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Resolution of mobiles in characteristic zero

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RESOLUTION OF MOBILES IN CHARACTERISTIC ZERO

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Lecture I: Axiomatic outset

This is a compact introduction for reading [EH]. Technicalities are omitted. Exact details can be found in [EH]. For background and motivations see [H].

Mobiles : Let W be a regular ambient scheme of finite type over a field K. A *singular mobile* of dimension n in $V \subseteq W$ is a quadruple $\mathcal{M} = (J, c, D, E)$ with

> J a coherent ideal sheaf on a regular locally closed n-dimensional subscheme V of W,
>
> c a non negative integer, the *control*,
>
> $D = (D_n, \ldots, D_1)$ a collection of (not necessarily reduced) normal crossings divisors D_i in W with D_n transversal to V, the *combinatorial handicap* of $\mathcal{M}$,
>
> $E = (E_n, \ldots, E_1)$ a collection of (reduced) normal crossings divisors E_i in W with E_n transversal to V, the *transversal handicap* of $\mathcal{M}$,

such that $J = M \cdot I$ with M the ideal $I_V(D_n \cap V)$ defining $D_n \cap V$ in V and I an ideal in V. The mobile $\mathcal{M}$ is *resolved* if $\mathrm{ord}_a J < c$ for all $a \in V$. Here, $\mathrm{ord}_a J$ denotes the minimal order of vanishing of the elements of the ideal J at points a of V.

Our purpose will be to resolve mobiles by a sequence of blowups in regular centers Z, dropping eventually the order of J below c. As a preliminary stage, we will decrease the order of the second factor I of J until it becomes 0, in which case $J = M \cdot 1 = M$ is locally a monomial ideal (principalization of ideals). Once J is a monomial ideal, it is relatively easy to make its order drop (combinatorial resolution).

Examples : (1) Let $X \subseteq W$ be a closed singular hypersurface of ideal $J = I_W(X)$, set $V = W$ and $c = 2$ and let all D_i and E_i be empty. Then $\mathcal{M}$ is resolved if X is regular (resolution of schemes).

(2) Let X and Y be regular in W, let J be the ideal $I_W(X \cap Y)$ of their intersection, and set $V = W$ and $c = 1$ and all D_i and E_i empty. Then $\mathcal{M}$ is resolved if X and Y do not meet (separation of schemes).

(3) Let J be generated by a product $x^\alpha \cdot f^o$, where x^α is the monomial defining D_n in W and f defines a regular hypersurface in W (o is some positive integer). Assume that E_n is D_n and equals the exceptional locus in W produced by previous blowups. To monomialize J, the appropriate center would be the support Z of f. But it may happen that Z is not transversal to E_n, in which case the factor x^α in J may no longer be a monomial after the blowup. Therefore, a smaller center has to be chosen to achieve by auxiliary blowups first the transversality of Z with the exceptional locus (transversality problem).

Remarks : Mobiles are the minimal datum to define and prove resolution by blowups. The ideal J is the object one wishes to improve. The control c fixes the goal and prescribes the transformation rule for J. The first entry D_n of the combinatorial handicap D determines the factorization of J into a monomial part and a singular part. This corresponds to collecting in J after each blowup the exceptional factor. The union of the E_i equals the exceptional locus produced by earlier blowups. The first entry E_n collects those exceptional components which may cause a transversality problem with the center when trying to resolve J (the dangerous components). The remaining entries D_i and E_i are only relevant for induction purposes and will appear later on.

$Transform\ of\ mobile$: If Z is a regular closed subscheme of W, we denote by $\pi : W' \to W$ the blowup of W with center Z and exceptional divisor $Y' = \pi^{-1}(Z)$. Let $\mathcal{M} = (J, c, D, E)$ be a mobile in $V \subseteq W$ and assume that V contains Z. Let $\pi_V : V' \to V$ be the restriction of π over V; it coincides with the blowup of V in Z with exceptional divisor $Y' \cap V'$. Assume that Z lies in the locus $\mathrm{top}(J, c)$ of points where $J = M \cdot I$ has order at least c (if $\mathrm{top}(J, c)$ is empty, $\mathcal{M}$ is already resolved), and that Z is transversal to all D_i. We define

$J^* = \pi^{-1}(J)$	the *total* transform of J in V';
$J^! = J^* \cdot I_{V'}(Y' \cap V')^{-c}$	the *controlled* transform of J w.r.t. c;
$I^\curlyvee = I^* \cdot I_{V'}(Y' \cap V')^{-\mathrm{ord}_Z I}$	the *weak* transform of J;
$D'_n = D^*_n + (\mathrm{ord}_Z I - c) \cdot Y'$	the transform of D_n.

It follows from general properties of blowups that all these objects are well defined. Observe that D'_n is a normal crossings divisor in W' transversal to V' because Z is transversal to D_n and contained in V. Setting $J' = J^!$, $I' = I^\curlyvee$ and $M' = I_{V'}(D'_n \cap V')$ we get the factorization $J' = M' \cdot I'$ in V' with locally monomial factor M'.

The definition of E'_n is more subtle, because E_n and E'_n will be *stratified* divisors (i.e., coherent only on the strata of a stratification of W). We shall assume in addition that $Z \subseteq \mathrm{top}(I)$, i.e. that Z lies in the locus of points where I has maximal order in V. This will hold by the choice of Z (which was not specified yet). Then $\mathrm{ord}_a I = \mathrm{ord}_Z I$ for all $a \in Z$, and $\mathrm{ord}_{a'} I' \leq \mathrm{ord}_a I$ for all $a' \in Y'$ (because I' is the weak transform of I).

Outside Y', we chose E'_n equal to E_n, since $\pi : W' \setminus Y' \to W \setminus Z$ is an isomorphism. On the closed subscheme O of Y' where $\mathrm{ord}_{a'} I' = \mathrm{ord}_a I$ we set $E'_n = E^\curlyvee_n = E^*_n \cdot I_{W'}(Y')^{-\mathrm{ord}_Z E_n}$ (weak transform), outside O we set $E'_n = E^\curlyvee_n + Y'$. It turns out that only if the order of I drops the new exceptional component Y' can become dangerous for J. This definition takes care of the transversality problem in dimension n. For the definitions of D'_i and E'_i for $i < n$, see [EH].

We set $\mathcal{M}' = (J', c', D', E')$ with $c' = c$. This is a mobile in $V' \subseteq W'$, the *transform* of $\mathcal{M}$ under $\pi : W' \to W$.

$Resolving\ mobiles$: A $strong\ resolution$ of a mobile $\mathcal{M} = (J, c, D, E)$ in $V \subseteq W$ is a sequence of blowups of W in regular closed centers such that

(e 1) *Embeddedness*: The centers are transversal to the handicaps D and E (and hence also to the exceptional loci).

(e 2) *Equivariance*: The sequence commutes with smooth morphisms $W^- \to W$ and embeddings $W \to W^+$ (taking fibre products).

(e 3) *Excision*: The induced sequence of blowups of V does not depend on the embedding of V in W.

(e 4) *Economy*: The centers of blowup lie over the top locus $\mathrm{top}(J, c)$ of J.

(e 5) *Effectiveness* (optional): The centers are given as the top locus of an upper-semicontinuous local invariant attached to the mobile.

(e 6) *Exit*: The final transform of $\mathcal{M}$ is resolved (i.e., the respective J has everywhere order less than c).

Theorem. *Mobiles* $\mathcal{M} = (J, c, D, E)$ *with* $D_i = \emptyset$ *for* $i < n$ *admit strong resolutions.*

Idea of proof : By induction on n, the dimension of the scheme V where J lives. Set $W_n = V$ and write $\mathcal{M} = (J_n, c_{n+1}, D, E)$. Associate to $\mathcal{M}$ locally at points of W_n mobiles $\mathcal{N} = (J_{n-1}, c_n, T, F)$ in (a collection of) locally closed hypersurfaces W_{n-1} of W_n such that the resolution of each $\mathcal{N}$ in $W_{n-1} \subset W_n$ (which can be assumed to exist by induction on n) does not depend on the choice of W_{n-1}. In particular, the centers in the various W_{n-1} will define a global center Z in W_n which, in turn, induces a resolution of $\mathcal{M}$.

Observe the shift in the index of the controls. The handicaps T and F of $\mathcal{N}$ are obtained from the handicaps of $\mathcal{M}$ by deleting the entries with index n, $T = (D_{n-1}, \ldots, D_1)$, $F = (E_{n-1}, \ldots, E_1)$. Thus it only remains to choose suitable hypersurfaces W_{n-1} and to construct ideals J_{n-1} in W_{n-1} with controls c_n. Before doing so, we collect properties we wish to be satisfied.

Properties : In order to make the descent in dimension work, a few “functorial” properties are required. In particular, the choice of local hypersurfaces is subject to certain restrictions. Any descent with the properties below will allow to establish the induction.

(f 1) *Factorization* : $J_{n-1} = M_{n-1} \cdot I_{n-1}$ with $M_{n-1} = I_{W_{n-1}}(D_{n-1} \cap W_{n-1})$ and I_{n-1} an ideal in W_{n-1}.

(f 2) *Transversality* : Setting $Q_n = I_{W_n}(E_n \cap W_n)$ the top locus $\mathrm{top}(Q_n)$ is contained in the exceptional components which may fail to be transversal to W_{n-1}.

(f 3) *Top loci* : The following inclusions hold:

$$\mathrm{top}(I_{n-1}) \cap \mathrm{top}(Q_{n-1}) \subseteq \mathrm{top}(J_{n-1}, c_n) \subseteq \mathrm{top}(I_n) \cap \mathrm{top}(Q_n) \subseteq \mathrm{top}(J_n, c_{n+1}).$$

(In practice, the ideals I_n and I_{n-1} have to be replaced by slightly modified ideals P_n and P_{n-1} in order to ensure these inclusions.)

(f 4) *Independence* : The orders of I_{n-1} and Q_{n-1} do not depend on the choice of W_{n-1}.

(f 5) *Commutativity* : Let $W' \to W$ be the blowup of W with center Z contained in $\mathrm{top}(I_n) \cap \mathrm{top}(Q_n)$, and let $\mathcal{M}'$ be the transform of $\mathcal{M}$. Assume that locally at a point a of some W_{n-1} one has $Z \subseteq W_{n-1}$, with induced blowup $W'_{n-1} \to W_{n-1}$. Let $a' \in Y'$ be a point above a.

(a) If $\mathrm{ord}_{a'} I'_n = \mathrm{ord}_a I_n$, then $a' \in W'_{n-1}$.

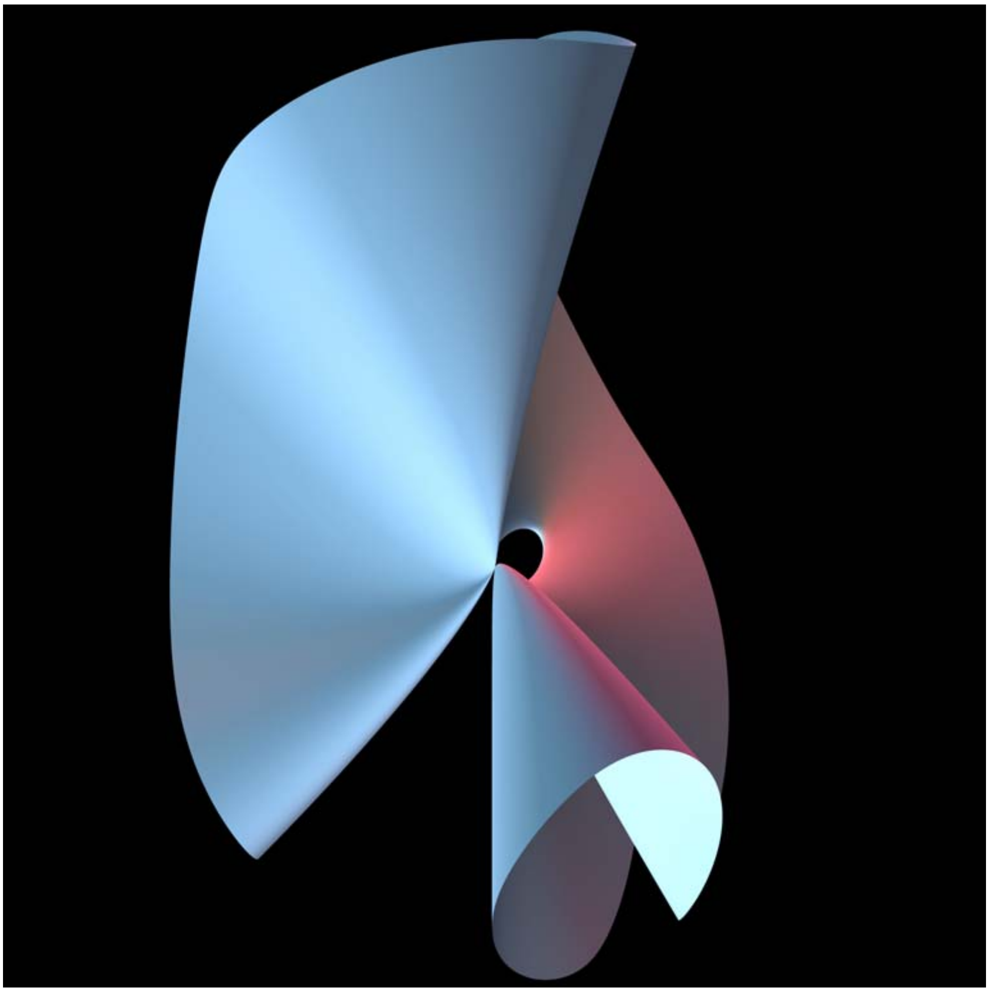
(b) If $\mathcal{N}'$ is the mobile associated to $\mathcal{M}'$ in W'_{n-1} at a point a' with $\mathrm{ord}_{a'} I'_n = \mathrm{ord}_a I_n$ and $\mathrm{ord}_{a'} Q'_n = \mathrm{ord}_a Q_n$, then J'_{n-1} is the controlled transform of J_{n-1} with respect to the control $c_n = \mathrm{ord}_a J_n$.

(c) In this case, I'_{n-1} equals the weak transform $I^{\curlyvee}_{n-1}$ of I_{n-1}. And, if $\mathrm{ord}_{a'} I'_{n-1} = \mathrm{ord}_a I_{n-1}$, then Q'_{n-1} equals the weak transform $Q^{\curlyvee}_{n-1}$ of Q_{n-1}.

Remarks : These are only the main properties one needs to build up the induction. The clue is of course the commutativity of the descent in dimension with blowups in case the orders of I_n and Q_n remain constant:

$$\begin{array}{ccc} J'_n & \rightsquigarrow & J'_{n-1} \\ \downarrow & & \downarrow \\ J_n & \rightsquigarrow & J_{n-1} \end{array}$$

.



Nadelöhr: $(x^2 - y^3)^2 = (x + y^2)z^3$

Lecture II: Construction of descent

Invariant : The properties (f 1) – (f 5) essentially suffice to define the local resolution invariant and to apply induction. Let $\mathcal{M}$ be the mobile in $V \subset W$, let a be a point in V and let $\mathcal{N}$ be a mobile associated to $\mathcal{M}$ locally at a in one dimension less. We set

$$i_a\mathcal{M} = (\mathrm{ord}_a I_n, \mathrm{ord}_a Q_n, i_a\mathcal{N}) \in \mathbb{N}^{2n}.$$

This vector of integers is considered with respect to the lexicographic ordering. The components of $i_a\mathcal{N} = (\mathrm{ord}_a I_{n-1}, \mathrm{ord}_a Q_{n-1}, \ldots)$ can be assumed to be defined by induction on n. By property (f 3) *independence*, the invariant will not depend on the choice of the local hypersurfaces.

We are left to deduce from these properties that the top locus of $i_a\mathcal{M}$ defines a suitable (regular and transversal) center in W and that under the blowup of W in Z the invariant $i_{a'}(\mathcal{M}')$ of the transformed mobile $\mathcal{M}'$ has decreased at all points in the exceptional divisor Y'.

In addition, it has to be shown that there exists a construction of mobiles in dimension $n-1$ which satisfy the above properties.

The center: The center is defined as the top locus of the invariant $i_a\mathcal{M}$. It is closed, because orders of ideals are upper-semicontinuous functions. To show that it is regular, we may place ourselves locally at a point a of W. By exhaustion, there must be an index d between 0 and $n-1$ for which the (local) ideal J_{d+1} is bold regular, i.e., the power of a variable (e.g., any non-zero ideal in one variable is of this type). Its support is a regular hypersurface W_d in the scheme W_{d+1} where J_{d+1} lives, and this hypersurface is also the top locus of J_{d+1} (which, in turn, equals $\mathrm{top}(J_{d+1}, c_{d+2})$). It is clear how to decrease the order of J_{d+1}: Just blow up W_{d+1} in the center $Z = W_d$ (notice that then the weak transform of the factor I_{d+1} becomes the trivial ideal 1).

In this case, of course, it is no longer necessary to descend in dimension. This describes the choice of our center: Locally, it equals some W_d. Hence it is regular. And this local definition extends to a global center in W by what was said earlier.

The center is transversal to the exceptional locus by property (f 2) *transversality* since it is contained in $\mathrm{top}(Q_i)$ for all $i \geq d+1$.

Construction of descent: There are various options how to define – in characteristic zero – local hypersurfaces W_{n-1} and ideals J_{n-1} so as to satisfy properties (f 1) – (f 5). Also, the transformation rules for mobiles allow some flexibility. Each choice yields a distinct resolution algorithm, with different features and advantages. We shall describe the construction of the descent as in [EH].

Let be given the mobile $\mathcal{M} = (J_n, c_{n+1}, D, E)$ with factorization $J_n = M_n \cdot I_n$ locally at a point a of $W_n = V$. Let P_n be the modification of I_n as mentioned in (f 2) (in most cases it equals I_n) and let $Q_n = I_{W_n}(E_n \cap W_n)$ be the associated transversality ideal.

For $W_{n-1} \subseteq W_n$ a regular local hypersurface at a, let J_{n-1} be the *coefficient ideal* of the product $K_n = P_n \cdot Q_n$

$$J_{n-1} = \mathrm{coeff}_{W_{n-1}}(K_n).$$

It is defined by expanding the elements of K_n as power series in the variable (called x_n) defining W_{n-1} in W_n and taking (suitable powers of) the coefficients of these series (which are hence series in the coordinates of W_{n-1}). If K_n is a principal ideal generated by a series of form $f = x_n^o + g_0(x_{n-1}, \ldots, x_1)$, the coefficient ideal is just the ideal generated by g. In the more general case $f = x_n^o + g_{o-1}x_n^{o-1} + \ldots + g_0$, the ideal J_{n-1} is generated by equilibrated powers of $g_{o-1}, \ldots, g_0$ (for the precise definition, cf. [EH]).

The coefficient ideal, and in particular its order, depend on the choice of W_{n-1}. The order can be made independent of this choice by allowing only hypersurfaces which maximize the order (hypersurfaces of *weak maximal contact*, cf. (f 4) *independence*). Don't care that J_{n-1} may still depend on W_{n-1}.

In characteristic zero, there is an explicit construction of hypersurfaces of weak maximal contact, so called *osculating* hypersurfaces. They enjoy three key properties. First, they contain locally the top locus of the ideal K_n (thus the same W_{n-1} can be chosen locally along $\mathrm{top}(K_n)$). Second, their transform under blowup contains all points a' where the order of K_n (hence of I_n and Q_n) may have remained constant (these are precisely the points where we wish to perform the descent in dimension). And, finally, their transforms are again osculating for the weak transform K_n' of K_n.

In positive characteristic, osculating hypersurfaces need not exist. It seems that it is not possible to find a substitute satisfying the first and third property (the second already follows from weak maximal contact).

Now, with the choice of W_{n-1} as osculating hypersurface and the definition of J_{n-1} as a coefficient ideal it is a half page computation to show that the key property (f 5) *commutativity* holds. The other properties are immediate. The control c_n has to be set equal to the order of K_n at a. It is constant locally at a along $\mathrm{top}(K_n) = \mathrm{top}(P_n) \cap \mathrm{top}(Q_n)$.

This is all what had to be done to construct $\mathcal{N}$ from $\mathcal{M}$.

Induction : By construction, the center Z lies (locally at a point a of W) in all $\mathrm{top}(I_i)$ and $\mathrm{top}(Q_i)$ for $d+1 \leq i \leq n$, where d is maximal so that J_{d+1} is bold regular. By general properties of blowups, the orders of the weak transform of ideals does not increase under blowup if the center is contained in their top locus. As $I_n' = I_n^\curlyvee$ is the weak transform of I_n, we get $\mathrm{ord}_{a'} I_n' \leq \mathrm{ord}_a I_n$. If strict inequality holds, we are done, $i_{a'}\mathcal{M}' <_{lex} i_a\mathcal{M}$ for $a \in Z$ and $a' \in Y'$.

If equality holds, we have by property (f 5) *commutativity* that $Q_n' = Q_n^\curlyvee$ is the weak transform of Q_n, hence $\mathrm{ord}_{a'} Q_n' \leq \mathrm{ord}_a Q_n$. Now the argument repeats, yielding $i_{a'}\mathcal{M}' \leq_{lex} i_a\mathcal{M}$. But at some instance, at least at index $i = d+1$, the order of I_i must drop under blowup to 0 (because I_{d+1} is bold regular). If its order was positive, we are done $i_{a'}\mathcal{M}' <_{lex} i_a\mathcal{M}$. If it was zero, $I_{d+1} = 1$ and hence $J_{d+1} = M_{d+1}$ is a monomial. In this case, one applies a combinatorial resolution argument (as mentioned at the beginning) to reduce the order of J_{d+2} below c_{d+3}. Induction applies.

[EH] Encinas, S., Hauser, H.: Strong resolution of singularities in characteristic zero. Comment. Math. Helv. **77** (2002), 821-845.

[H] Hauser, H.: The Hironaka Theorem on resolution of singularities. (Or: A proof we always wanted to understand.) Bull. Amer. Math. Soc. **40** (2003), 323-403.

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