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Graph 3-Manifolds, splice diagrams, singularities

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GRAPH 3-MANIFOLDS, SPLICE DIAGRAMS, SINGULARITIES

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ABSTRACT. We describe how a coarse classification of graph manifolds can give clearer insight into their structure, and we relate this particularly to the manifolds that can occur as the links of points in normal complex surfaces. We relate this discussion to a special class of singularities; those of "splice type", which turn out to play a central role among singularities of complex surfaces.

An appendix gives a brief introduction to classical 3-manifold theory.

This paper was written to serve as notes for a short course at ICTP Trieste.

1. INTRODUCTION

The early study of 3-manifolds and knots in 3-manifolds was motivated to a large extent by the theory of complex surfaces. For example, Poul Heergaard's 1898 thesis [7], in which he introduced the fundamental tool of 3-manifold theory now called a "Heegaard splitting," was on the topology of complex surfaces. For a thread from Heergaard's thesis through knot theory to the "splice diagrams" that will play a central role in this paper, see the survey [23] on topology of complex surface singularities.

The local topology of a normal complex surface ("normal" roughly means that any "inessential" singularities have been removed) at any point is the cone on a closed oriented 3-manifold. The manifold is called the "link" of the point. We call it a "singularity-link," even though we allow S^3, which can only be the link of a nonsingular point (Mumford [16]).

Singularity links and other 3-manifolds that arise in the study of complex surfaces are of a special type, namely "graph manifolds." Graph manifolds were defined and classified by Waldhausen in his thesis [40]. The motivation was certainly that the set of graph manifolds includes all singularity-links, and Waldhausen's work together with Grauert's criterion effectively gave a description of exactly what 3-manifolds are singularity-links. This description was put in a more convenient algorithmic form in [17]. More elegant versions have emerged since, which depend on taking a coarser look at the classification of graph manifolds. These coarse classifications are a central theme of this paper. They will also lead us to a special class of singularities, the singularities of "splice type" which encompasses several important classes of singularities.

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An appendix to this paper provides a convenient reference for some of the basic 3-manifold theory that we use.

This paper was written to serve as notes for a short course at ICTP Trieste. It is based in part also on lectures the author gave at CIRM (Luminy) in March 2005.

2. THE PLACE OF GRAPH MANIFOLDS IN 3-MANIFOLD THEORY

Throughout this paper, 3-manifolds will be compact and oriented unless otherwise stated. They will also be *prime* — not decomposable as a nontrivial connected sum. One forms the *connected sum* of two 3-manifolds by removing the interior of a disk from each and then gluing the resulting punctured 3-manifolds along their S^2 boundaries. Kneser and Milnor [12, 15] showed that any oriented 3-manifold has an essentially unique decomposition into prime 3-manifolds. Singularity links are always prime ([17]).

Definition 2.1. A *graph-manifold* is a 3-manifold M that can be cut along a family of disjoint embedded tori to decompose it into pieces $S_i \times S^1$, where each S_i is a compact surface (i.e., 2-manifold) with boundary.

The JSJ-decomposition is a natural decomposition of any prime 3-manifold into Seifert fibered and simple non-fibered pieces (see the appendix for relevant definitions and more detail). Its existence was proved in the mid-1970's independently by by Jaco and Shalen [9] and by Johannson [11], although it had been sketched earlier by Waldhausen [41]. From the point of view of JSJ-decomposition, a graph manifold is simply a 3-manifold which has no non-Seifert-fibered JSJ-pieces. There are various modifications of the JSJ decomposition, depending on the intended application, and they differ in essentially elementary ways (see e.g., [25]). One version is the "geometric decomposition" — a minimal decomposition along tori and Klein bottles into pieces that admit geometric structures in the sense of Thurston (finite volume locally homogeneous Riemannian metrics). The relevant geometry for simple non-Seifert-fibered pieces is hyperbolic geometry [1]. From this geometric point of view, graph manifolds are manifolds that have no hyperbolic pieces in their geometric decompositions.

In summary, a graph manifold is a 3-manifold that can be glued together from pieces of the form (surface)$\times S^1$, or more efficiently, from pieces which are Seifert fibered. Both points of view will be useful in the sequel.

3. SEIFERT MANIFOLDS

Let $M^3 \to F$ be a Seifert fibration of a closed 3-manifold. It is classified up to orientation preserving homeomorphism (or diffeomorphism) by the following data:

[1] The existence of the hyperbolic structure when M is simple non-Siefert fibered and the JSJ decomposition is trivial was still conjectural until recently; although proved in many cases by Thurston, it is probably now proved in general by Perelman's work.

- The homeomorphism type of the base surface F, which we can encode by its *genus* g. We use the convention that $g < 0$ refers to a nonorientable surface, so $g = -1, -2, \dots$ means F is a projective place, Klein bottle, etc.
- A collection of rational numbers $0 < q_i/p_i < 1$, $i = 1, \dots, n$, that encode the types of the singular fibers. Here p_i is the multiplicity of the ith singular fiber and q_i encodes how nearby fibers twist around this singular fiber.
- A rational number $e = e(M \to F)$ called the *Euler number* of the Seifert fibration. Its only constraint is that $e + \sum_{i=1}^{n} \frac{q_i}{p_i}$ should be an integer.

It is most natural to think of the base surface F as an orbifold rather than a manifold, with orbifold points of degrees $p_1, \dots, p_n$. As such, it has an *orbifold Euler characteristic*

$$\chi^{orb}(F) = \chi_g - \sum_i (1 - \frac{1}{p_i})$$

where χ_g is the Euler characteristic of the surface of genus g:

$$\chi_g = \begin{cases} 2 - 2g\,, & g \geq 0\,, \\ 2 + g\,, & g < 0\,. \end{cases}$$

Note that an oriented 3manifold M^3 may be Seifert fibered with nonorientable base. However, we do not need to consider this for links of singularities: a Seifert fibered 3manifold is a singularity link if and only if it has a Seifert fibration over an orientable base and the Euler number $e(M \to F)$ is negative.

From the point of view of geometric structures and geometric decomposition, there are exactly six geometries that occur for Seifert fibered manifolds and the type of the geometry is determined by whether $\chi^{orb}(F)$ is $> 0, = 0, < 0$ and whether $e(M \to F)$ is $= 0$ or $\neq 0$ ([19, 34]). These two invariants, which we will abbreviate simply as χ and e, are thus fundamental invariants for a Seifert fibered M^3. If $e \neq 0$ then M^3 has a unique orientation that makes $e < 0$, and we call this its "natural orientation," since it is the orientation that makes it (or a double cover of it if the base surface is nonorientable) into a singularity link.

The above discussion was for a closed 3manifold M^3. If M^3 is allowed to have boundary (but is still compact) then the Euler number e is indeterminate unless one has extra data. The additional data consists of a choice of a simple closed curve in each boundary torus of M^3, transverse to the fibers of the Seifert fibration.

Definition 3.1. We call this collection of curves a *system of meridians for* M.

Given a system of meridians, we can form a closed Seifert fibered manifold $\bar{M}^3$ by gluing a solid torus onto each boundary component, matching a meridian of the solid torus with the chosen "meridian" on the boundary T^2. The Euler invariant $e(\bar{M})$ is called the *Euler invariant of* M *with its system of meridians.*

4. Decomposition graphs, decomposition matrices

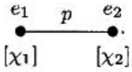
We now return to a general graph manifold M, considering it from the point of view of JSJdecomposition. So M can be cut along tori so that if breaks into pieces that are Seifert fibered 3manifolds. "JSJ decomposition" means that no smaller collection of cutting tori will work (see the Appendix for a proof of existence and uniqueness of JSJ decomposition).

If M fibers over the circle with torus fiber or is double covered by such a manifold then M admits a geometric structure, so the geometric version of JSJ decomposition would not decompose it, even though the standard JSJ usually cuts it along a torus. Such manifolds are completely understood (for a discussion close to the current point of view see [20]) so:

Assumption. From now on we assume that M cannot be fibered over S^1 with T^2 fiber.

Each piece M_i in the JSJ decomposition comes with a system of meridians (Definition 3.1) by choosing the meridian in each boundary torus of M_i to be a Seifert fiber of the piece across the torus from M_i. Thus the orbifold Euler characteristic and Euler number invariants are both defined for the ith piece M_i, and we call them χ_i and e_i.

The *decomposition graph* is the graph with a vertex for each piece M_i and an edge for each gluing torus. The edge connects the vertices corresponding to the pieces of M that meet along the torus. We decorate this graph with weights as follows: At the vertex i corresponding to M_i we give the numbers χ_i and e_i, writing χ_i in square brackets to distinguish it. And for an edge E corresponding to a torus T^2 we record the absolute value of the intersection number $F.F'$, where F and F' are fibers in T^2 of the Seifert fibrations on the pieces M_i and M_j that meet along T^2. For example, if M is glued from two Seifert fibered pieces, each of which has one boundary component, then the decomposition graph has the form



There is one problem with the definition of the decomposition graph. If a piece M_i is the total space SMb of the unit tangent bundle of the Möbius band, then weights on adjacent edges of the decomposition graph are not welldefined. This is because SMb has two different Seifert fibrations, one as this circle bundle and another by orbits of the action of the circle on SMb induced by the nontrivial S^1 action on the Möbius band Mb. For this reason we always use the latter Seifert fibration if such a piece occurs. However, we will usually want to go further and avoid SMb pieces altogether. This can always be done by replacing M by a double cover. In fact:

Proposition 4.1 ([20]). *M always has a double cover whose JSJ decomposition satisfies:*

- *Every piece of M has a Seifert fibration over a orientable base surface.*

- *The fibers of each piece can be oriented so that the fiber intersection number in each torus is positive*[2].
- *No SMb pieces occur and no piece is glued to itself across a torus.*

If the first condition holds we say M is *good* and if the first two conditions hold M is *very good.*

Remark 4.2. The third of the above conditions is a condition on the decomposition graph: the absence of SMb pieces says that the χ–weight at each vertex is negative (unless the graph consists of a single vertex with $\chi = 0$), and the absence of "self-gluings" is absence of edges that have both ends at the same vertex.

Even though the decomposition graph carries much less information than is needed to reconstruct M, it determines M up to finite ambiguity:

Proposition 4.3. *There are only finitely many different manifolds for any given decomposition graph.*

The proof of this is an exercise, based on the fact that there are only finitely many 2orbifolds with given orbifold Euler characteristic χ. But the number can grow quite rapidly with χ, so already for simple decomposition graphs the number of manifolds can be large. Nevertheless, the decomposition graph does determine M up to commensurability (recall that manifolds are *commensurable* if they have diffeomorphic finite covers):

Theorem 4.4 ([24]). *If M_1 and M_2 are graph manifolds with no SMb pieces and their decomposition graphs are isomorphic then there exist dfold covers $\tilde{M}_1$ and $\tilde{M}_2$ of M_1 and M_2 for some $d \in \mathbb{N}$ such that $\tilde{M}_1 \cong \tilde{M}_2$.*

For many properties of M even less information suffices. Namely, the *decomposition matrix* is the matrix $A = (a_{ij})$ with entries

$$\begin{aligned} a_{ij} &= e_i + 2\sum_{iEi} \frac{1}{|p(E)|} \quad \text{if } i = j \\ &= \quad \sum_{iEj} \frac{1}{|p(E)|} \quad \text{if } i \neq j\,, \end{aligned}$$

where iEj means E is an edge joining i and j, and $p(E)$ is the fiber intersection weight on this edge. So the decomposition matrix no longer retains the invariants χ_i nor the exact number of edges joining a vertex to another.

It turns out that a variety of questions about M are answered in the literature completely in terms of the decomposition matrix (in some cases variations of "good" or "very good" are needed, that are always achieved in some double cover):

- Is M a singularity link ([20])?

[2]To get a well defined intersection number we view the separating torus from one side and intersect the Seifert fibers in the torus in the order (fiber from the near side).(fiber from the far side). If we look from the other side we reverse the orientation of the torus and reverse the order of the fibers, so the intersection number stays the same.

- Does M fiber over the circle ([20])?
- Does some cover of M fiber over the circle ([20])?
- Does M have an immersed incompressible surface of negative Euler number ([21])?
- Does some cover of M have an embedded incompressible surface of negative Euler number ([21])?
- Does M admit a metric of nonpositive curvature ([1])?

For the first of these the answer is as follows:

Theorem 4.5. *M is a singularity link if and only if it is very good and the decomposition matrix is negative definite.*

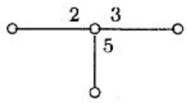
This is proved in [20] by a combinatorial argument, but we can give a geometric reason why it might be expected. Grauert's criterion [5] characterizes singularity links among "plumbed manifolds" (another way of looking at graph manifolds) by the negative definiteness of the intersection matrix of a resolution of the singularity. Our decomposition matrix is the intersection matrix of a resolution, but not a full resolution. The socalled logcanonical resolution of a surface singularity resolves the singularity to the point where only cyclic quotient singularities remain. Although we then do not yet have a smooth manifold, it is a $\mathbb{Q}$homology manifold, so intersection numbers are still defined (they are rational numbers rather than integers). The resulting intersection matrix is the decomposition matrix. The theorem can be interpreted to say that Grauert's criterion still holds in this situation.

5. Splice diagrams for rational homology spheres

This section and the next describe joint work of the author and J. Wahl. We will describe a different encoding of graph manifolds, that again brings focus to some information by throwing away other information. We now restrict to graph manifolds M which are rational homology spheres, that is $H_1(M;\mathbb{Z})$ is finite. We say, briefly, that M is a $\mathbb{Q}$HS. For the JSJ decomposition this implies that the decomposition graph must be a tree, and, moreover, that the base of each Seifert fibered piece is of genus zero. However, instead of using the JSJ decomposition we now use the Waldhausen decomposition — the minimal decomposition into pieces of the form (surface)$\times S^1$

We again form a graph for this decomposition. This graph, with weights on edges to be described, is called a *splice diagram.*

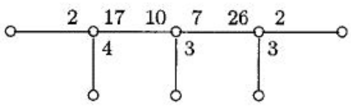
The Waldhausen decomposition differs from the JSJ decomposition in that for each singular fiber of a Seifert fibered piece we must cut out a $D^2 \times S^1$ neighborhood of that singular fiber. For example, the JSJ decomposition graph for a Seifert fibered manifold consists of a single vertex (decorated with two numerical weights e and χ), while the splice diagram is a starshaped graph: a central node with an edge sticking out for each singular fiber of the Seifert fibration. We weight the edges by the degrees of the singular fibers. For example a Seifert fibered manifold with exactly three singular fibers of degrees $2, 3, 5$ would have splice diagram



(There are infinitely many such Seifert manifolds, all with the same splice diagram; the corresponding decomposition graphs consist of a single vertex with weights $[\chi = 1/30]$, $e = q/30$, with q an arbitrary integer prime to 30.)

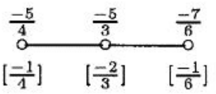
In general a slice diagram is a finite tree with vertices only of valence 1 ("leaves") or ≥ 3 ("nodes") and with nonnegative integer weights decorating the edges around each node, and such that the weights on edges from nodes to leaves are ≥ 2. In addition, we decorate a node with an additional "$-$" sign if the linking number in M of two fibers of the corresponding Seifert piece is negative (this never occurs for splice diagrams of links of singularities).

Here is an example of a splice diagram for a certain singularity link M.



We describe the meaning of the weights by example of the weight 17. It is on an edge joining the two nodes, and this edge corresponds to a torus T^2 which cuts M into two pieces M_1 and M_2 (M_1 is Seifert fibered and M_2 is not). We look at the piece M_2 at the far side of T^2 and form a closed manifold $\bar{M}^2$ by gluing a solid torus into its boundary, matching – as in the previous section – meridian of the solid torus with the "meridian curve" on the boundary of M^2 (recall that this is a fiber in T^2 of the Seifert fibration across T^2 from M_2). The weight 17 is the order $|H_1(\bar{M}_2; \mathbb{Z})|$. This procedure weights an edge leading to a leaf with degree of the singular Seifert fiber corresponding to that leaf.

(It turns out that there is just one manifold with the above splice diagram. Its JSJ decomposition graph, obtained from the splice diagram by removing all the leaves and decorating with appropriate χ_i and e_i weights, is



)

Definition 5.1. The *edge determinant* of an edge connecting two nodes in a splice diagram is defined to be the product of the two weights on that edge minus the product of the weights adjacent to that edge.

For example, both edge determinants in the above splice diagram are 2 since $10 \times 17 - 2 \times 4 \times 3 \times 7 = 2$ and $7 \times 26 - 10 \times 3 \times 2 \times 2 = 2$.

Theorem 5.2. *M is the link of a singularity if and only if no $+$ decorations occur in the diagram and every edge determinant is positive.*

Again, although the splice diagram does not determine M uniquely in general, it does determine M up to commensurability. In fact, recall that the universal abelian cover of a space M is the Galois cover whose covering transformation group is $H_1(M;\mathbb{Z})$.

Theorem 5.3 (??). *If M_1 and M_2 are QHS graph manifolds with the same splice diagram then the universal abelian covers of M_1 and M_2 are diffeomorphic.*

The question marks are because we have not yet carefully written up a full proof of this theorem in the generality claimed. It is certainly correct when M_1 and M_2 are singularity links (the case that interests us most here).

In general the universal abelian cover of a $\mathbb{Q}$HS graph manifold M may be something quite horrible, with a complicated decomposition graph and lots of homology. But there is a case when we can describe it very nicely. For any splice diagram with pairwise coprime weights around each node there is a unique integral homology sphere ($\mathbb{Z}$HS) with the given splice diagram. It is its own universal abelian cover, so the theorem implies that this $\mathbb{Z}$HS is diffeomorphic to the universal abelian cover of any other graph manifold with the same splice diagram.

The splice diagram of a $\mathbb{Z}$HS graph manifold always has pairwise coprime weights around each node, so such diagrams classify $\mathbb{Z}$HS graph manifolds (see [3]).

We saw that the decomposition graph determines M up to finite ambiguity. The same is true for the splice diagram, except in the case of onenode splice diagrams (a onenode splice diagram always has infinitely many different manifolds associated with it). This is a consequence of Proposition 4.3 and the following:

Proposition 5.4 ([24]). *The splice diagram of M and the order of $H_1(M;\mathbb{Z})$ together determine the decomposition graph of M. The order of $H_1(M;\mathbb{Z})$ is a common divisor of the edge determinants of the splice diagram.*

6. Singularities of splice type

In general it has been very difficult to give explicit analytic realizations of singularities with given topology, but when the link is a $\mathbb{Q}$HS the recently discovered "singularities of splice type" [28] often do this.

Singularities of splice type have very strong properties: the universal abelian cover of a splice type singularity (by which we mean the maximal abelian cover that is ramified only at the singular point) is a complete intersection, defined by a quite elegant system of equations, and the covering transformation group acts diagonally in the coordinates. So the singularity is described by explicit equations and an explicit diagonal group action.

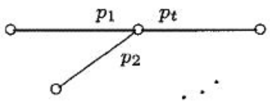
But, despite these strong properties, splice type singularities seem surprisingly common. For example, it has long been known that weighted homogeneous singularities with $\mathbb{Q}$HS link are of splice type ([18]), we (J. Wahl and the author) showed in [27] that Hirzebruch's quotienteusp singularities are, and recently Okuma [32] has confirmed our conjecture that every rational singularity is of splice type and every minimally elliptic singularities with $\mathbb{Q}$HS link also is. Very recently we have proved a conjecture we had struggled with for some time, the "End Curve Conjecture", which postulated a characterization of this class of singularities in terms of curves through the singular point, and which has Okuma's theorem as a consequence.

To describe this result we need some terminology. Let (V, o) be a normal complex surface singularity and $\pi\colon (Y, E) \to (V, o)$ a good resolution. Recall that this means that $\pi^{-1}(o) = E$, π is biholomorphic between $Y - E$ and $V - o$, and E is a union of smooth curves E_j that intersect each other transversally, no three through a point. The link of the singularity is a $\mathbb{Q}$HS if and only if the resolution graph (the graph with a vertex for each E_j and an edge for each intersection of two E_j's) is a tree and each E_j is a rational curve. Let E_j correspond to a leaf j of the tree, so E_j intersects the rest of E in a single point x. An *endcurve* for j is a smooth curve germ cutting E_j transversally in a point other than x. An *endcurve function* for this leaf j is an analytic function germ $z_j\colon (V, o) \to (\mathbb{C}, 0)$ that "cuts out" an endcurve for j, in the sense that its zero set is the image in V of an endcurve for j (with some multiplicity).

Theorem 6.1 (End Curve Theorem, [30]). *Suppose (V, o) is a normal complex surface singularity with $\mathbb{Q}$HS link. It is of splice type if and only if an endcurve function exists for each leaf of the resolution graph. In this case appropriate roots of the endcurve functions can be used as coordinates on the universal abelian cover.*

The existence of endcurve functions is well known for rational singularities and for $\mathbb{Q}$HS-link minimally elliptic ones, so Okuma's theorem that these are of splice type follows.

To give the analytic description of splice type singularities we start with the weighted homogeneous case. Then there is a $\mathbb{C}^*$–action on the singularity which induces an S^1–action on the link M, so the the link is Seifert fibered. The splice diagram thus has the form



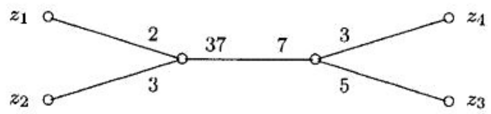
In this case it was shown in [18] that the universal abelian cover of the singularity is a Brieskorn complete intersection

$$\tilde V^{ab} \cong \{(z_1, \dots, z_t) \in \mathbb{C}^k \mid a_{i1} z_1^{p_1} + \dots + a_{ik} z_t^{p_t} = 0, \quad i = 1, \dots, t-2\},$$

for suitable coefficients a_{ij}. Moreover, an explicit action of $H_1(M; \mathbb{Z})$ on this Brieskorn complete intersection was given, with quotient the original singularity. Note that the Brieskorn equations are weighted homogeneous of total weight $p_1 \dots p_t$ if we give the jth variable weight $p_1 \dots \hat p_j \dots p_t$.

General splice type singularities generalize this situation. A variable z_i is associated to each leaf of the splice diagram, and for each node j of the diagram one associates a collection of $\delta - 2$ equations (δ the valence of the node) which are weighted homogeneous with respect to a system of weights associated to the node (one also allows higher weight perturbations of these equations). Doing this for all nodes gives a total of $t-2$ equations, where t is the total number of leaves.

To describe these weights, fix the node v. The v-weight of the variable z_i corresponding to leaf i is the product of the weights directly adjacent to but not on the path from v to i in the splice diagram. We denote this number ℓ_{vi}. For example, if v is the left node in the splice diagram

$\Delta =$ 

then the v-weights of the variables z_1, z_2, z_3, z_4 are:

$$\ell_{v1} = 3 \times 37 = 111, \quad \ell_{v2} = 74, \quad \ell_{v3} = 18, \quad \ell_{v4} = 30\,.$$

The weight of the equations that we want to write down is the product of the weights at the node v, we denote this d_v; in our example $d_v = 222$. For each of the edges e departing v we choose a monomial M_e of total weight d_v in the variables corresponding to leaves beyond e from v. In this example the monomials z_1^2, z_2^3, and $z_3^4 z_4^5$ are suitable. Our equations will be equations which equate $\delta - 2$ generic linear combinations of these monomials to zero, where δ is the valence of v. So in this case a there would be a single equation for the node v, of the form $az_1^2 + bz_2^3 + cz_3^4 z_4^5 = 0$, for example

$$z_1^2 + z_2^3 + z_3^4 z_4^5 = 0\,. \tag{1}$$

Note that a monomial M_e as above may not exist in general. The monomial $M_e = \prod_i z_i^{\alpha_i}$ has weight $\sum_i \alpha_i \ell_{vi}$, and the equation

$$d_v = \sum_i \alpha_i \ell_{vi}$$

may not have a solution in nonnegative integers α_i as i runs through the leaves beyond e. The solubility of these equations gives a condition on the splice diagram that we call the *semigroup condition*. It is a fairly weak condition; for example the fact that rational and QHS-link minimally elliptic singularities are of splice type says that the semigroup condition is satisfied for the splice diagrams of the links of such singularities.

If the semigroup condition is satisfied, then we can write down equations as above for all nodes of the splice diagram, and we get a complete intersection singularity whose topology is the desired topology of a universal abelian cover. The other ingredient in defining splice type singularities is the group action that gives the covering transformations for the universal abelian cover. This group action in computed in a simple way from the desired topology of the singularity (as encoded by a resolution diagram; we describe this later), but the above equations will not necessarily be respected by it. Being able to choose the monomials so that the equations are respected is a further condition (on the resolution diagram for the singularity rather than just the splice diagram) which we call the *congruence condition*. Again, it is a condition that is satisfied for the classes of singularities that we mentioned above.

If both the semigroup condition and congruence condition are satisfied, so that the monomials can be chosen appropriately, then the complete intersection singularity we have described is the universal abelian cover of a singularity with the desired resolution diagram and the covering transformations are given by the group action in question.

We will carry this out for the explicit example of the splice diagram above. This is the splice diagram for a singularity with ZHS link. The universal abelian cover is a trivial cover in this case, so the equations we construct will actually give such a singularity.

We have already seen that a possible equation for the left node is given by equation (1). In a similar way, we see that a possible equation for the right node is

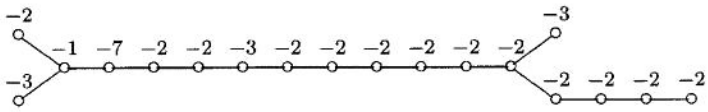
(2) $$z_1z_2^2 + z_3^5 + z_4^3 = 0\,.$$

The variety

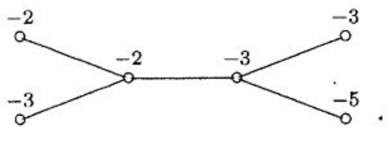
$$V = \{(z_1, z_2, z_3, z_4) \mid z_1^2 + z_2^3 + z_3^4z_4^5 = 0, \quad z_1z_2^2 + z_3^5 + z_4^3 = 0\}$$

thus has an isolated singularity at 0 whose link is the ZHS corresponding to the above splice diagram.

However, suppose the singularity we are really interested in is not the singularity with ZHS link, which has resolution graph



but instead the singularity with resolution graph

$\Gamma =$ 

which has the same splice diagram, but its link M has first homology

$$H_1(M;\mathbb{Z}) = \mathbb{Z}/169\,.$$

By what we have already said, we expect the above variety V to be the universal abelian cover of what we want, so we want the $\mathbb{Z}/169$ action on V. As we describe in more detail below, the action of $\mathbb{Z}/169$ is generated by the map

$$(z_1, z_2, z_3, z_4) \mapsto (\xi^9 z_1, \xi^6 z_2, \xi^{38} z_3, \xi^7 z_4)\,,$$

where ξ is a primitive 169th root of unity. This multiplies the first equation by ξ^{18} and the second by ξ^{21}, so it respects the equations and gives a free action of $\mathbb{Z}/169$ on the variety. The theory developed in [28] proves that $V' = V/(\mathbb{Z}/169)$ has the desired topology and that V is its universal abelian cover.

To describe the action of $H_1(M;\mathbb{Z})$ on $\mathbb{C}^t$ in general we first recall from [28] how to construct the splice diagram from the resolution graph Γ. Denote the incidence matrix of Γ by $A(\Gamma)$ — this is the intersection matrix of the resolution: the matrix whose diagonal entries are the selfintersection weights of Γ and which has has an entry 1 or 0 in the kl position with $k \neq l$ according as Γ does or does not have an edge connecting vertices k and l. The cokernel of $A(\Gamma)$ is isomorphic to $H_1(M\ \mathbb{Z})$, so $\det(-A(\Gamma)) = |H_1(M;\mathbb{Z})|$.

The splice diagram Δ has the same shape as Γ but with vertices of valence 2 suppressed. The splice diagram weights can be computed as follows. If one removes a node v of Γ and adjacent edges then Γ breaks into δ subgraphs, where δ is the valence of v. The weights adjacent to v are the number $\det(-A(\Gamma'))$ as Γ' runs through these subgraphs.

This allows us to define a weight also adjacent to leaves of the splice diagram Δ, namely $\det(-A(\Gamma'))$ where Γ' is obtained by removing the leaf and adjacent edge. We now define ℓ_{ij} for any pair of leaves as the product of weights adjacent to the direct path from i to j (or just the weight adjacent to i if $i = j$).

With the leaves of Δ numbered $j = 1, \dots, t$ we define for each leaf i a diagonal matrix

$$g_i = \operatorname{diag}(e^{2\pi i \ell_{ij}/d}; j = 1, \dots, t)$$

where $d = |H_1(M;\mathbb{Z})|$. These matrices generate a diagonal subgroup of $\mathrm{GL}(\mathbb{C}^t)$ which is isomorphic to $H_1(M;\mathbb{Z})$. This gives the desired action of $H_1(M;\mathbb{Z})$ on $\mathbb{C}^t$ (see [28]).

In our particular example above, any one of $g_1, \dots, g_4$ generates the cyclic group $H_1(M;\mathbb{Z}) = \mathbb{Z}/169$ and the actual element we gave above was g_3.

The congruence condition is the condition that for any node of the splice diagram we can choose the monomials M_e so that they all transform the same way under this group action. In this example the congruence condition turns out to be satisfied for any choice of monomials.

APPENDIX: CLASSICAL 3-MANIFOLD THEORY

This appendix gives a quick survey of some "classical" 3manifold theory. It is adapted from Chapter 2 of the notes [22]. Manifolds are always assumed to be smooth (or at least piecewise smooth).

APPENDIX A. SOME BASICS

This section describes some fundamental classical tools of 3manifold theory. The proofs of the results in this section can be found in several books on 3-manifolds, for example [8].

Theorem A.1 (Dehn's Lemma). *If M^3 is a 3manifold and $f\colon D^2 \to M^3$ a map of a disk such that for some neighborhood N of ∂D^2 the map $f|N$ is an embedding and $f^{-1}(f(N)) = N$. Then $f|\partial D^2$ extends to an embedding $g\colon D^2 \to M^3$.*

Dehn's proof of 1910 [4] had a serious gap which was pointed out in 1927 by Kneser. Dehn's Lemma was finally proved by Papakyriakopoulos in 1956, along with two other results, the loop and sphere theorems, which have been core tools ever since. These theorems have been refined by various authors since then. The following version of the loop theorem contains Dehn's lemma. It is due to Stallings [36].

Theorem A.2 (Loop Theorem). *Let F^2 be a connected submanifold of ∂M^3, N a normal subgroup of $\pi_1(F^2)$ which does not contain $\ker(\pi_1(F^2) \to \pi_1(M^3))$. Then there is a proper embedding $g\colon (D^2, \partial D^2) \to (M^3, F^2)$ such that $[g|\partial D^2] \notin N$.*

Theorem A.3 (Sphere Theorem). *If N is a $\pi_1(M^3)$invariant proper subgroup of $\pi_2(M^3)$ then there is an embedding $S^2 \to M^3$ which represents an element of $\pi_2(M^3) - N$.*

(These theorems also hold if M^3 is nonorientable except that in the Sphere Theorem we must allow that the map $S^2 \to M^3$ may be a degree 2 covering map onto an embedded projective plane.)

Definition A.4. An embedded 2sphere $S^2 \subset M^3$ is *essential* or *incompressible* if it does not bound an embedded ball in M^3. M^3 is *irreducible* if it contains no essential 2sphere.

Note that if M^3 has an essential 2sphere that separates M^3 (i.e., M^3 falls into two pieces if you cut along S^2), then there is a resulting expression of M as a *connected sum* $M = M_1 \# M_2$ (to form connected sum of two manifolds, remove the interior of a ball from each and then glue along the resulting boundary components S^2). If M^3 has no essential separating S^2 we say M^3 is *prime*

Exercise 1. M^3 prime $\Leftrightarrow$ Either M^3 is irreducible or $M^3 \simeq S^1 \times S^2$. Hint[3].

[3]If M^3 is prime but not irreducible then there is an essential nonseparating S^2. Consider a simple path γ that departs this S^2 from one side in M^3 and returns on the other. Let N be a closed regular neighborhood of $S^2 \cup \gamma$. What is ∂N? What is $M^3 - N$?

Theorem A.5 (Kneser and Milnor). *Any 3manifold has a unique connected sum decomposition into prime 3manifolds (the uniqueness is that the list of summands is unique up to order).*

We next discuss embedded surfaces other than S^2. Although we will mostly consider closed 3manifolds (i.e., compact without boundary), it is sometimes necessary to consider manifolds with boundary. If M^3 has boundary, then there are two kinds of embeddings of surfaces that are of interest: embedding F^2 into ∂M^3 or embedding F^2 so that $\partial F^2 \subset \partial M^3$ and $(F^2 - \partial F^2) \subset (M^3 - \partial M^3)$. The latter is usually called a "proper embedding." Note that ∂F^2 may be empty. In the following we assume without saying that embeddings of surfaces are of one of these types.

Definition A.6. If M^3 has boundary, then a properly embedded disk $D^2 \subset M^3$ is *essential* or *incompressible* if it is not "boundaryparallel" (i.e., it cannot be isotoped to lie completely in ∂M^3, or equivalently, there is no ball in M^3 bounded by this disk and part of ∂M^3). M^3 is *boundary irreducible* if it contains no essential disk.

If F^2 is a connected surface $\neq S^2, D^2$, an embedding $F^2 \subset M^3$ is *incompressible* if $\pi_1(F^2) \to \pi_1(M^3)$ is injective. An embedding of a disconnected surface is incompressible if each component is incompressibly embedded.

It is easy to see that if you slit open a 3manifold M^3 along an incompressible surface, then the resulting pieces of boundary are incompressible in the resulting 3manifold. The loop theorem then implies:

Proposition A.7. *If $F^2 \neq S^2, D^2$, then a twosided embedding $F^2 \subset M^3$ is compressible (i.e., not incompressible) if and only if there is an embedding $D^2 \to M^3$ such that the interior of D^2 embeds in $M^3 - F^2$ and the boundary of D^2 maps to an essential simple closed curve on F^2.*

(For a onesided embedding $F^2 \subset M^3$ one has a similar conclusion except that one must allow the map of D^2 to fail to be an embedding on its boundary: ∂D^2 may map 21 to an essential simple closed curve on F^2. Note that the boundary of a regular neighborhood of F^2 in M^3 is a twosided incompressible surface in this case.)

Exercise 2. Show that if M^3 is irreducible then a torus $T^2 \subset M^3$ is compressible if and only if either

- it bounds an embedded solid torus in M^3, or
- it lies completely inside a ball of M^3 (and bounds a knot complement in this ball).

A 3manifold is called *sufficiently large* if it contains an incompressible surface, and is called *Haken* if it is irreducible, boundaryirreducible, and sufficiently large. Fundamental work of Haken and Waldhausen analyzed Haken 3manifolds by repeatedly cutting along incompressible surfaces until a collection of balls was reached (it is a theorem of Haken that this always happens). A main result is

Theorem A.8 (Waldhausen). *If M^3 and N^3 are Haken 3manifolds and we have an isomorphism $\pi_1(N^3) \to \pi_1(M^3)$ that "respects peripheral structure" (that is, it takes each subgroup represented by a boundary component of N^3 to a a conjugate of a subgroup represented by a boundary component of M^3, and similarly for the inverse homomorphism). Then this isomorphism is induced by a homeomorphism $N^3 \to M^3$ which is unique up to isotopy.*

The analogous theorem for surfaces is a classical result of Nielsen.

We mention one more "classical" result that is a key tool in Haken's approach.

Definition A.9. Two disjoint surfaces $F_1^2, F_2^2 \subset M^3$ are *parallel* if they bound a subset isomorphic to $F_1 \times [0, 1]$ between them in M^3.

Theorem A.10 (KneserHaken finiteness theorem) . *For given M^3 there exists a bound on the number of disjoint pairwise nonparallel incompressible surfaces that can be embedded in M^3.*

Appendix B. JSJ Decomposition

We shall give a quick proof, originating in an idea of Swarup (see [23]), of the main "JSJ decomposition theorem" which describes a canonical decomposition of any irreducible boundaryirreducible 3manifold along tori and annuli. The characterization of this decomposition that we actually use in these notes is here an exercise (Exercise 3 at the end of this section). F. Costantino gives a nice exposition in [2] of a proof, based on this proof and ideas of Matveev, that directly proves this characterization.

We shall just describe the decomposition in the case that the boundary of M^3 is empty or consists of tori, since that is what is relevant to these notes. Then only tori occur in the JSJ decomposition (see section C.5). An analogous proof works in the general torusannulus case (see [23]), but the general case can also be deduced from the case we prove here.

The theory of such decompositions for Haken manifolds with toral boundaries was first outlined by Waldhausen in [41]; see also [42] for his later account of the topic. The details were first fully worked out by Jaco and Shalen [9] and independently Johannson [11].

Definition B.1. M is *simple* if every incompressible torus in M is boundary-parallel.

If M is simple we have nothing to do, so suppose M is not simple and let $S \subset M$ be an essential (incompressible and not boundaryparallel) torus.

Definition B.2. S will be called *canonical* if any other properly embedded essential torus T can be isotoped to be disjoint from S.

Take a disjoint collection $\{S_1, \dots, S_s\}$ of canonical tori in M such that

- no two of the S_i are parallel;
- the collection is maximal among disjoint collections of canonical tori with no two parallel.

A maximal system exists because of the KneserHaken finiteness theorem. The result of splitting M along such a system will be called a *JSJ decomposition* of M. The maximal system of pairwise nonparallel canonical tori will be called a *JSJsystem* .

The following lemma shows that the JSJsystem $\{S_1, \dots, S_s\}$ is unique up to isotopy.

Lemma B.3. *Let $S_1, \dots, S_k$ be pairwise disjoint and nonparallel canonical tori in M. Then any incompressible torus T in M can be isotoped to be disjoint from $S_1 \cup \dots \cup S_k$. Moreover, if T is not parallel to any S_i then the final position of T in $M - (S_1 \cup \dots \cup S_k)$ is determined up to isotopy.*

By assumption we can isotop T off each S_i individually. Writing $T = S_0$, the lemma is thus a special case of the stronger:

Lemma B.4. *Suppose $\{S_0, S_1, \dots, S_k\}$ are incompressible surfaces in an irreducible manifold M such that each pair can be isotoped to be disjoint. Then they can be isotoped to be pairwise disjoint and the resulting embedded surface $S_0 \cup \ldots \cup S_k$ in M is determined up to isotopy.*

Proof. We just sketch the proof. We start with the uniqueness statement. Assume we have $S_1, \dots, S_k$ disjointly embedded and then have two different embeddings of $S = S_0$ disjoint from $T = S_1 \cup \ldots \cup S_k$. Let $f \colon S \times I \to M$ be a homotopy between these two embeddings and make it transverse to T. The inverse image of T is either empty or a system of closed surfaces in the interior of $S \times I$. Now use Dehn's Lemma and Loop Theorem to make these incompressible and, of course, at the same time modify the homotopy (this procedure is described in Lemma 1.1 of [40] for example). We eliminate 2spheres in the inverse image of T similarly. If we end up with nothing in the inverse image of T we are done. Otherwise each component T' in the inverse image is a parallel copy of S in $S \times I$ whose fundamental group maps injectively into that of some component S_i of T. This implies that S can be homotoped into S_i and its fundamental group $\pi_1(S)$ is conjugate into some $\pi_1(S_i)$. It is a standard fact (see, e.g., [37]) in this situation of two incompressible surfaces having comparable fundamental groups that, up to conjugation, either $\pi_1(S) = \pi_1(S_j)$ or S_j is onesided and $\pi_1(S)$ is the fundamental group of the boundary of a regular neighborhood of T and thus of index 2 in $\pi_1(S_j)$. We thus see that either S is parallel to S_j and is being isotoped across S_j or it is a neighborhood boundary of a onesided S_j and is being isotoped across S_j. The uniqueness statement thus follows.

A similar approach to proves the existence of the isotopy using Waldhausen's classification [39] of proper incompressible surfaces in $S \times I$ to show that S_0 can be isotoped off all of $S_1, \dots, S_k$ if it can be isotoped off each of them. □

The thing that makes decomposition along incompressible annuli and tori special is the fact that they have particularly simple intersection with other incompressible surfaces.

Lemma B.5. *If a properly embedded incompressible torus T in an irreducible manifold M has been isotoped to intersect another properly embedded incompressible surface F with as few components in the intersection as possible, then the intersection consists of a family of parallel essential simple closed curves on T.*

Proof. Suppose the intersection is nonempty. If we cut T along the intersection curves then the conclusion to be proved is that T is cut into annuli. Since the Euler characteristics of the pieces of T must add to the Euler characteristic of T, which is zero, if not all the pieces are annuli then there must be at least one disk. The boundary curve of this disk bounds a disk in F by incompressibility of F, and these two disks bound a ball in M by irreducibility of M. We can isotop over this ball to reduce the number of intersection components, contradicting minimality. □

Let $M_1, \dots, M_m$ be the result of performing the JSJdecomposition of M along the JSJsystem $\{S_1 \cup \cdots \cup S_s\}$.

Theorem B.6. *Each M_i is either simple or Seifert fibered by circles (or maybe both).*

Proof. Suppose N is one of the M_i which is nonsimple. We must show it is Seifert fibered by circles.

Since N is nonsimple it contains essential tori. Consider a maximal disjoint collection of pairwise nonparallel essential tori $\{T_1, \dots, T_r\}$ in N. Split N along this collection into pieces $N_1, \dots, N_n$. We shall analyze these pieces and show that they are of one of nine basic types, each of which is evidently Seifert fibered. Moreover, we will see that the fibered structures match together along the T_i when we glue the pieces N_i together again to form N.

Consider N_1, say. It has at least one boundary component that is a T_j. Since T_j is not canonical, there exists an essential torus T' in N which essentially intersects T_1. We make the intersection of T' with the union $T = T_1 \cup \cdots \cup T_r$ minimal, and then by Lemma B.5 the intersection consists of parallel essential curves on T'.

Let s be one of the curves of $T_j \cap T'$. Let P be the part of $T' \cap N_1$ that has s in its boundary. P is an annulus. Let s' be the other boundary component of P. It may lie on a T_k with $k \neq j$ or it may lie on T_j again. We first consider the case

Case 1: s' lies on a different T_k.

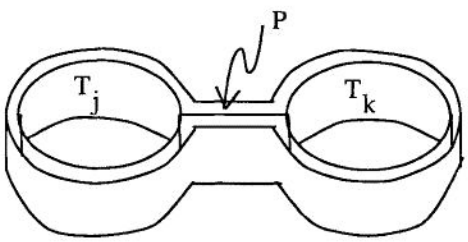


FIGURE 1.

In Fig. 1 we have drawn the boundary of a regular neighborhood of the union $T_j \cup T_k \cup P$ in N_1. The top and the bottom of the picture should be identified, so that the whole picture is fibered by circles parallel to s and s'. The boundary torus T of the regular neighborhood is a new torus disjoint from the T_i's, so it must be parallel to a T_i or nonessential. If T is parallel to a T_i then N_1 is isomorphic to $X \times S^1$, where X is a the sphere with three disks removed. Moreover all three boundary tori are T_i's. If T is nonessential, then it is either parallel to a boundary component of N or it is compressible in N. In the former case N_1 is again isomorphic to $X \times S^1$, but with one of the three boundary tori belonging to ∂N. If T is compressible then it must bound a solid torus in N_1 and the fibration by circles extends over this solid torus with a singular fiber in the middle (there must be a singular fiber there, since otherwise the two tori T_j and T_k are parallel).

We draw these three possible types for N_1 in items 1,2, and 3 of Fig. 2, suppressing the circle fibers, but noting by a dot the position of a possible singular fiber. Solid lines represent part of ∂N while dashed lines represent T_i's. We next

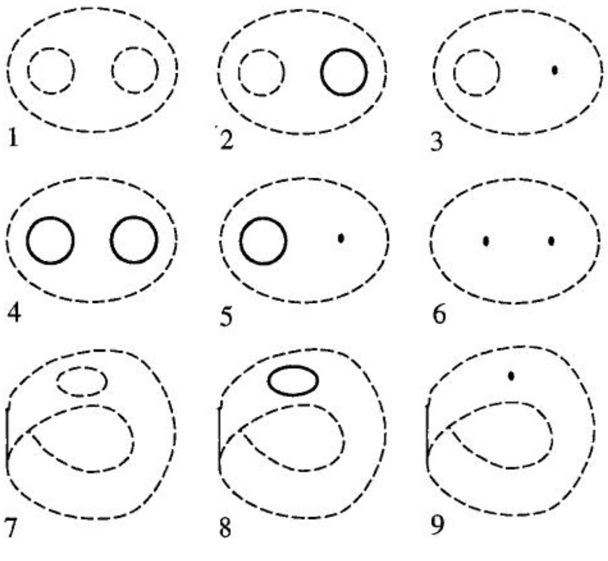


FIGURE 2.

consider

Case 2. s' also lies on T_j, so both boundary components s and s' of P lie on T_j.

Now P may meet T_j along s and s' from the same side or from opposite sides, so we split Case 2 into the two subcases:

Case 2a. P meets T_j along s and s' both times from the same side;

Case 2b. P meets T_j along s and s' from opposite sides.

It is not hard to see that after splitting along T_j, Case 2b behaves just like Case 1 and leads to the same possibilities. Thus we just consider Case 2a. This case has two subcases 2a1 and 2a2 according to whether s and s' have the same or opposite orientations as parallel curves of T_j (we orient s and s' parallel to each other in P). We have pictured these two cases in Fig. 3 with the boundary of a regular neighborhood of $T_j \cup P$ also pictured.

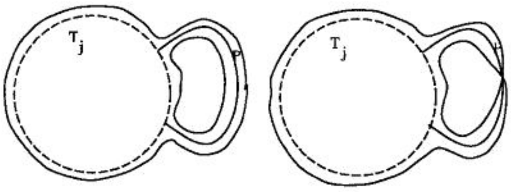


FIGURE 3.

In Case 2a1 the regular neighborhood is isomorphic to $X \times S^1$ and there are two tori in its boundary, each of which may be parallel to a T_i, parallel to a boundary component of N, or bound a solid torus. This leads to items 1 through 6 of Fig. 2.

In Case 2b the regular neighborhood is a circle bundle over a möbius band with one puncture (the unique such circle bundle with orientable total space). The torus in its boundary may be parallel to a T_i, parallel to a component of ∂N, or bound a solid torus. This leads to cases 7, 8, and 9 of Fig. 2. In all cases but case 9 a dot signifies a singular fiber, but in case 9 it signifies a fiber which may or may not be singular.

We now know that N_1 is of one of the types of Fig. 2 and thus has a Seifert fibration by circles, and therefore similarly for each piece N_i. Moreover, on the boundary component T_j that we are considering, the fibers of N_1 are parallel to the intersection curves of T_j and T' and therefore match up with fibers of the Seifert fibration on the piece on the other side of T_j. We must rule out the possibility that, if we do the same argument using a different boundary component T_k of N_1, it would be a different Seifert fibration which we match across that boundary component. In fact, it is not hard to see that if N_1 is as in Fig. 2 with more than one boundary component, then its Seifert fibration is unique. To see this up to homotopy, which is all we really need, one can use the fact that the fiber generates a normal cyclic subgroup of $\pi_1(N_1)$, and verify by direct calculation that $\pi_1(N_1)$ has a unique such subgroup in the cases in question.

(In fact, the only manifold of a type listed in Fig. 2 that does not have a unique Seifert fibration is case 6 when the two singular fibers are both degree 2 singular fibers and case 9 when the possible singular fiber is in fact not singular. These are in fact two Seifert fibrations of the same manifold $T^1 Mb$, the unit tangent bundle of the Möbius band Mb. This manifold can also be fibered by lifting the fibration of the Möbius band by circles to a fibration of the total space of the tangent bundle of Mb by circles.) □

An alternative characterization of the JSJ decomposition is as a minimal decomposition of M along incompressible tori into Seifert fibered and simple pieces. In particular, if some torus of the JSJsystem has Seifert fibered pieces on both sides of it, the fibrations do not match up along the torus.

Exercise 3. Verify the last statement.

APPENDIX C. SEIFERT FIBERED MANIFOLDS

In this section we describe all threemanifolds that can be Seifert fibered with circle or torus fibers.

Seifert's original concept of what is now called "Seifert fibration" referred to 3manifolds fibered with circle fibers, allowing certain types of "singular fibers." For orientable 3manifolds this gives exactly fibrations over 2orbifolds, so it is reasonable to use the term "Seifert fibration" more generally to mean "fibration of a manifold over an orbifold." So we start by recalling what we need about orbifolds.

C.1. **Orbifolds.** An $n$orbifold is a space that looks locally like $\mathbb{R}^n/G$ where G is a finite subgroup of $GL(n,\mathbb{R})$. Note that G varies from point to point, for example, a neighborhood of $[x] \in \mathbb{R}^n/G$ looks like $\mathbb{R}^n/G_x$ where $G_x = \{g \in G \mid gx = x\}$.

We will restrict, for simplicity, to locally orientable 2orbifolds (i.e., the above G preserves orientation). Then the only possible local structures are $\mathbb{R}^2/C_p$, $p = 1, 2, 3, \dots$, where C_p is the cyclic group of order p acting by rotations. the local structure is then a "cone point" with "cone angle $2\pi/p$" (Fig. 4).

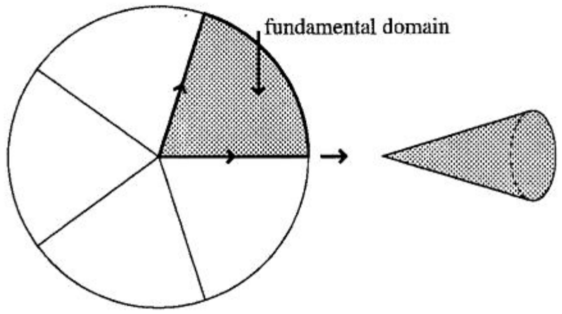


FIGURE 4.

Topologically, a 2orbifold is thus simply a 2dimensional manifold, in which certain points are singled out as being "orbifold points" where the total angle around the point is considered to be $2\pi/p$ instead of 2π. The underlying 2manifold is classified by its genus g (we use negative numbers to refer to nonorientable surfaces, so genus $-1, -2, \dots$ mean projective plane, Klein bottle, etc.). We can thus characterize a 2–orbifold by a tuple of numbers $(g; p_1, \dots, p_k)$ where g is the genus and $p_1, \dots, p_k$ describe the orbifold points.

C.2. **General concept of Seifert fibrations via orbifolds.** A map $M \to N$ is a Seifert fibration if it is locally isomorphic to maps of the form $(U \times F)/G \to U/G$, with U/G an orbifold chart in N (so U is isomorphic to an open subset of $\mathbb{R}^n$ with an action of the finite group G) and F a manifold with Gaction such that the diagonal action of G on $U \times F$ is a free action. The freeness of the action is to make M a manifold rather than just an orbifold.

C.3. **Seifert circle fibrations.** We start with "classical" Seifert fibrations, that is, fibrations with circle fibers, but with some possibly "singular fibers." We first describe what the local structure of the singular fibers is. This has already been suggested by the proof of JSJ above.

We have a manifold M^3 with a map $\pi\colon M^3 \to F^2$ to a surface such that all fibers of the map are circles. Pick one fiber f_0 and consider a regular neighborhood N of it. We can choose N to be a solid torus fibered by fibers of π. To have a reference, we will choose a longitudingal curve l and a meridian curve m on the boundary torus $T = \partial N$. The typical fiber f on T is a simple closed curve, so it is homologous to $pl + rm$ for some coprime pair of integers p, r. We can visualize the solid torus N like an onion, made up of toral layers parallel to T (boundaries of thinner and thinner regular neighborhoods) plus the central curve f_0. Each toral shell is fibered just like the boundary T, so the typical fibers converge on pf_0 as one moves to the center of N.

Exercise 4. Let s be a closed curve on T that is a section to the boundary there. Then (with curves appropriately oriented) one has the homology relation $m = ps + qf$ with $qr \equiv 1 \pmod p$.

The pair $(p.q)$ is called the *Seifert pair* for the fiber f_0. It is important to note that the section s is only well defined up to multiples of f, so by changing the section s we can alter q by multiples of p. If we have chosen things so $0 \le q < p$ we call the Seifert pair *normalized.*

By changing orientation of f_0 if necessary, we may assume $p \geq 0$. In fact:

Exercise 5. If M^3 contains a fiber with $p = 0$ then M^3 is a connected sum of lens spaces. (A lens space is a 3manifold obtained by gluing two solid tori along their boundaries; it is classified by a pair of coprime integers (p, q) with $0 \le q < p$ or $(p, q) = (0, 1)$. One usually writes it as $L(p, q)$. Special cases are $L(0, 1) = S^2 \times S^1$, $L(1, 0) = S^3$. For $p \geq 0$ $L(p, q)$ can also be described as the quotient of $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ by the action of $\mathbb{Z}/p$ generated by $(z, w) \mapsto (e^{2\pi i/p}z, e^{2\pi i q/p}w)$.)

We therefore rule out $p = 0$ and assume from now on that every fiber has $p > 0$. Note that $p = 1$ means that the fiber f_0 is a nonsingular fiber, i.e., the whole neighborhood N of f_0 is fibered as the product $D^2 \times S^1$. If $p > 1$ then f_0 is a singular fiber, but the rest of N consists only of nonsingular fibers. In particular, singular fibers are isolated, so there are only finitely many of them in M^3.

Now let $f_0, \dots, f_r$ be a collection of fibers which includes all singular fibers. For each one we choose a fibered neighborhood N_i and a section s_i on ∂N_i as above, giving a Seifert pair (p_i, q_i) with $p_i \geq 1$ for each fiber. Now on $M_0 := M^3 - \bigcup \int(N_i)$ we have a genuine fibration by circles over a surface with boundary. Such a fibration always has a section, so we can assume that our sections s_i on ∂M_0 have come from a global section on M_0. This section on M_0 is not unique. If we change it, then each s_i is replaced by $s_i + n_i f$ for some integers n_i, and a homological calculation shows that $\sum n_i$ must equal 0. The effect on the Seifert pairs (p_i, q_i) is to replace each by $(p_i, q_i - n_i p_i)$. In summary, we see that changing the choice of global section on M_0 changes the Seifert pairs (p_i, q_i) by changing each q_i, keeping fixed:

- the congruence class $q_i \pmod{p_i}$
- $e := \sum \frac{q_i}{p_i}$

The above number e is called the *Euler number* of the Seifert fibration. We have not been careful about describing our orientation conventions here. With a standard choice of orientation conventions that is often used in the literature, e is more usually defined as $e := -\sum \frac{q_i}{p_i}$.

Note that we can also change the collection of Seifert pairs by adding or deleting pairs of the form $(1, 0)$, since they correspond to nonsingular fibers with choice of local section that extends across this fiber. Up to these changes the topology of the base surface F and the collection of Seifert pairs is a complete invariant of M^3. A convenient normalization is to take f_0 to be a nonsingular fiber and $f_1, \dots, f_s$ to be all the singular fibers and normalize so that $0 < q_i < p_i$ for $i \geq 1$. This gives a complete invariant:

$$(g; (1, q_0), (p_1, q_1), \dots, (p_r, q_r)) \quad \text{with } g = \text{genus}(F)$$

which is unique up to permuting the indices $i = 1, \dots, r$. A common convention is to use negative g for the genus of nonorientable surfaces (even though we are assuming M^3 is oriented, the base surface F need not be orientable).

Exercise 6. Explain why the base surface F most naturally has the structure of an orbifold of type $(g; p_1, \dots, p_r)$.

Seifert manifolds can be given locally homogeneous Riemannian metrics (briefly "geometric structures"). There are six underlying types for the geometric structure. The orbifold Euler characteristic of this base orbifold and the Euler number e of the Seifert fibration together determine the type of natural geometric structure that can be put on M^3.

There exist a few manifolds M^3 that have more than one Seifert fibration. For example, the lens space $L(p, q)$ has infinitely many, all of them with base surface S^2 and at most two singular fibers (but if one requires the base to be a "good orbifold"– one that is globally the quotient of a group action on a manifold), then $L(p, q)$ has only one Seifert fibration up to isomorphism).

C.4. **"Seifert fibrations" with torus fiber.** There are two basic ways a 3manifold M^3 can fiber with torus fibers. The base must be 1dimensional so it is either the circle, or the 1orbifold that one obtains by factoring the circle by the involution $z \mapsto \bar{z}$. The latter is the unit interval $[0, 1]$ considered as an orbifold.

In the case M^3 fibers over the circle, we can obtain it by taking $T^2 \times [0, 1]$ and then pasting $T^2 \times \{0\}$ to $T^2 \times \{1\}$ by an automorphism of the torus. Thinking of the torus as $\mathbb{R}^2/\mathbb{Z}^2$, it is clear that an automorphism is given by a 2×2 integer matrix of determinant 1 (it is orientation preserving since we want M^3 orientable), that is, by an element $A \in \text{SL}(2, \mathbb{Z})$.

Exercise 7. Show the resulting M^3 is Seifert fibered by circles if $|\operatorname{tr}(A)| \leq 2$. Work out the Seifert invariants.

If $|\operatorname{tr}(A)| > 2$ then the natural geometry for a geometric structure on M is the $\mathbb{S}ol$ geometry.

In case M^3 fibers over the orbifold $[0, 1]$ we can construct it as follows. The manifold SMb mentioned in Section 4 of this paper can also be described as the total space of the unique interval bundle over the Klein bottle with oriented total space. From this point of view, SMb is fibered by tori that are the boundaries of thinner versions of SMb obtained by shrinking the interval I, with the Klein bottle zerosection as special fiber. Gluing two copies of SMb by some identification of their torus boundaries gives M^3. This M^3 has a double cover that fibers over the circle, and it is Seifert fibered by circles if and only this double cover is Seifert fibered by circles, otherwise it again belongs to the $\mathbb{S}ol$ geometry.

C.5. **Simple Seifert fibered manifolds.** We said earlier that if M^3 is irreducible and all its boundary components are tori then only tori occur in the JSJ decomposition. This is essentially because of the following:

Exercise 8. Let M^3 be an orientable manifold, all of whose boundary components are tori, which is simple (no essential tori) and suppose M^3 contains an essential embedded annulus (i.e., incompressible and not boundary parallel). Then M^3 is Seifert fibered over D^2 with two singular fibers, or over the annulus or the Möbius band with at most one singular fiber.

For manifolds with boundary, "simple" is often defined by the absence of essential annuli and tori, rather than just tori. The difference between these definitions is just the manifolds of the above exercise. $D^2 \times S^1$ is simple by either definition. The only other simple Seifert fibered manifolds are those that are Seifert fibered over S^2 with at most three singular fibers or over $\mathbb{P}^2$ with at most one singular fiber and which moreover satisfy $e(M^3 \to F) \neq 0$.

Appendix D. Geometric versus JSJ decomposition

The JSJ decomposition does not give exactly the decomposition of M^3 into pieces with geometric structure. This is because of the fact that the manifold SMb (that caused us problems in Section 4 of this paper) may occur as a Seifert fibered piece in the decomposition, but it does not admit a geometric structure.

Recall (subsection C.4) that SMb has an embedded Klein bottle, and splitting it along this Klein bottle gives $T^2 \times I$. Thus, whenever SMb occurs as a piece in the JSJ decomposition, instead of including the boundary of this piece as one of the surfaces to split M^3 along, we include its core Klein bottle. The effect of this is simply to eliminate all such pieces without affecting the topology of any other piece. The modified version of JSJdecomposition that one gets this way is called *geometric decomposition.*

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