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Metric Theory of Singularities
Lipschitz Geometry of Singular Spaces

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METRIC THEORY OF SINGULARITIES.
LIPSCHITZ GEOMETRY OF SINGULAR SPACES.

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1. Metric viewpoint. Comparison of metrics. Normal embedding.

Metric Geometry is a geometry of metric spaces. The collection of metric spaces is considered as a category. In our setting Lipschitz maps are the morphisms in this category. Remind that a map $F\colon X \to Y$ is called *Lipschitz* if there exists a positive constant $K \in \mathbf{R}$ such that, for every $x_1, x_2 \in X$, we have: $d_Y(F(x_1), F(x_2)) \leq K d_X(x_1, x_2)$. A map F is called *bi-Lipschitz* if F is bijective, Lipschitz and F^{-1} is Lipschitz. Clearly, bi-Lipschitz maps are isomorphisms in this category. We apply a natural classification question to "singular spaces". During these lectures by "singular space" we mean a compact semialgebraic or subanalytic (or definable in some o-minimal structure) set in $\mathbf{R}^n$ which is not a smooth submanifold of $\mathbf{R}^n$.

Let $X \subset \mathbf{R}^n$ be a compact connected subanalytic set. We consider two natural metrics on X. The first one is an euclidean metric $d_e(x_1, x_2) = ||x_1 - x_2||$. The second one is a so-called inner (or intrinsic) metric $\mathrm{d}_i(x_1, x_2) = \inf_{\gamma \in \Gamma(x_1, x_2)} l(\gamma)$ where $\Gamma(x_1, x_2)$ is a set of piece-wise smooth arcs connecting x_1 and x_2 and $l(\gamma)$ is a length of γ.

Remark. In fact, $d(x_1, x_2) = \min_{\gamma \in \Gamma(x_1, x_2)} l(\gamma)$ because Hopf-Rinov theorem is true for compact singular spaces.

These two metrics define the same topology on X but they are not necessary bi-Lipschitz isomorphic. To see it, consider the following semialgebraic subset of $\mathbf{R}^2$:

$$X = \{(x_1, x_2)| \quad x_1^3 - x_2^3 = 0, \quad |x_1| \le 1, \quad |x_2| \le 1\}$$

Fig.1

Clearly, $d_i(x_1, x_2)/d_e(x_1, x_2) \to \infty$, for x_1, x_2 sufficiently closed to 0.

This example motivates the following definition. A set $A \subset \mathbf{R}^n$ is called *normally embedded* if (X, d_e) and (X, d_i) are bi-Lipschitz isomorphic. All compact smooth submanifolds of $\mathbf{R}^n$ are normally embedded. A so-called β-horn H_β gives an example of a normally embedded singular set

$$H_\beta = \{(x_1, x_2, y) \in \mathbf{R}^3| \quad 0 \le y \le 1, \quad (x_1^2 + x_2^2)^q = y^{2p}\},$$

where $\beta = p/q \ge 1$ is a rational number. **Fig.2**

These notes are devoted to first steps of bi-Lipschitz classification of subanalytic sets with respect to the inner metric. The following theorem shows that this question is equivalent to a question of bi-Lipschitz classification of normally embedded singular sets.

Theorem. (Normal Embedding Theorem.)[BM]. *Let $X \subset \mathbf{R}^n$ be a compact subanalytic set. Then there exists another subanalytic set $\widetilde{X} \subset \mathbf{R}^m$ such that $\widetilde{X}$ is bi-Lipschitz equivalent to X with respect to the inner metric.*

Question. Let $X \subset \mathbf{R}^n$ be a compact real algebraic set. Is it possible to construct a real algebraic set $\widetilde{X} \subset \mathbf{R}^m$ with the same properties as in the theorem above? Namely, $\widetilde{X}$ must be normaly embedded and bi-Lipschitz equivalent to X with respect to the inner metric.

2. Finiteness results.

There are many classification problems in Singularity Theory. If one restricts a classification problem to an algebraic setting the following question is important. Consider the set of singular spaces (singular maps, singular germs) defined by polynomials of degree (complexity) bounded from above by some number K. Is a set of equivalence classes (by the given equivalence relation) finite? Positive answers for this question are called Finiteness rezults. Fortunately, Lipschitz Geometry of Singularities admits finiteness rezult.

Definition. *Upper complexity* of a semialgebraic set $X \subset \mathbf{R}^n$ is a number defined as follows: $U(X) = \min(N + D + E)$, where N is a number of variables, E is a number of equations and inequalities from a given formula defining X and D is a maximal degree of all the polynomials appear in this formula. The minimum is taken on the set of all the presentations X.

Theorem (Mostowski [Mo], Parusinski [Pa]). *Let K be a positive integer number. Consider the set of all semialgebraic sets with upper complexity bounded from above by K. Then the number of equivalence classes for the bi-Lipschitz equivalence with respect to the euclidean metric is finite.*

If two semialgebraic (or subanalytic) sets are bi-Lipschitz isomorphic with respect to the euclidean metric they are bi-Lipschitz isomorphic with respect to the inner metric. Hence, the bi-Lipschitz equivalence with respect to the inner metric also admits a finiteness result. For subanalytic sets, A. Parusinski proved the following generalization of finiteness theorem.

Theorem [Pa]. *Consider a finite-dimensional subanalytic family of subanalytic sets. Then the number of equivalence classes with respect to a bi-Lipschitz isomorphism with respect to the euclidean metric is finite.*

For definable sets in o-minimal structures (especially, if a structure is not polynomially bounded), the finiteness result cited above does not take place. To see it one can consider the following family T_λ depending on λ.

$$T_\lambda = \{(x_1, x_2) \in \mathbf{R}^2 | \quad 0 \le x_1 \le 1, \quad 0 \le x_2 \le x_1^\lambda\},$$

for $\lambda \in [1, \infty)$. For different λ_1, λ_2, the sets $T_{\lambda_1}, T_{\lambda_2}$ are not bi-Lipschitz equivalent. This family is definable in $\log - \exp$ o-minimal structure.

3. Germs of subanalytic surfaces.

A subanalytic surface $X \subset \mathbf{R}^n$ is a subanalytic set of dimension 2. This section is devoted to a local bi-Lipschitz classification of subanalytic surfaces. Remined that two germs of subanalytic sets are called bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz isomorphism $h\colon (X, x_0) \to (Y, y_0)$. Remined that a standard β-horn is a semialgebraic set defined as follows:

$$H_\beta = \{(x_1, x_2, y) \in \mathbf{R}^3 | \quad (x_1^2 + x_2^2)^q = y^{2p}, \quad 0 \leq y \leq 1\},$$

where $\beta = p/q$ is a rational number and $\beta \geq 1$.

We start our classification from the following result.

Theorem 3.1. **[L1]** *Let $X \subset \mathbf{R}^n$ be a subanalytic surface and let $x_0 \in X$ be a singular point such that, for sufficiently small $\varepsilon > 0$, the intersection $X \cap S_{x_0,r}$ is connected. Then there exists a rational $\beta \geq 1$ such that the germ of X at x_0 is bi-Lipschitz isomorphic to the germ of H_β at $0 \in \mathbf{R}^3$. Moreover, for $\beta_1, \beta_2 \geq 0$, the germs of H_{β_1} and H_{β_2} are not bi-Lipschitz isomorphic.*

Here we say that x_0 is a singular point of X if X is not a smooth submanifold without boundary of $\mathbf{R}^n$ near x_0. It means thet $0 \in T_\beta$ is not an isolated singular point.

In fact, this theorem is a corollary of a more general result so-called Hölder Complex Theorem. Hölder Complex Theorem is a Lipschitz version of the Triangulation Theorem (see notes of the course of M.Coste). In 2-dimensional case a triangulation can be chosen canonically and this canonical triangulation presents a complete Lipschitz invariant for germs of semialgebraic surfaces.

Definition. An *Abstract Hölder Complex* is a pair (Γ, β) where Γ is a finite graph without loops and $\beta\colon E_\Gamma \to Q$ (where E_Γ is the set of edges of Γ) is a rational function such that, for all $g \in E_\Gamma$, we have: $\beta(g) \geq 1$.

Hölder triangle T_β is a semialgebraic set in $\mathbf{R}^2$ defined as follows:

$$T_\beta = \{(x_1, x_2) \in \mathbf{R}^2 | \quad 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq x_1\}.$$

Let (Γ, β) be an Abstract Hölder Complex. A germ of subanalytic surface X at a point $x_0 \in X$ is called *a Geometric Hölder Complex associated to* (Γ, β) if there exists a subanalytic triangulation such that x_0 is a vertex of this triangulation, Γ is isomorphic to the star of x_0 in this triangulation and, for any $\beta \in E_\Gamma$, the corresponding 2-dimensional simplex is bi-Lipschitz isomorphic with respect to the intrinsic metric to $T_{\beta(g)}$. Moreover, the triangulation map maps $0 \in T_{\beta(y)}$ to x_0:

Fig.3

The following result was proved by several authors independently (using different notations and terminology).

Theorem 3.2. ([Ku],[BRi],[Pa] and [L1],[L2] for the present version). *Let X be a subanalytic surface and let $x_0 \in X$. Then there exists an Abstract Hölder Complex (Γ, β) such that the germ of X at x_0 is a Geometric Hölder Complex associated to (Γ, β).*

Note that this Abstract Hölder Complex is not unique. Now we are going to correct this Abstract Hölder Complex in order to obtain a complete bi-Lipschitz invariant.

Let (Γ, β) be a Hölder complex. A vertex $a \in V_\Gamma$ (here V_Γ is the set of vertices of Γ) is called *smooth* if a is connected with two other vertices by only two edges correspondentes. Suppose that a is connected by edges g_1 and g_2 with vertices a_1 and a_2. Let us construct a new graph $\widetilde{\Gamma}$ such that the vertex a and the edges g_1, g_2 are removed, the vertices a_1 and a_2 are connected by a new edge g. Set $\beta(g) = \min\{\beta(g_1), \beta(g_2)\}$.

Fig.4

This operation is called an *elimination of a smooth vertex*.

A vertex a is called *loop vertex* if it is connected with only two edges g_1 and g_2 and these edges connect a with the same vertex a_1. A loop vertes is called *simple* if $\beta(g_1) = \beta(g_2)$. Let (Γ, β) be a Hölder complex with a nonsimple loop vertex a. Let us construct a new Hölder complex $(\widetilde{\Gamma}, \tilde{\beta})$ in the following way. We make this loop vertex simple just redefining $\tilde{\beta}(g_1) = \tilde{\beta}(g_2) = \min(\beta(g_1), \beta(g_2))$. This operation is called *a correction near a loop vertex*.

Abstract Hölder Complex is called *simplified* if it has no smooth vertices and all the loop vertices are simple. Abstract Hölder Complex $(\widetilde{\gamma}, \tilde{\beta})$ is called *a simplification* of (Γ, β) if it is simplified and can be obtained from (Γ, β) using the operations described above.

Theorem 3.3. *Let a germ of a subanalytic surface X at x_0 be a Geometric Hölder Complex associated to two Abstract Hölder Complexes (Γ_1, β_1) and (Γ_2, β_2). Then the simplifications of them are isomorphic.*

It motivates the following definition. *A Canonical Hölder Complex associated to a germ of a subanalytic surface* (X, x_0) is a simplification of any Abstract Hölder Complex (Γ, β) such that (X, x_0) is a Geometric Hölder Complex associated to (Γ, β). By Theorem 3.3, Canonical Hölder Complex is well defined.

Theorem 3.4. (Classification Theorem of Subanalytic Surfaces)[L1]. *Two germs of subanalytic surfaces (X, x_0) and (Y, y_0) are bi-Lipschitz isomorphic if and only if the corresponding Canonical Hölder Complexes are isomorphic.*

We finish this section by the following Realization Theorem.

Theorem 3.5. **[BS]** *Let (Γ, β) be Abstract Hölder Complex. Then there exists Geometric Hölder Complex (X, x_0) associated to (Γ, β). Moreover, (X, x_0) can be germ of a semialgebraic set.*

Question. When Abstract Hölder Complex admits a real algebraic realization?

Remark. R.Benedetti with M.Dedo and independently S.Akbulut with H.King gave conditions for a topological realization of a 2-dimensional simplicial complex. Our problem is a geometric version of their results.

4. Metric Homology.

A theory presented in section 3 is a 2-dimensional theory. Here we are going to discuss some invariants of similar nature for higher dimensions. Let Y and Z be two bounded subanalytic subsets of $\mathbf{R}^n$. Let $U_\varepsilon(Z)$ be an ε-neighbourhood of Z. We define a function $f(\varepsilon)$ as follows:

$$f(\varepsilon) = \mathrm{vol}_{\dim Y} Y \cap U_\varepsilon(Z).$$

Let

$$\alpha(r) = \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon^r}.$$

By results of Lion-Rolin [LR] for volume-functions on subanalytic sets, there exists a rational number μ such that

$$\alpha(r) = \begin{cases} 0, & \text{if} \quad 0 < r < \mu, \\ \infty, & \text{if} \quad \mu < r < \infty. \end{cases}$$

This number μ is called *a volume grouth number of Y with respect to Z*. We use the notation $\mu(Y, Z)$.

Remark. If Z is a point and Y is a smooth manifold then $\mu(Y, Z) = \dim Y$. If Y and Z are submanifolds of $\mathbf{R}^n$ and if they intersect transversally such that $Y \cap Z \neq \emptyset$ then $\mu(Y, Z) = n - \dim Z$.

Let X be a subanalytic set. A partition $\{X_i\}$ of X is called *Lipschitz trivial stratification* if

1. All X_i are Lipschitz submanifolds of $\mathbf{R}^n$.

2. $\{X_i\}$ is a topological stratification of X.
3. For any i and for any two points $x_1, x_2 \in X_i$, there exist a pair of neighbourhoods U_{x_1} and U_{x_2} and a bi-Lipschitz homeomorphism $h: U_{x_1} \to U_{x_2}$ such that, for all other stratum X_j, we have: $h(U_{x_1}) = U_{x_2} \cap X_j$.

Now we are going define Metric Homology ([BB1],[BB2]) in "Intersection Homology" [GM] style. Let $\bar{p}: \{0, 1, \dots, n-1\} \to Q \cap [1, \infty[$ be a function called *volume-perversity*. Now we fix a field of coefficients L and consider, for each k, the set of subanalytic k-chains, i.e. the set of expressions

$$\eta = \sum_{i=1}^{p} F_i(\Delta^k) a_i,$$

where a_i are elements of L. A k-chain η is called *admissible with respect to a stratification* $\{X_j\}$ *and a perversity function* $\bar{p}$ if, for each stratum X_j, one has:

$$\mu(\text{Supp } \eta,\, X_j) \geq \bar{p}(codim\, X_j), \quad \mu(\text{Supp } \partial\eta,\, X_j) \geq \bar{p}(codim\, X_j).$$

Admissible chains form a chain complex and the homology of this chain complex is called *Metric Homology* with respect to the stratification $\{X_j\}$ and the perversity function $\bar{p}$. We use a notation $MH^{\bar{p}}(X, \{X_j\})$.

Basic properties of Metric Homology.

1. Let $\bar{p}$ be a volume-perversity function satisfying the following condition:

$$\bar{p}(i) \leq \bar{p}(i+1) \leq \bar{p}(i) + 1.$$

This condition is called G-M condition. Then $MH^{\bar{p}}(X, \{X_j\})$ does not depend on a Lipschitz trivial stratification $\{X_j\}$. Thus, one can use the notation $MH^{\bar{p}}(X)$.

2. Let $F: X \to Y$ be a subanalytic bi-Lipschitz map. Then $MH^{\bar{p}}(X) = MH^{\bar{p}}(Y)$.

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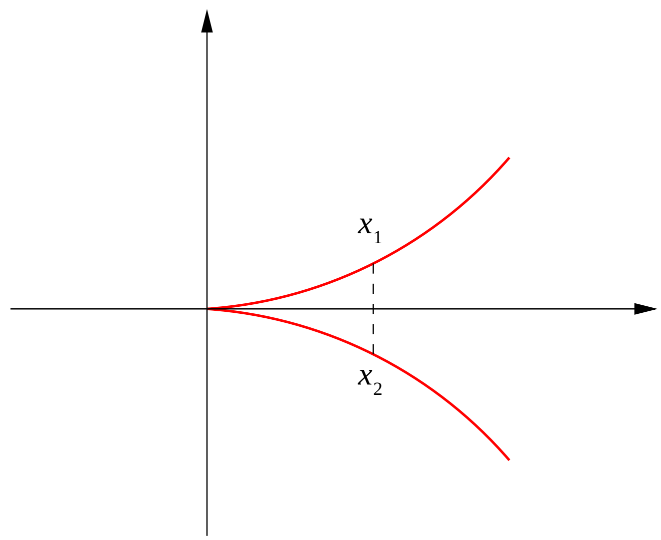
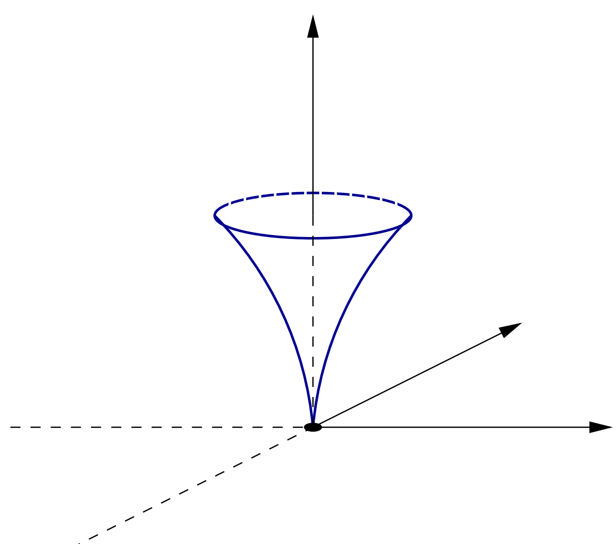
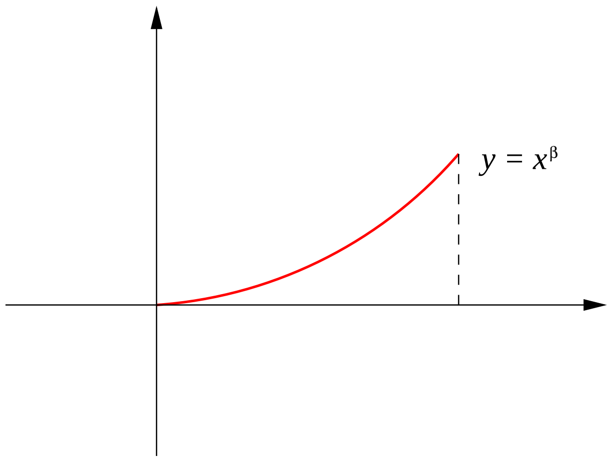

x_1
x_2





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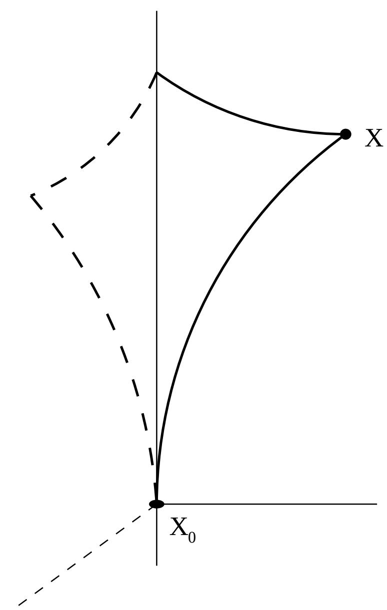

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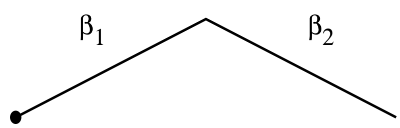

β_1
β_2

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