] \in H_2(\mathbf{P}_2)$. One has, by Poincaré duality

$$c_*(Y) = [Y] + 3[d] + 3[a]$$

where a is a point in Y. One has

$$K(c_*(Y)) = [KY] + 3[Kd] + 3[Ka]$$

such that, in KY, [BG1], 3.b, $Y \sim 2Kd$, $d \sim 2Ka$ and $a \sim s$. One has

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{5[Kd]}_{H_4(KY)} + \underbrace{9[Ka]}_{H_2(KY)} + \underbrace{4[s]}_{H_0(KY)}$$

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