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Hodge Theory: the search for purity

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1 Introduction

These notes aim at providing a summary of mixed Hodge theory, starting with the origin of weights on the cohomology of algebraic varieties in étale cohomology and ending with the discussion of mixed Hodge modules. Big parts of the text are extracted from the book [21] in preparation.

I spent the academic year 1974–1975 at the Institut des Hautes Études Scientifiques in France. Many things were discussed there which occur in these notes. Goresky and MacPherson already had the basic ideas of intersection homology. With Shi-Wei-Shu we had a mini-seminar on D-modules, discussing the Bernstein polynomial. And of course, Deligne was there, the founder of mixed Hodge theory, and John Morgan, who extended this theory to homotopy groups.

On hindsight one can say that all ingredients of mixed Hodge modules were present at the time. However, these ingredients seemed totally unrelated, and it required the joint effort of many people to achieve the complete picture of Hodge theory which we have now: Deligne, proving the Weil conjectures, and providing a sheaf-theoretic framework for intersection homology; Kashiwara and Mebkhout connecting D-modules and constructible sheaves, Malgrange who discovered how to formulate the nearby and vanishing cycle functors on the level of D-modules. But the great synthesis was accomplished by Morihiko Saito in the years 1981–1988. His work is very complicated, but the results are very powerful.

These notes contain these ingredients of mixed Hodge modules, starting with the notion of weights. Then pure and mixed Hodge structures are defined. We proceed with perverse complexes, deal with the foundations of D-module theory and formulate the Riemann-Hilbert correspondence. Finally an axiomatic treatment of mixed Hodge modules is given.

I hope that these notes (and my talks) will be a useful introduction to this fascinating but until now rather inaccessible topic. I presuppose a knowledge of algebraic geometry including sheaf theory; in the last three sections knowledge of derived categories is very useful. See e.g. [10].

2 Weights in ℓ -adic cohomology

Let X_0 be an algebraic variety over a finite field k with q elements and let $X = X_0 \times_k \bar{k}$. The zeta function of X_0 is given by

$$Z(X_0, t) = \prod_i \det(1 - F^*t, H_c^i(X, \mathbb{Q}_\ell))^{(-1)^{i+1}} \quad (1)$$

where ℓ is a prime not dividing q and F is the Frobenius morphism on X , given in coordinates by $(x_1, \dots, x_n) \mapsto (x_1^q, \dots, x_n^q)$.

This definition involves the etale cohomology groups of X with values in \mathbb{Q}_ℓ . However, the zeta function can also be obtained by point counting:

$$t \frac{d}{dt} \log Z(X_0, t) = \sum_{n>0} a_n t^n \quad (2)$$

where a_n is the number of points of X_0 with values in \mathbb{F}_{q^n} . The equality of (1) and (2) is a consequence of the Lefschetz fixed point theorem in ℓ -adic cohomology, applied to the action of F^n on X . Deligne showed [6, Thm. (1.6)]

Theorem 1 *Suppose that X_0 is smooth projective. Then for each i , the characteristic polynomial $\det(t - F^*, H_c^i(X, \mathbb{Q}_\ell))$ has integer coefficients independent of ℓ . The complex roots α of this polynomial (the complex conjugates of the eigenvalues of F^*) have absolute value $|\alpha| = q^{i/2}$.*

This result, one of the Weil Conjectures, can be reformulated as: the i -th ℓ -adic cohomology group of a smooth projective variety over a finite field is *pure of weight i* as a module over the Galois group of the ground field.

Next consider a complex algebraic variety X . It is defined by a finite number of polynomial equations. The coefficients of these polynomials generate a subalgebra R of \mathbb{C} , of finite type over \mathbb{Z} , and X is obtained from a scheme \mathcal{X} over R by extension of scalars to \mathbb{C} .

Take a maximal ideal m of R , let $k = R/m$ and let $q = |k|$. Write $X_m = \mathcal{X} \times_R \bar{k}$ and let F_m denote its Frobenius endomorphism.

Theorem 2 *There exists non-zero $f \in R$ such that for all m with $f \notin m$, and all primes $\ell \notin m$ the eigenvalues of F_m on $H^i(X_m, \mathbb{Q}_\ell)$ are algebraic integers, and for each eigenvalue α there exists an integer $w(\alpha)$ such that all complex conjugates of α have absolute value $q^{\frac{w(\alpha)}{2}}$.*

One has a natural identification of $H^i(X, \mathbb{Q}) \otimes \mathbb{Q}_\ell$ with $H^i(X_m, \mathbb{Q}_\ell)$, which gives a rational structure to the latter vector space. If ${}_m W_j$ denotes the sum of the generalized eigenspaces corresponding to the eigenvalues α with $w(\alpha) \leq j$, then the increasing filtration ${}_m W$ is defined over \mathbb{Q} and its intersection W with $H^i(X, \mathbb{Q})$ is independent of ℓ and m .

See [7, Thm. 14].

3 Mixed Hodge theory

Let us again consider a complex algebraic variety. The filtration W on the cohomology of X obtained from Theorem 2 is Deligne's *weight filtration*, and is one of the main ingredients for his theory of mixed Hodge structures on the cohomology of complex algebraic varieties. For the i -th cohomology group of a smooth projective variety X the weight filtration is trivial:

$$W_{i-1}H^i(X) = 0, \quad W_i H^i(X) = H^i(X)$$

which we rephrase as: $H^i(X)$ is *pure of weight i* .

The other ingredient of Deligne's mixed Hodge theory is the *Hodge filtration*. This is not defined on the cohomology with rational coefficients, but one needs to pass to complex coefficients to see it. For a smooth projective variety we can choose a Kähler metric so that we are in the realm of compact Kähler manifolds. The complex cohomology of such a compact Kähler manifold admits a Hodge decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X) \tag{3}$$

where we consider the cohomology as de Rham cohomology, i.e. as the space of (complex valued) closed differential forms modulo the exact forms.

The Kähler metric provides us with an adjoint d^* of the operator d , and the Laplacian δ is defined as $\delta = dd^* + d^*d$. A form ω is called *harmonic* if $\delta\omega = 0$.

Each cohomology class is represented by a unique *harmonic form*. On the other hand, complex valued differential forms on X admit a decomposition according to type (a form is of type (p, q) if in local holomorphic coordinates it is given by an expression containing p factors dz_i and q factors $d\bar{z}_j$). A cohomology class is called of type (p, q) if its harmonic representative is of type (p, q) . The Kähler identities imply that a form is harmonic if and only if all of its (p, q) -components are, hence we have the Hodge decomposition. One can show that a cohomology class is of type (p, q) if and only if it can be represented by a closed form of type (p, q) . Hence the Hodge decomposition does not depend on the choice of Kähler metric.

The *Hodge filtration* on $H^i(X, \mathbb{C})$ is defined in this case by

$$F^p H^i(X, \mathbb{C}) = \bigoplus_{r \geq p} H^{r, i-r} . \quad (4)$$

Note that it is a *decreasing* filtration and that one needs the de Rham complex (differential forms) to define it. It reflects the analytic structure in contrast with the weight filtration, which is rather reflecting the topology.

Two reasons to work with the Hodge filtration rather than with the Hodge decomposition is that the Hodge filtration has better behaviour in families of varieties (it varies holomorphically) and that it has a good generalization to arbitrary complex algebraic varieties.

The Hodge decomposition of the cohomology of a compact Kähler manifold gives rise to the following concept.

Definition 1 A *pure weight i (rational) Hodge structure* on a finite dimensional rational vector space V consists of a direct sum decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=i} V^{p,q}, \text{ with } V^{p,q} = \overline{V^{q,p}}$$

on its complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$. The numbers

$$h^{p,q}(V) := \dim V^{p,q}$$

are the *Hodge numbers* of the Hodge structure. The polynomial

$$\begin{aligned} P_{\text{hn}}(V) &= \sum_{p,q \in \mathbb{Z}} h^{p,q}(V) u^p v^q \\ &= \sum h^{p,i-p}(V) u^p v^{i-p} \in \mathbb{Z}[u, v, u^{-1}, v^{-1}] \end{aligned} \quad (5)$$

its associated *Hodge number polynomial*. The corresponding *Hodge filtration* is given by

$$F^p(V) = \bigoplus_{r \geq p} V^{r, i-r}.$$

The classical example of a weight i Hodge structure is furnished by the rank i (singular) cohomology group $H^i(X)$ (with \mathbb{Q} -coefficients) of a compact Kähler manifold X .

Various multilinear algebra operations can be applied to Hodge structures. Suppose that V and W are two real vector spaces with a Hodge structure of weight k and ℓ respectively. Then:

1. $V \otimes W$ has a Hodge structure of weight $k + \ell$ given by

$$F^p(V \otimes W)_{\mathbb{C}} = \sum_m F^m(V_{\mathbb{C}}) \otimes F^{p-m}(W_{\mathbb{C}}) \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}$$

and with Hodge number polynomial given by

$$P_{\text{hn}}(V \otimes W) = P_{\text{hn}}(V)P_{\text{hn}}(W). \quad (6)$$

2. On $\text{Hom}(V, W)$ we have a Hodge structure of weight $\ell - k$:

$$F^p \text{Hom}(V, W)_{\mathbb{C}} = \{f : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}} \mid fF^n(V_{\mathbb{C}}) \subset F^{n+p}(W_{\mathbb{C}}) \quad \forall n\}$$

with Hodge number polynomial

$$P_{\text{hn}}(\text{Hom}(V, W))(u, v) = P_{\text{hn}}(V)(u^{-1}, v^{-1})P_{\text{hn}}(W)(u, v). \quad (7)$$

In particular, taking $W = \mathbb{Q}$ with $W_{\mathbb{C}} = W^{0,0}$ we get a Hodge structure of weight $-k$ on the dual V^* of V with Hodge number polynomial

$$P_{\text{hn}}(V^*)(u, v) = P_{\text{hn}}(V)(u^{-1}, v^{-1}). \quad (8)$$

The *Hodge structure* $\mathbb{Q}(\ell)$ of Tate is the \mathbb{Q} -vector space $(2\pi i)^\ell \mathbb{Q} \subset \mathbb{C}$ with Hodge structure of pure type $(-\ell, -\ell)$. This Hodge structure can be used to reduce weights of a given pure Hodge structure V to 0 or 1 by *Tate twisting*: if V has weight 2ℓ , $V(\ell) := V \otimes \mathbb{Q}(\ell)$ is a Hodge structure of weight 0 and if V has weight $2\ell + 1$, its twist $V(\ell)$ has weight 1. Note also

$$e_{\text{hn}}(V(\ell)) = e_{\text{hn}}(V)(uv)^{-\ell}. \quad (9)$$

Definition 2 Let HS be the category of pure \mathbb{Q} -Hodge structures (of varying weights). The *Grothendieck ring* $K_0(\text{HS})$ is the free group on the isomorphism classes $[V]$ of Hodge structures V modulo the subgroup generated by $[V] - [V'] - [V'']$ where

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

is an exact sequence of pure Hodge structures where the complexified maps preserve the Hodge decompositions.

Because the Hodge number polynomial (5) is clearly additive and by (6) behaves well on products, we have:

Lemma 3 *The Hodge number polynomial defines a ring homomorphism*

$$P_{\text{hn}} : K_0(\text{HS}) \rightarrow \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

Pure Hodge structures in algebraic geometry arise as the cohomology groups of smooth projective varieties: for such X the cohomology group $H^k(X, \mathbb{Q})$ carries a Hodge structure of weight k . We put

$$\begin{aligned} \chi_{\text{Hdg}}(X) &:= \sum (-1)^k [H^k(X)] \in K_0(\text{HS}); \\ e_{\text{hn}}(X) &:= P_{\text{hn}}(\chi_{\text{Hdg}}(X)) = \sum (-1)^k P_{\text{hn}}(H^k(X)) \in \mathbb{Z}[u, v, u^{-1}, v^{-1}] \end{aligned}$$

which we call the Hodge-Grothendieck class and the *Hodge-Euler polynomial* of X respectively.

Lemma 4 *Suppose that X is a smooth projective variety and $Y \subset X$ is a smooth closed subvariety. Let $\pi : Z \rightarrow X$ be the blowing-up with center Y and let $E = \pi^{-1}(Y)$ be the exceptional divisor. Then*

$$\begin{aligned} \chi_{\text{Hdg}}(X) - \chi_{\text{Hdg}}(Y) &= \chi_{\text{Hdg}}(Z) - \chi_{\text{Hdg}}(E); \\ e_{\text{hn}}(X) - e_{\text{hn}}(Y) &= e_{\text{hn}}(Z) - e_{\text{hn}}(E). \end{aligned}$$

Proof: By [12, p. 605]

$$0 \rightarrow H^k(X) \rightarrow H^k(Z) \oplus H^k(Y) \rightarrow H^k(E) \rightarrow 0$$

is exact.

We recall the definition of the naive Grothendieck group $K_0(\text{Var})$ of (complex) varieties. It is the quotient of the free abelian group on isomorphism classes $[X]$ of algebraic varieties over \mathbb{C} with relations $[X] = [X - Y] + [Y]$ for $Y \subset X$ a closed subvariety.

Theorem 5 *The group $K_0(\text{Var})$ is isomorphic to the free abelian group generated by the isomorphism classes of smooth complex projective varieties subject to the relations $[\emptyset] = 0$ and $[Z] - [E] = [X] - [Y]$ where X, Y, Z, E are as in Lemma 4.*

Proof: See [2, Theorem 3.1].

Remark 6 In particular, for every complex variety X there exist projective smooth varieties $X_1, \dots, X_r, Y_1, \dots, Y_s$ such that

$$[X] = \sum_i [X_i] - \sum_j [Y_j] \text{ in } K_0(\text{Var}).$$

For compact X , the construction of cubical hyperresolutions $(X_I)_{\emptyset \neq I \subset A}$ of X from [13] leads to such an expression:

$$[X] = \sum_{\emptyset \neq I \subset A} (-1)^{\#I-1} [X_I].$$

Corollary 7 *The Hodge Euler polynomial extends to a group homomorphism*

$$e_{\text{hn}} : K_0(\text{Var}) \rightarrow \mathbb{Z}[u, v, u^{-1}, v^{-1}]$$

However, one cannot expect a pure Hodge structure on the cohomology of singular or non-compact algebraic varieties. For example, if a vector space V carries a Hodge structure of odd weight, then its dimension must be even. So if X is an algebraic variety such that $H^1(X, \mathbb{Q})$ carries a Hodge structure of weight one, then the first Betti number of X had better be even. However, if X is an irreducible algebraic curve with one node, then the first Betti number is odd.

Deligne's first discovery is, that the graded parts for the weight filtration

$$\mathrm{Gr}_m^W H^i(X, \mathbb{Q})$$

underly *pure Hodge structures* of weight k . His second discovery is, that the Hodge filtrations

$$F^\bullet \mathrm{Gr}_m^W H^i(X, \mathbb{C})$$

on all the graded quotients are induced by a canonical Hodge filtration F^\bullet on $H^i(X, \mathbb{C})$. The data of the weight and Hodge filtrations on the cohomology of X constitute what has been called a *mixed Hodge structure*, constructed by Deligne in [4], [5].

We let V be a finite dimensional \mathbb{Q} -vector space.

Definition 3 A *mixed Hodge structure* on V consists of two filtrations, an increasing filtration on V , the *weight filtration* W_\bullet and a decreasing filtration F^\bullet on $V_{\mathbb{C}} = V \otimes \mathbb{C}$, the *Hodge filtration* which has the additional property that it induces a pure Hodge structure of weight k on each graded piece

$$\mathrm{Gr}_k^W(V) = W_k/W_{k-1}.$$

Mixed Hodge structures form an abelian category. Every morphism of mixed Hodge structures is strictly compatible with the Hodge and weight filtrations. As a consequence, an exact sequence of mixed Hodge structures remains exact if at each place one applies the functor $V \mapsto \mathrm{Gr}_k^W(V)$.

Deligne's main result is [4] [5]:

Theorem 8 *Homology, cohomology, Borel-Moore homology and cohomology groups with compact supports of algebraic varieties carry functorial mixed Hodge structures. Virtually all natural maps like cup product and Poincaré morphisms are morphisms of mixed Hodge structures.*

The weight filtration on these groups has the following properties:

1. If X is compact, then $W_i H^i(X) = H^i(X)$.
2. If X is smooth, then $W_{i-1} H^i(X) = 0$.

Moreover, the mixed Hodge structures on $H_i(X)$ and on $H^i(X)$ are dual to each other, and the same holds for $H_c^i(X)$ and $H_i^{\mathrm{BM}}(X)$.

It is clear that the Grothendieck ring of the category of mixed Hodge structures is the same as for Hodge structures. Moreover, Lemma 4 has the following generalization to the context of singular varieties:

Lemma 9 Let $f : \tilde{X} \rightarrow X$ be a proper modification with discriminant D . Put $E = f^{-1}(D)$. Let $g : f|_E : E \rightarrow D$ and let $i : D \rightarrow X$ and $\tilde{i} : E \rightarrow \tilde{X}$ denote the inclusions. Then one has a long exact sequence of mixed Hodge structures

$$\dots \rightarrow H^k(X) \rightarrow H^k(\tilde{X}) \oplus H^k(D) \rightarrow H^k(E) \rightarrow H^{k+1}(X) \rightarrow \dots$$

It is called the Mayer-Vietoris sequence for the discriminant square associated to f . One has

$$\chi_{\text{Hdg}}(\tilde{X}) = \chi_{\text{Hdg}}(X) + \chi_{\text{Hdg}}(E) - \chi_{\text{Hdg}}(D).$$

Here the discriminant D is the minimal closed subset of X with the property that f is an isomorphism when restricted to the inverse image complement of D .

4 From groups to sheaves

In this section we describe Deligne's construction of the Hodge and weight filtrations on the cohomology of smooth algebraic varieties.

Let U be a smooth complex algebraic variety. By [20] U is Zariski open in some compact algebraic variety X , which by [14] one can assume to be smooth and for which $D = X - U$ locally looks like the crossing of coordinate hyperplanes. It is called a *normal crossing divisor*. If the irreducible components D_k of D are smooth, we say that D has *simple* or *strict normal crossings*.

Definition 4 We say that X is a *good compactification* of $U = X - D$ if X is smooth and D is a simple normal crossing divisor.

We return for the moment to the situation where $D \subset X$ is a hypersurface (possibly with singularities and reducible) inside a smooth n -dimensional complex manifold X and we set

$$j : U = X - D \hookrightarrow X$$

Recall that a holomorphic differential form ω on U is said to have *logarithmic singularities along D* if ω and $d\omega$ have at most a pole of order one along D .

It follows that these holomorphic differential forms constitute a subcomplex $\Omega_X^\bullet(\log D) \subset j_*\Omega_U^\bullet$.

Suppose now that D has simple normal crossings, $p \in D$ and $V \subset X$ is an open neighbourhood with coordinates (z, \dots, z_n) in which D has equation $z_1 \cdots z_k = 0$. One can show [12, p. 449]

$$\begin{aligned}\Omega_X^1(\log D)_p &= \mathcal{O}_{X,p} \frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{X,p} \frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p} dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{X,p} dz_n, \\ \Omega_X^p(\log D)_p &= \bigwedge^p \Omega_X^1(\log D)_p.\end{aligned}$$

An essential ingredient in the proof of the following theorem is the residue map which is defined as follows. We set $X_k = \{z_k = 0\}$ and we let D' be the divisor on X_k traced out by D . Then writing $\omega = \eta \wedge (dz_k/z_k) + \eta'$ with η, η' not containing dz_k , the *residue map* can be defined as

$$\begin{aligned}\text{res} &: \Omega_X^p(\log D) \rightarrow \Omega_{X_k}^{p-1}(\log D') \\ \omega &\mapsto \eta|_{X_k}.\end{aligned}$$

As a special case we have the Poincaré residues $R_k : \Omega_X^1(\log D) \rightarrow \mathcal{O}_{X_k}$.

Theorem 10 *Let U be a complex algebraic manifold and let X be a good compactification, i.e. $D = X - U$ is a divisor with simple normal crossings. Then the following is true.*

1. $H^k(U; \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log D))$;
2. The trivial filtration F on the complex $\Omega_X^\bullet(\log D)$ given by

$$F^p \Omega_X^\bullet(\log D) = [0 \rightarrow \cdots \rightarrow \Omega_X^p(\log D) \rightarrow \Omega_X^{p+1}(\log D) \rightarrow \cdots]$$

together with the filtration W defined by

$$W_m \Omega_X^p(\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \Omega_X^p(\log D) & \text{for } m \geq p \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D) & \text{if } 0 \leq m \leq p. \end{cases}$$

induce in cohomology two filtrations

$$\begin{aligned}F^p H^k(U, \mathbb{C}) &= \text{Im} \left(\mathbb{H}^k(X, F^p \Omega_X^\bullet(\log D)) \rightarrow H^k(U; \mathbb{C}) \right), \\ W_m H^k(U; \mathbb{C}) &= \text{Im} \left(\mathbb{H}^k(X, W_{m-k} \Omega_X^\bullet(\log D)) \rightarrow H^k(U; \mathbb{C}) \right).\end{aligned}$$

which put a mixed Hodge structure on $H^k(U)$.

A few words about the symbol \mathbb{H} occurring in this theorem. It stands for hypercohomology, of which we give here a simple treatment.

Let \mathcal{F} be a sheaf on a topological space X . We view it as the presheaf $U \mapsto \mathcal{F}(U) := \Gamma(U, \mathcal{F})$, where Γ is the functor “taking global continuous sections”. If instead, we consider $\prod_{x \in U} \mathcal{F}_x$, the “discontinuous sections” over U we obtain a presheaf $\mathcal{C}_{\text{Gdm}}^0 \mathcal{F}$ which is in fact a sheaf. By definition it comes equipped with an injective homomorphism $\mathcal{F} \hookrightarrow \mathcal{C}_{\text{Gdm}}^0 \mathcal{F}$. Following [9, II.4.3] one inductively defines

$$\left. \begin{aligned} \mathcal{Z}^0 \mathcal{F} &= \mathcal{F} \\ \mathcal{Z}^p \mathcal{F} &:= \mathcal{C}_{\text{Gdm}}^{p-1} \mathcal{F} / \mathcal{Z}^{p-1} \mathcal{F} \\ \mathcal{C}_{\text{Gdm}}^p \mathcal{F} &:= \mathcal{C}_{\text{Gdm}}^0 (\mathcal{C}_{\text{Gdm}}^{p-1} \mathcal{F} / \mathcal{Z}^{p-1} \mathcal{F}). \end{aligned} \right\} \quad (\text{Godement's resolution})$$

The sheaves $\mathcal{C}_{\text{Gdm}}^p \mathcal{F}$ are *flabby*, i.e. any section over an open set extends to the entire space X . The natural maps

$$d : \mathcal{C}_{\text{Gdm}}^p \mathcal{F} \rightarrow \mathcal{Z}^{p+1} \mathcal{F} \hookrightarrow \mathcal{C}_{\text{Gdm}}^{p+1} \mathcal{F}$$

fit into an exact complex whose global sections by definition yields the cohomology groups of \mathcal{F} :

$$H^p(X, \mathcal{F}) := H^p(\Gamma(X, \mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F})).$$

From the definition of the Godement resolution it follows that any morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$ induces a morphism of complexes $\mathcal{C}_{\text{Gdm}}^\bullet(f)$ between the respective Godement resolutions. Moreover, for two such morphisms f and g , we have:

$$\mathcal{C}_{\text{Gdm}}^\bullet(f \circ g) = \mathcal{C}_{\text{Gdm}}^\bullet(f) \circ \mathcal{C}_{\text{Gdm}}^\bullet(g).$$

If $f : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on X , the induced morphism $\mathcal{C}_{\text{Gdm}}^\bullet(f)$ induce maps $H^q(f) : H^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{G})$ which therefore behave functorially.

Secondly, any exact sequence of sheaves of R -modules

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

induces short exact complexes on the level of their Godement resolutions and hence long exact sequences

$$\dots H^q(X, \mathcal{F}') \rightarrow H^q(X, \mathcal{F}) \rightarrow H^q(X, \mathcal{F}'') \rightarrow H^{q+1}(X, \mathcal{F}') \dots$$

Next, we pass to a bounded below complex of sheaves \mathcal{F}^\bullet on X . For every \mathcal{F}^p take its Godement resolution $\mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^p$. The derivatives $d^p : \mathcal{F}^p \rightarrow \mathcal{F}^{p+1}$ induce maps of complexes $d^p : \mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^p \rightarrow \mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^{p+1}$ and by functoriality, $d^{p+1} \circ d^p = 0$ so that we have a double complex $\mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^\bullet$. Since the Godement sheaves are flabby, the associated simple complex $s(\mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^\bullet)$ gives a *flabby resolution* of \mathcal{F}^\bullet .

The hypercohomology groups $\mathbb{H}^k(X, \mathcal{F}^\bullet)$ are now defined as the cohomology groups of the complex of global sections of $s(\mathcal{C}_{\text{Gdm}}^\bullet \mathcal{F}^\bullet)$.

5 Intersection homology and perverse sheaves

Consider the cohomology of a possibly singular projective variety X which is irreducible of dimension d . Suppose that the cup product pairing

$$H^i(X, \mathbb{Q}) \otimes H^{2d-i}(X, \mathbb{Q}) \rightarrow H^{2d}(X, \mathbb{Q}) \simeq \mathbb{Q}(-d)$$

is a perfect pairing. This pairing is compatible with the filtration W . On the other hand, $H^i(X)$ has weights $\leq i$ and $H^{2d-i}(X)$ has weights $\leq 2d - i$ because X is compact. This implies that the image of $W_{i-1}H^i(X) \otimes H^{2d-i}(X)$ under cup product is contained in $W_{2d-1}H^{2d}(X) = 0$. Hence we obtain that $W_{i-1}H^i(X) = 0$ i.e. $H^i(X)$ is pure of weight i . So we see that purity is a consequence of Poincaré duality.

The search for purity is therefore intimately connected with the search for Poincaré duality. In this direction, the main development of the previous decades is the discovery of *intersection homology* by Goresky and MacPherson [11]. Its definition involves the choice of a *perverseity*, but for complex analytic spaces, which admit a stratification with only even-dimensional strata, there is a canonical “middle” perverseity which is commonly used. Intersection homology has as its input furthermore a local system \mathbb{V} of rational or complex vector spaces on a dense open subset of the regular part of X . From these data a sheaf complex ${}^\pi\mathbb{V}$ is constructed, the *minimal perverse extension* of \mathbb{V} . Assuming X compact of dimension d_X , we have

$$IH_q(X, \mathbb{V}) = \mathbf{H}^{d_X - q}(X, {}^\pi\mathbb{V}).$$

The characterization of ${}^\pi\mathbb{V}$ involves the notion of *perverse sheaf* on X , which we now introduce.

Definition 5 Let R be a commutative ring with 1 and let \mathcal{F} be a sheaf of R -modules of finite rank on a Whitney-stratifiable analytic space X . We say that \mathcal{F} is *(analytically) constructible* if X admits an analytic stratification such that \mathcal{F} restricts to a locally constant sheaf on any of the strata. We say that a bounded complex \mathcal{F}^\bullet of sheaves of R -modules is *(analytically) cohomologically constructible* if the sheaves $H^q(\mathcal{F}^\bullet)$ have finite rank and are analytically constructible. We set

$$D_c^b(X; R) := \left\{ \begin{array}{l} \text{derived category of bounded} \\ \text{cohomologically constructible complexes} \\ \text{of sheaves of } R\text{-modules on } X \end{array} \right\} \quad (10)$$

On this category one has the *Verdier duality functor* \mathbb{D}_X .

Definition 6 1. A bounded analytically cohomologically constructible complex \mathcal{F}^\bullet of sheaves of R -vector spaces is *perverse* if the following two conditions hold:

$$\begin{array}{ll} \dim_{\mathbb{C}} \text{Supp } H^j(\mathcal{F}^\bullet) \leq -j & \forall j < d_X \quad \text{(support condition);} \\ \dim_{\mathbb{C}} \text{Supp } H^j(\mathbb{D}(\mathcal{F}^\bullet)) \leq -j & \forall j < d_X \quad \text{(cosupport condition).} \end{array}$$

2. In the derived category, perverse complexes make up a subcategory

$$\text{Perv}_R(X) \subset D_c^b(X; R)$$

Then $\text{Perv}_R(X)$ is an abelian category.

Note that a local system \mathbb{V} on a complex manifold, considered as a complex concentrated in degree zero, is not perverse, because its Verdier dual is $\mathbb{V}^\vee[2d_X]$. However, $\mathbb{V}[d_X]$ is perverse.

Definition 7 Let X be a complex variety of pure dimension d_X and \mathbb{V} a local system over a dense open subset of X . With π the middle perversity, the (analytic) intersection complex for \mathbb{V} is the unique perverse complex ${}^\pi\mathbb{V}^\bullet$ on X which restricts to $\mathbb{V}[d_X]$ on U and has no sub- or quotient object in $\text{Perv}_R(X)$ supported on $X - U$.

We have

Lemma 11

$$\begin{array}{l} \mathbb{H}^{d_X - q}(X, {}^\pi\mathbb{V}^\bullet) = IH_q^{\text{BM}}(X, \mathbb{V}) = IH^{2d_X - q}(X, \mathbb{V}), \\ \mathbb{H}_c^{d_X - q}(X, {}^\pi\mathbb{V}^\bullet) = IH_q(X, \mathbb{V}) = IH_c^{2d_X - q}(X, \mathbb{V}). \end{array}$$

Remark 12

- 1) Note that the intersection (co)homology is nonzero only in the interval $[0, 2d_X]$ as it should.
- 2) The complex $\mathbb{Q}_X[d_X]$ is perverse on any complete intersection variety X . However, it need not be the minimal extension of its restriction to the regular locus.

6 D-modules

Let X be an n -dimensional complex manifold. A germ at $x \in X$ of a holomorphic vectorfield on X is the same thing as a \mathbb{C} -linear derivation $D : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$. As such it is an example of a germ of a differential operator of order 1. Germs of functions acting by multiplication on the left give germs of differential operators of order 0. Together they generate a sub-algebra of germs of \mathbb{C} -linear endomorphisms of $\mathcal{O}_{X,x}$. This is the germ at x of the *sheaf of differential operators* on X , denoted by

$$\mathcal{D}_X \subset \text{Hom}_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X).$$

The *order filtration* F_m^{ord} , $m = 0, 1, \dots$ is defined recursively as follows. One sets $F_0^{\text{ord}}\mathcal{D}_X = \mathcal{O}_X$ and for any open set $U \subset X$, one sets

$$F_m^{\text{ord}}\mathcal{D}_X(U) = \{P \in \mathcal{D}_X(U) \mid Pf - fP \in F_{m-1}^{\text{ord}}\mathcal{D}_X(U) \forall f \in \mathcal{O}_X(U)\}. \quad (11)$$

This defines a presheaf on X which then needs to be sheafified to obtain $F_m^{\text{ord}}\mathcal{D}_X$. To see this concretely, let $(U, (z_1, \dots, z_n))$ be a holomorphic chart. Putting $\partial_i = \partial/\partial z_i$, $i = 1, \dots, n$ and using multi-index notation $I = (i_1, \dots, i_n)$, $|I| = \sum_k i_k$, sections P of $F_m^{\text{ord}}\mathcal{D}_X$ over U can be uniquely written as

$$P = \sum_{|I| \leq m} P_I \partial^I, \quad P_I \in \mathcal{O}_X(U), \quad \partial^I = \partial_1^{i_1} \dots \partial_n^{i_n}.$$

This shows that the sheaves of m -th order operators are locally free of finite rank and that

$$\text{Gr}_m^{F^{\text{ord}}}\mathcal{D}_X \cong \text{Sym}^m(T(X)).$$

In this way we may compare the non-commutative algebra \mathcal{D}_X with its commutative graded, the symmetric algebra on $T(X)$:

$$\text{Gr}\mathcal{D}_X := \bigoplus_{m=0}^{\infty} \text{Gr}_m^{F^{\text{ord}}}\mathcal{D}_X \cong \text{Sym}(T(X)). \quad (12)$$

We observe:

Lemma 13 *The sheaf $\text{Gr}\mathcal{D}_X$ can be identified with the sheaf of holomorphic functions*

$$\sigma : T^\vee X \rightarrow \mathbb{C}$$

which restrict polynomially to each cotangent space.

A sheaf \mathcal{M} of left \mathcal{D}_X -modules is called a \mathcal{D}_X -module, or, if no confusion is possible, a D -module. So \mathcal{M} admits a left multiplication with germs of vector fields, or, in other words, we obtain a Lie-algebra representation

$$\rho : T(X) \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{M}). \quad (13)$$

Definition 8 A \mathcal{D}_X -module \mathcal{M} is *coherent* if it is first of all locally finitely generated, i.e. every point has a neighbourhood U over which there exists a surjection

$$\mathcal{D}_U^p \rightarrow \mathcal{M}|_U \rightarrow 0,$$

and secondly if every homomorphism $\mathcal{D}_U^q \rightarrow \mathcal{M}|_U$ has a kernel which is locally finitely generated.

From the fact that \mathcal{O}_X is coherent it is not hard to see [3, II.§3] that \mathcal{D}_X is coherent (as a left- \mathcal{D}_X -module) and from this one deduces the following lemma.

Lemma 14 *A D -module is coherent if and only if it is locally finitely presented: locally over an open subset $U \subset X$ we have an exact sequence of $\mathcal{D}(U)$ -modules*

$$\mathcal{D}(U)^p \rightarrow \mathcal{D}(U)^q \rightarrow \mathcal{M}(U) \rightarrow 0.$$

Examples

1. The structure sheaf \mathcal{O}_X is a left \mathcal{D}_X -module, generated globally by the section 1. In local coordinates (z_1, \dots, z_n) on an open set $U \subset X$ the kernel of the sheaf homomorphism $\text{ev} : \mathcal{D}_X \rightarrow \mathcal{O}_X$ given by $P \mapsto P(1)$ is generated by the vector fields $\partial_1, \dots, \partial_n$. Hence \mathcal{O}_X is a \mathcal{D}_X -module locally of finite presentation, and therefore a coherent \mathcal{D}_X -module. A coordinate invariant description of $\ker(\text{ev})$ can be given as follows. The sheaf $T(X)$ of germs of holomorphic tangent vectors is locally free of rank n over \mathcal{O}_X . Hence the tensor product $\mathcal{D}_X \otimes_{\mathcal{O}_X} T(X)$ is a locally

free left \mathcal{D}_X -module. The map $P \otimes \theta \mapsto P\theta$ defines a homomorphism of left \mathcal{D}_X -modules $\mathcal{D}_X \otimes T(X) \rightarrow \mathcal{D}_X$ and it represents $\ker(\text{ev})$. This shows that \mathcal{O}_X is a coherent \mathcal{D}_X -module.

2. Every locally free \mathcal{D}_X -module is coherent.
3. Every \mathcal{O}_X -coherent \mathcal{D}_X -module \mathcal{M} is locally free as an \mathcal{O}_X -module. To see this, it suffices to show that \mathcal{M}_x is a free $\mathcal{O}_{X,x}$ -module for any $x \in X$. Let \mathfrak{m}_x denote the maximal ideal of $\mathcal{O}_{X,x}$ and choose elements e_1, \dots, e_r in \mathcal{M}_x which map to a \mathbb{C} -basis of the fibre

$$\mathcal{M}(x) := \mathcal{M}_x / \mathfrak{m}_x \mathcal{M}_x.$$

By Nakayama's lemma, \mathcal{M}_x is generated by e_1, \dots, e_r . These generators form a free basis. Indeed, if not there would be a relation $\sum_{i=1}^r f_i e_i = 0$ such that not all the f_i are zero. Let k to be the minimum of the orders of vanishing at x of f_i . We call it the *order of the relation*. For simplicity, assume that f_1 realizes this minimum. We cannot have $k = 0$, since in that case the classes of the e_i in $\mathcal{M}(x)$ become dependent. But if $k > 0$, we can reduce order of the relation: choose i such that in local coordinates, $\partial^i f_1$ vanishes to order k at x . Then, writing $\partial^i e_j = \sum_k b_{jk} e_k$, we find

$$0 = \partial^i \left(\sum_{j=1}^s f_j e_j \right) = \left(\partial^i f_1 + \sum_{k=1}^s f_k b_{k1} \right) e_1 + \sum_{j=2}^s \left(\partial^i f_j + \sum_{k=1}^s f_l b_{kj} \right) e_j$$

which is a relation of lower order. This contradiction indeed shows that \mathcal{M} is locally free as an \mathcal{O}_X -module.

4. Let \mathcal{M}_X denote the sheaf of germs of meromorphic functions on X . This is a \mathcal{D}_X -module which is not locally of finite type.

6.1 Good Filtrations and Characteristic Varieties

Let X be a complex manifold of dimension n and \mathcal{M} be a \mathcal{D}_X -module.

Definition 9 A *filtration* on \mathcal{M} is an increasing and exhaustive ($\mathcal{M} = \bigcup_p F_p \mathcal{M}$) sequence of submodules $(F_p \mathcal{M})_{p \in \mathbb{Z}}$ such that $F_r^{\text{ord}} \mathcal{D}_X [F_p \mathcal{M}] \subset F_{p+r} \mathcal{M} \forall r, s \in \mathbb{Z}$. It is called a *good filtration* if moreover

1. locally on X this filtration is bounded below (in the sense that locally $F_p\mathcal{M} = 0$ for $p \in \mathbb{Z}$ small enough) and above (in the sense that locally for $r \in \mathbb{Z}$ large enough we have $F_r^{\text{ord}}\mathcal{D}_X[F_p\mathcal{M}] = F_{p+r}\mathcal{M}$;
2. each $F_p\mathcal{M}$ is a coherent \mathcal{O}_X -module.

A *filtered D -module* is a D -module equipped with a good filtration.

For every coherent \mathcal{D}_X -module \mathcal{M} such a good filtration exists locally on X : starting from a local presentation $\bigoplus^a \mathcal{D}_X \xrightarrow{v} \bigoplus^b \mathcal{D}_X \xrightarrow{u} \mathcal{M} \rightarrow 0$ one can put $F_p\mathcal{M} := u(\bigoplus^b F_p^{\text{ord}}\mathcal{D}_X)$ for $p \geq 0$ while $F_p\mathcal{M} = 0$ for $p < 0$. Then $F_r^{\text{ord}}\mathcal{D}_X[F_p\mathcal{M}] = F_{p+r}\mathcal{M}$ for all $r, p \in \mathbb{Z}$. Conversely, if \mathcal{M} locally possesses a good filtration, \mathcal{M} is coherent.

To test if a given filtration is good, the following Lemma is useful (see [3, II.4]). To state it, recall (12) that $\text{Gr}\mathcal{D}_X$ is the graded module associated to \mathcal{D}_X with respect to the order filtration. Similarly, we set

$$\text{Gr}_F\mathcal{M} = \bigoplus_k \text{Gr}_F^k\mathcal{M}.$$

Lemma 15 *Let (\mathcal{M}, F) be a \mathcal{D}_X -module equipped with a filtration. Then F is good precisely when $\text{Gr}_F\mathcal{M}$ is coherent as a $\text{Gr}\mathcal{D}_X$ -module.*

It is also important (and easy to show) that any two good filtrations F and G on a given \mathcal{D}_X -module \mathcal{M} are locally commensurable in the sense that there exist two integers a and b such that locally for all $p \in \mathbb{Z}$ we have $F_{p-a}\mathcal{M} \subset G_p\mathcal{M} \subset F_{p+b}\mathcal{M}$. Using this, one proves

Proposition 16 *Let $\mathcal{I}(\mathcal{M}, F)$ be the annihilator of $\text{Gr}_F\mathcal{M}$, i.e. the ideal of $\text{Gr}\mathcal{D}_X$ consisting of w with $w\bar{m} = 0$ for all $\bar{m} \in \text{Gr}_F\mathcal{M}$. Put*

$$\sqrt{\mathcal{I}(\mathcal{M}, F)} = \{a \in \text{Gr}(\mathcal{D}_X) \mid \exists k \in \mathbb{N}, a^k \in \mathcal{I}(\mathcal{M}, F)\}.$$

Then $\sqrt{\mathcal{I}(\mathcal{M}, F)}$ does not depend on the choice of the good filtration F on \mathcal{M} .

Since locally good filtrations F exist, we deduce from this that there exists a globally defined sheaf of ideals $\sqrt{\mathcal{I}(\mathcal{M})} \subset \text{Gr}(\mathcal{D}_X)$ which locally coincides with $\sqrt{\mathcal{I}(\mathcal{M}, F)}$. Recall (Lemma 13) that $\text{Gr}(\mathcal{D}_X)$ consists of the sheaf of functions on the total space $T^\vee X$ of the cotangent bundle of X which are polynomial on each fibre $T_x^\vee X$. The ideal $\sqrt{\mathcal{I}(\mathcal{M})}$ thus defines a subvariety

of $T^\vee X$, the *characteristic variety* of \mathcal{M} , which in each fibre $T_x^\vee X$ is a cone. It will be denoted

$$\text{Char}(\mathcal{M}) := \bigcup_{x \in X} V(\sqrt{\mathcal{I}_x}) \subset T(X)^\vee. \quad (14)$$

We finally remark that if we have a good filtration F on \mathcal{M} , the characteristic variety can also be seen as the support of the ideal $\text{Gr}^F(\mathcal{M}) \subset \text{Gr}(\mathcal{D}_X)$ (inside the cotangent bundle).

Examples

1. Let $\mathcal{M} = \mathcal{O}_X$. Then a good filtration is given by $F_p \mathcal{M} = 0$ for $p < 0$ and $F_p \mathcal{M} = \mathcal{M}$ for $p \geq 0$. The same procedure works if \mathcal{M} is a \mathcal{D}_X -module which is coherent as an \mathcal{O}_X -module. The characteristic variety of such a \mathcal{D}_X -module is the zero section of the cotangent bundle. Conversely, suppose that the characteristic variety of \mathcal{M} consists of the zero section. Then for local coordinates (z_1, \dots, z_n) on $U \subset X$, considering the differentials dz_j as local functions w_j on the total space of the cotangent bundle, $(z_1, \dots, z_n, w_1, \dots, w_n)$ give a set of local coordinates on $T^\vee(U) \simeq U \times \mathbb{C}^n$. Then $\sqrt{\mathcal{I}(\mathcal{M})}$ is generated by (w_1, \dots, w_n) . This means that $\text{Gr}_F \mathcal{M}$ is killed by a power of the ideal (w_1, \dots, w_n) and hence is a finitely generated \mathcal{O}_U -module. Hence \mathcal{M} is itself a finitely generated \mathcal{O}_X -module i.e. \mathcal{M} is a coherent \mathcal{O}_X -module and hence free.
2. Let $D \subset X$ be a submanifold of codimension one. Recall that

$$\mathcal{O}_X(*D) := \bigcup_m \mathcal{O}_X(mD),$$

the sheaf of meromorphic functions on X , holomorphic on $X - D$ and having a pole along D . Let $\mathcal{M} = \mathcal{O}_X(*D)/\mathcal{O}_X$ and put $F_p \mathcal{M} = 0$ for $p < 0$ and $F_p \mathcal{M} = \mathcal{O}_X(pD)/\mathcal{O}_X$ if $p \geq 0$. This defines a good filtration on \mathcal{M} . If $\mathcal{N}_{D|X} = \mathcal{O}_X(D)/\mathcal{O}_X$ is the normal bundle of D in X , then $\text{Gr}_p^F(\mathcal{M}) = 0$ for $p \leq 0$ and $\text{Gr}_p^F(\mathcal{M}) \simeq \mathcal{N}_{D|X}^{\otimes p}$ for $p > 0$. Let (z_1, \dots, z_n) be local coordinates on X such that D is given by $z_1 = 0$. Let $\delta(z_1)$ be the class of z_1^{-1} modulo \mathcal{O}_X . Then $\delta(z_1)$ locally generates $\text{Gr}^F(\mathcal{M})$ over $\text{Gr}(\mathcal{D}_X) \simeq \mathcal{O}_X[w_1, \dots, w_n]$. The annihilator ideal of this generator is generated by z_1, w_2, \dots, w_n . Hence $\text{Char}(\mathcal{M})$ is the *conormal bundle* of D in X , i.e. the subspace of $T^\vee(X)$ consisting of pairs (x, α) such that the covector α vanishes on tangent vectors to D .

3. Let

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$$

be an exact sequence of \mathcal{D}_X -modules. If two of these are coherent, the third one is coherent too. In that case, we have

$$\text{Char}(\mathcal{M}) = \text{Char}(\mathcal{M}') \cup \text{Char}(\mathcal{M}'').$$

Applying this to the defining sequence for $\mathcal{O}_X(*D)/\mathcal{O}_X$ it follows that the characteristic variety of $\mathcal{O}_X(*D)$ is the union of the zero section and the conormal bundle of D .

4. The order filtration on \mathcal{D}_X is a good filtration. We see that $I(\mathcal{M}, F)$ is the zero ideal, so the characteristic variety of \mathcal{M} is the whole cotangent bundle.

6.2 Basics on Holonomic D -Modules

The tangent bundle $T(X)$ is a symplectic manifold: its tangent bundle carries a canonical symplectic form. A linear subspace A of a vector space endowed with an alternating nondegenerate bilinear form is called *isotropic* if $A \subset A^\perp$ and *involutive* if $A \supset A^\perp$. If $A = A^\perp$ then A is called *Lagrangian*. These definitions have their analogs for subspaces $C \subset T(X)$: it is called isotropic iff each tangent space to C at a regular point is isotropic, and similarly for involutive and Lagrangian.

It is a deep theorem that for a coherent \mathcal{D}_X -module \mathcal{M} the characteristic variety $\text{Char}(\mathcal{M}) \subset T(X)^\vee$ is involutive. See [17]. Hence the following definition makes sense.

Definition 10 A coherent \mathcal{D}_X -module is *holonomic* if its characteristic variety is Lagrangian, or, equivalently if

$$\dim \text{Char}(\mathcal{M}) = d_X.$$

In that case $\text{Char}(\mathcal{M})$ consists of the union of closures of normal bundles to regular loci of irreducible subvarieties of X .

Definition 11 A coherent \mathcal{D}_X -module \mathcal{M} is called *regular* if it has global good filtration whose annihilator ideal is equal to its radical, i.e. such that the components of the characteristic variety have multiplicity one.

6.3 De Rham functor and Riemann-Hilbert correspondence

For a coherent \mathcal{D}_X -module \mathcal{M} we define its *de Rham complex* as

$$\mathrm{DR}_X(\mathcal{M}) = \left[\left(\mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \cdots \rightarrow \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{M} \right) \right] [d_X], \quad (15)$$

where the derivatives in the complex are given in local coordinates by

$$d(\omega \otimes m) = d\omega \otimes m - \sum_{i=1}^n (dz_i \wedge \omega) \otimes \partial_i m.$$

The link between D -modules and perverse complexes is given by Kashiwara's theorem, one of the central ingredients of the Riemann-Hilbert Correspondence: [15]

Theorem 17 *Let X be a complex analytic manifold. The de Rham complex of a holonomic \mathcal{D}_X -module is a perverse complex.*

We let $D_{\mathrm{rh}}^b(\mathcal{D}_X)$ denote the derived category of bounded complexes of \mathcal{D}_X -modules whose cohomology sheaves are regular holonomic. The de Rham complex can equally be defined for complexes of \mathcal{D}_X -modules and we have the celebrated

Theorem 18 (RIEMANN-HILBERT CORRESPONDENCE (II)) *Let X be a complex algebraic manifold. The de Rham functor establishes an equivalence of categories*

$$D_{\mathrm{rh}}^b(\mathcal{D}_X) \longleftrightarrow D_c^b(\mathbb{C}_X).$$

It induces an equivalence of categories

$$\{\text{regular holonomic } \mathcal{D}_X\text{-modules}\} \longleftrightarrow \{\text{perverse complexes on } X\},$$

i. e. the cohomology sheaves of a regular holonomic complex \mathcal{M}^\bullet is concentrated in degree 0 if and only if $\mathrm{DR}_X(\mathcal{M}^\bullet)$ is perverse.

See [18, 19, 16].

7 Mixed Hodge modules

7.1 Motivating example

Let X be a projective manifold of dimension n and let $Y \subset X$ be a smooth hypersurface. We are going to define some sheaf data on X whose ingredients are a perverse sheaves and a D-module, both filtered, which together determine the mixed Hodge structure on the cohomology of $U = X - Y$. Let $j : U \rightarrow X$ and $i : Y \rightarrow X$ denote the inclusion maps.

First we consider the derived direct image $K^\bullet = \mathbb{R}j_*\mathbb{Q}_U[n]$. Its cohomology sheaves are given by $H^{-n}(K^\bullet) = \mathbb{Q}_X$ and $H^{-n+1}(K^\bullet) = i_*\mathbb{Q}_Y(-1)$ where (-1) refers to a Tate twist. As $\mathbb{Q}_U[n]$ is self-dual the Verdier dual of K^\bullet is $j_!\mathbb{Q}[n]$ and we see that K^\bullet is an object of $\text{Perv}_X(\mathbb{Q})$. It carries a weight filtration, given by

$$0 \subset W_n K^\bullet = \tau_{\leq -n} K^\bullet \subset W_{n+1} K^\bullet = K^\bullet.$$

This will take care of the rational structure, and the weight filtration which is defined over \mathbb{Q} .

On the other hand we have the logarithmic de Rham complex $\Omega_X^\bullet(\log Y)$ with its filtrations W and F . Consider the \mathcal{D}_X -module $\mathcal{M} = \mathcal{O}_X(*Y)$ whose sections are meromorphic functions on X with only poles along Y . It has the submodule $W_m \mathcal{M} = \mathcal{O}_X$, and we put $W_{m-1} \mathcal{M} = 0$, $W_{m+1} \mathcal{M} = \mathcal{M}$. This filtration is a filtration by \mathcal{D}_X -modules.

Lemma 19 *We have an isomorphism of filtered objects in $\text{Perv}_X(\mathbb{C})$:*

$$(K^\bullet, W) \otimes \mathbb{C}_X \cong \text{DR}(\mathcal{M}, W).$$

Using this isomorphism we obtain the rational structure plus the weight filtration on the cohomology of U .

We also have a good filtration by \mathcal{O}_X -submodules given by $F_p \mathcal{M} = \mathcal{O}_X(pX)$. It induces a filtration on the de Rham complex, which in turn gives the Hodge filtration on the cohomology of U .

This is the simplest non-trivial example of a mixed Hodge module I know. The ingredients for a mixed Hodge module on a smooth projective variety X are:

1. an object K^\bullet of $\text{Perv}_X(\mathbb{Q})$ equipped with an increasing filtration W ;

2. a regular holonomic \mathcal{D}_X -module \mathcal{M} equipped with two filtrations: its weight filtration, which is a filtration by \mathcal{D}_X -submodules, and a Hodge filtration, which is a good filtration in the usual sense.
3. an isomorphism of filtered objects in $\text{Perv}_X(\mathbb{C})$:

$$(K^\bullet, W) \otimes \mathbb{C}_X \cong \text{DR}(\mathcal{M}, W).$$

One needs to formulate the right conditions which guarantee that the hypercohomology of such objects produce mixed Hodge structures. This formidable task was accomplished by Morihiko Saito in the eighties of the past century [22, 23].

7.2 Axioms for mixed Hodge modules

The definition of mixed Hodge modules is very involved. For this reason it is more suitable to start with an axiomatic introduction. This makes it possible to deduce important results rather painlessly, such as the existence of pure Hodge structures on the intersection cohomology groups.

It is not really necessary to understand the intricacies of the construction of mixed Hodge modules in order to be able to relate mixed Hodge modules to mixed Hodge structures. The reason is first of all that mixed Hodge modules on points are just mixed Hodge structures. Secondly, any mixed Hodge module has an underlying perverse complex, and, this is crucial, the Verdier duality operator and the basic direct image and inverse image functors can be extended to the level of the mixed Hodge module.

We recall that for any complex algebraic variety X the derived category of bounded complexes of cohomologically constructible complexes of sheaves of \mathbb{Q} -vector spaces on X is denoted $D_c^b(X; \mathbb{Q})$ and that it contains as a full subcategory the category $\text{Perv}_X(\mathbb{Q})$ of perverse \mathbb{Q} -complexes. The Verdier duality operator \mathbb{D} is an involution on $D_c^b(X; \mathbb{Q})$ preserving $\text{Perv}_X(\mathbb{Q})$.

Associated to a morphism $f : X \rightarrow Y$ between complex algebraic varieties, there are pairs of adjoint functors (f^{-1}, Rf_*) and $(f^!, Rf_!)$ between the respective derived categories of cohomologically constructible complexes which are interchanged by Verdier duality. We can now state the axioms:

- A)** For each complex algebraic variety X there exists an abelian category $\text{MHM}(X)$, the category of *mixed Hodge modules* on X , together with

a faithful functor

$$\text{rat}_X : D^b\text{MHM}(X) \rightarrow D_c^b(X; \mathbb{Q})$$

which sends $\text{MHM}(X)$ to $\text{Perv}_X(\mathbb{Q})$. We say that $\text{rat}_X M$ is the underlying perverse complex of M . Moreover, we say that

$$M \in \text{MHM}(X) \text{ is supported on } Z \iff \text{rat}_X M \text{ is supported on } Z.$$

- B)** The category of mixed Hodge modules supported on a point is the category of polarizable rational mixed Hodge structures; the functor rat associates to the mixed Hodge structure the underlying rational vector space.
- C)** Each object M in $\text{MHM}(X)$ admits a *weight filtration* W such that
- the object $\text{Gr}_k^W M$ is semisimple in $\text{MHM}(X)$;
 - the functors $M \mapsto W_k M$, $M \mapsto \text{Gr}_k^W M$ are exact;
 - if X is a point the W -filtration is the usual weight filtration for the mixed Hodge structure.

Since $\text{MHM}(X)$ is an abelian category, the cohomology groups of any complex of mixed Hodge modules on X is again a mixed Hodge module on X . With this in mind, we say that for a complex $M^\bullet \in D^b\text{MHM}(X)$ the *weight* satisfies

$$\text{weight}[M^\bullet] \begin{cases} \leq n, \\ \geq n \end{cases} \iff \text{Gr}_i^W H^j(M^\bullet) = 0 \begin{cases} \text{for } i > j + n \\ \text{for } i < j + n. \end{cases}$$

- D)** The duality functor \mathbb{D}_X of Verdier lifts to $\text{MHM}(X)$ as an involution, also denoted \mathbb{D}_X , in the sense that $\mathbb{D}_X \circ \text{rat}_X = \text{rat}_X \circ \mathbb{D}_X$.
- E)** For each morphism $f : X \rightarrow Y$ between complex algebraic varieties, there are induced functors $f_*, f_! : D^b\text{MHM}(X) \rightarrow D^b\text{MHM}(Y)$, $f^*, f^! : D^b\text{MHM}(Y) \rightarrow D^b\text{MHM}(X)$ exchanged under \mathbb{D}_X and which lift the functors with the same symbol on the level of perverse complexes.
- F)** The functors $f_!, f^*$ do not increase weights in the sense that if M^\bullet has weights $\leq n$, the same is true for $f_! M^\bullet$ and $f^* M^\bullet$.

G) The functors $f^!, f_*$ do not decrease weights in the sense that if M^\bullet has weights $\geq n$, the same is true for $f^!M^\bullet$ and f_*M^\bullet .

By way of terminology, we say that $M^\bullet \in D^b\text{MHM}(X)$ is *pure of weight n* if it has weight $\geq n$ and weight $\leq n$. We say that a morphism *preserves weights*, if it neither decreases or increases weights. Since for a proper map $f_* = f_!$ axioms F) and G) entail:

H) Proper maps between complex algebraic varieties preserve pure complexes.

7.3 First Consequences of the Axioms

From axiom A) and B) we see that there is a unique element

$$\mathbb{Q}^{\text{Hdg}} \in \text{MHM}(\text{pt}), \quad \text{rat}(\mathbb{Q}^{\text{Hdg}}) = \mathbb{Q}(0), \quad (16)$$

the unique Hodge structure on \mathbb{Q} of type $(0,0)$. The next lemma explains how the various cohomology groups can be expressed using direct and inverse functors. On the level of mixed Hodge modules this then naturally leads to mixed Hodge structures.

Lemma 20 *Let $a_X : X \rightarrow \text{pt}$ be the constant map to the point. Then we have:*

$$\begin{aligned} H^k(X; \mathbb{Q}) &= H^k(\text{pt}, (a_X)_* a_X^* \mathbb{Q}) \\ H_{-k}(X; \mathbb{Q}) &= H^k(\text{pt}, (a_X)! a_X^! \mathbb{Q}) \\ H_{-k}^{\text{BM}}(X; \mathbb{Q}) &= H^k(\text{pt}, (a_X)_* a_X^! \mathbb{Q}). \end{aligned}$$

Moreover, if $i : Z \hookrightarrow X$ is a closed subvariety, we have

$$H_Z^k(X; \mathbb{Q}) = H^k(\text{pt}, (a_X)_* i_* i^! a_X^* \mathbb{Q}).$$

Motivated by Lemma 20, using axiom D) and E) we do the same for the complex of mixed Hodge modules \mathbb{Q}^{Hdg} (16):

$$\left. \begin{aligned} \underline{\mathbb{Q}}_X^{\text{Hdg}} &:= a_X^* \mathbb{Q}^{\text{Hdg}} \in D^b\text{MHM}(X) \\ \underline{\mathbb{D}}_X^{\text{Hdg}} &:= a_X^! \mathbb{Q}^{\text{Hdg}} \in D^b\text{MHM}(X). \end{aligned} \right\} \quad (17)$$

By axiom E), applying the direct image functors associated to a_X produces complexes of mixed Hodge modules on the point p , hence, by axiom B), their cohomology groups have mixed Hodge structures. We deduce:

Corollary 21 *Let X be a complex algebraic variety and $i : Z \hookrightarrow X$ a subvariety. The complexes of mixed Hodge modules $(a_X)_* \underline{\mathbb{Q}}_X^{\text{Hdg}}$, $(a_X)_! \underline{\mathbb{D}}_X^{\text{Hdg}}$, $(a_X)_* \underline{\mathbb{D}}_X^{\text{Hdg}}$, respectively $i_* i^! \underline{\mathbb{Q}}_X^{\text{Hdg}}$ put mixed Hodge structures on cohomology, homology, Borel-Moore homology, and cohomology with support in Z respectively.*

These mixed Hodge structures coincide with the ones constructed by Deligne.

Referentias

- [1] Beilinson, A., J. Bernstein and P. Deligne: Faisceaux pervers, in: *Analyse et topologie sur les espaces singuliers I*, Astérisque **100** (1982).
- [2] Bittner, F.: The universal Euler characteristic for varieties of characteristic zero, *Comp. Math.* **140**(2004), 1011-1032.
- [3] Borel, A. et al.: *Algebraic D-modules*, Perspectives in Math. **2**, Academic Press, Boston, etc (1987).
- [4] Deligne, P.: Théorie de Hodge II, *Publ. Math. I.H.E.S.* **40** (1971), 5–58.
- [5] Deligne, P.: Théorie de Hodge III, *Publ. Math., I. H. E. S.* **44** (1974), 5-77.
- [6] Deligne, P.: La conjecture de Weil, *Publ. Math., I. H. E. S.* **43** (1974), 273–307.
- [7] Deligne, P.: Poids dans la cohomologie des variétés algébriques, *Proc. ICM Vancouver 1974*, Vol. 1, pp. 79–85. *Canad. Math. Congress*, Montreal, Que. (1975).
- [8] Deligne, P.: La conjecture de Weil II, *Publ. Math., I. H. E. S.* **52** (1980), 137–252.
- [9] Godement, R.: *Théorie des faisceaux*, Hermann, Paris (1964).
- [10] Dimca, A.: *Sheaves in Topology*, Universitext, Springer-Verlag, Berlin etc. (2004).
- [11] Goresky, M., MacPherson, R.: Intersection homology theory. *Topology* **19** (1980), no. 2, 135–162.

- [12] Griffiths, P. and J. Harris: *Principles of algebraic geometry*, John Wiley & Sons, New York etc. (1978).
- [13] Guillén, F., V. Navarro Aznar, P. Pascual-Gainza and F. Puerta: *Hyperresolutions cubiques et descente cohomologique*, Springer Lecture Notes in Math. **1335** (1988).
- [14] Hironaka, H.: Resolution of singularities of an algebraic variety of characteristic zero, *Ann. Math.* **79** (1964), 109–326.
- [15] Kashiwara, M.: On the maximally overdetermined system of linear differential equations I. *Publ. Res. Inst. Math. Kyoto Univ.* **10** (1974/75), 563–579.
- [16] Kashiwara, M.: The Riemann-Hilbert problem for holonomic systems. *Publ. Res. Inst. Math. Sci.* **20** (1984), no. 2, 319–365.
- [17] Malgrange, B.: L’involutivité des caractéristiques des systèmes différentiels et microdifférentiels. *Sém. Bourbaki* **522** (1977–1978), *Lect. Notes in Math.* **710**, 277–289, Springer Verlag, Berlin etc. (1979).
- [18] Mebkhout, Z. Une équivalence de catégories. *Compositio Math.* **51** (1984), no. 1, 51–62.
- [19] Mebkhout, Z. Une autre équivalence de catégories. *Compositio Math.* **51** (1984), no. 1, 63–88.
- [20] Nagata, M.: Imbedding of an abstract variety in a complete variety, *J. Math. Kyoto Univ.* **2**(1962), 1–10.
- [21] Peters, C.A.M., Steenbrink, J.H.M.: *Mixed Hodge Structures*, Book in preparation.
- [22] Saito, M.: Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **24** (1988), 849–995.
- [23] Saito, M.: Mixed Hodge Modules, *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **26** (1990), 221–333.