The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1671/17

Advanced School and Workshop on Singularities in Geometry and Topology

(15 August - 3 September 2005)

MCKAY correspondence for quotient surface singularities

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McKay correspondence

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Advanced School and Workshop on

Singularities in Geometry and Topology

Lectures on August 25 and 26, 2005
at the ICTP in Trieste

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Table of contents

0. Platonic triples and simple objects

The *Platonic triples* consist of integers p, q, r ($p, q, r \geq 2$) satisfying

$$(*) \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\,.$$

Normalizing to $r = 2$, we get the following pairs (p, q) and their well known relation to the (symmetry groups of the) *Platonic solids* or to *regular tesselations* of the sphere S^2 (by p–gons, q of them meeting at each corner).

p	q	(generalized) Platonic solid
2	n	Hosohedron
n	2	Dihedron
3	3	Tetrahedron
4	3	Hexahedron (Cube)
3	4	Octahedron
5	3	Dodecahedron
3	5	Icosahedron

Not distinguishing between $(p,\,q)$ and $(q,\,p)$, we find pairs of numbers attached to the corresponding *symmetry groups* in $\mathrm{SO}(3,\,\mathbb{R})$.

Theorem
Besides the cyclic groups, there exist - up to conjugacy - only 4 (classes of) *finite subgroups of* $\mathrm{SO}(3,\,\mathbb{R})$. *They are (as abstract groups) presented by generators* $\alpha,\,\beta$ *and relations*

$$\alpha^p = \beta^q = (\alpha\beta)^2 = e\,.$$

Under the group isomorphism

$$\mathrm{PSU}(2,\,\mathbb{C}) \cong \mathrm{SO}(3,\,\mathbb{R})$$

the finite subgroups of $\mathrm{SO}(3,\,\mathbb{R})$ can be lifted to subgroups of doubled order in

$$\mathrm{SU}(2,\,\mathbb{C}) \subset \mathrm{SL}(2,\,\mathbb{C})\,.$$

They are called the *binary polyhedral groups*.

Theorem
Besides the cyclic groups of odd order, there exist - up to conjugacy - only the binary polyhedral groups as subgroups of $\mathrm{SL}(2, \mathbb{C})$. *They have (as abstract groups) a presentation by generators and relations governed by the associated Platonic triple*.

Condition $(*)$ also appears in other contexts for certain quadratic forms to being positive or negative definite, in particular in *Lie theory*.

Theorem
The only simply laced simply connected simple complex Lie *groups are classified by their* Dynkin *diagrams of type* ADE *which, in the cases* D *and* E, *are also in 1 - to - 1 bijection to the Platonic triples*.

Thus, there seems to be a close connection between finite groups and LIE groups, and already FELIX KLEIN has speculated about this interaction of discrete mathematics and geometry. The main tool used nowadays for understanding the relation can be found in his famous book on the *Equation of the fifth degree and the Icosahedron* of 1884 where he introduced - of course not under this name - the *KLEIN singularities* (also known as *Rational Double Points, Simple Singularities* etc.).

In fact one can find any KLEIN singularity (and even its complete deformation theory) inside the geometry of the corresponding simple LIE group (work of GROTHENDIECK, BRIESKORN and SLODOWY).

1. KLEIN singularities

The theory of KLEIN *singularities* establishes a (formal) one-to-one correspondence between the *conjugacy classes of finite subgroups* of $\mathrm{SL}(2, \mathbb{C})$ (also called *binary polyhedral groups* by abuse of language) and the COXETER–DYNKIN–WITT *diagrams* (or DYNKIN–*diagrams*, as they are usually called) of type ADE via the following scheme:

$$\begin{gathered}
\{\,\text{finite subgroups } \Gamma \subset \mathrm{SL}\,(2, \mathbb{C})\,\}/\text{ conjugacy} \\
\updownarrow \\
\{\,\text{KLEIN singularities } X_\Gamma = \mathbb{C}^2/\,\Gamma\,\}/\sim \\
\updownarrow \\
\{\,\text{minimal resolutions } \widetilde{X}_\Gamma\,\}/\sim \\
\updownarrow \\
\{\,\text{CDW-diagrams of type ADE}\,\}
\end{gathered}$$

Here, the symbol $\sim$ in the second and third line denotes *complex–analytic equivalence*.

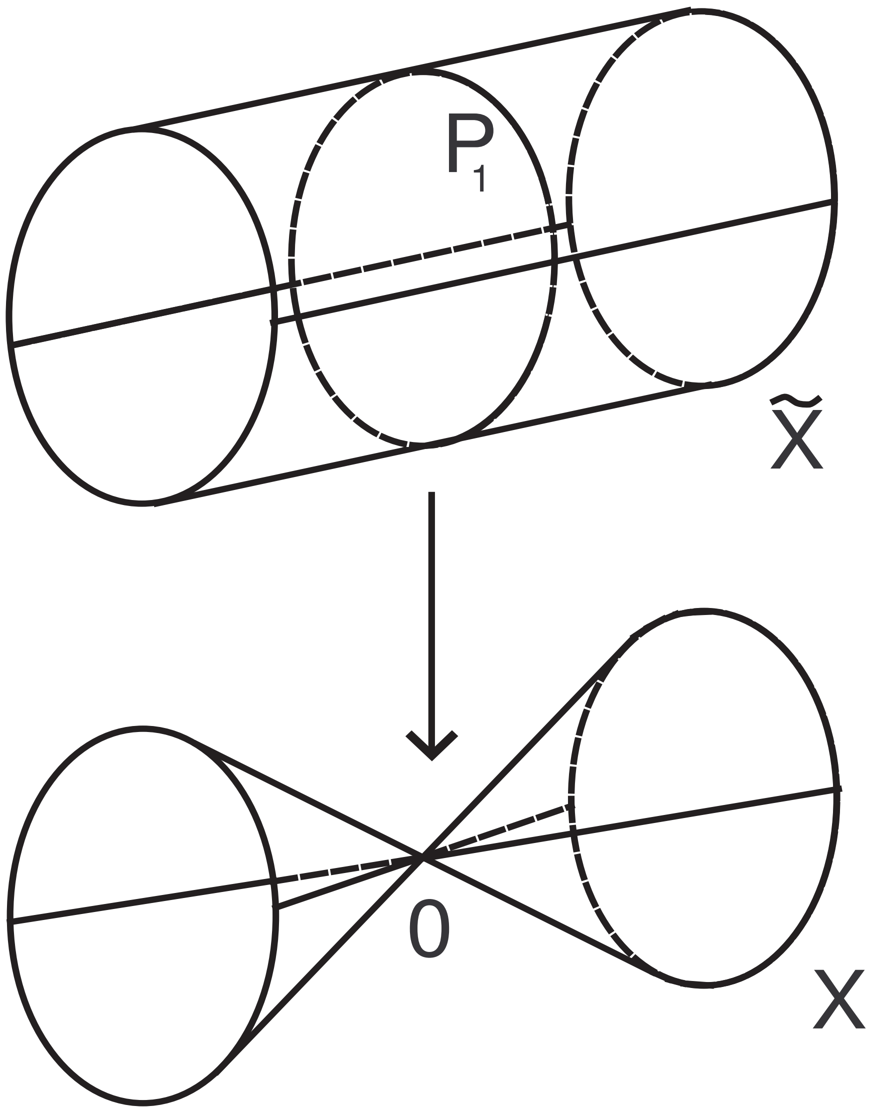


Figure 1

The last arrow is given in the downward direction by associating to a *minimal resolution* $\pi : \widetilde{X}_\Gamma \to X_\Gamma$ the *dual graph* of its *exceptional set* $E := \pi^{-1}(0) \subset \widetilde{X}_\Gamma$ (with the irreducible components $E_1, E_2, \ldots$).

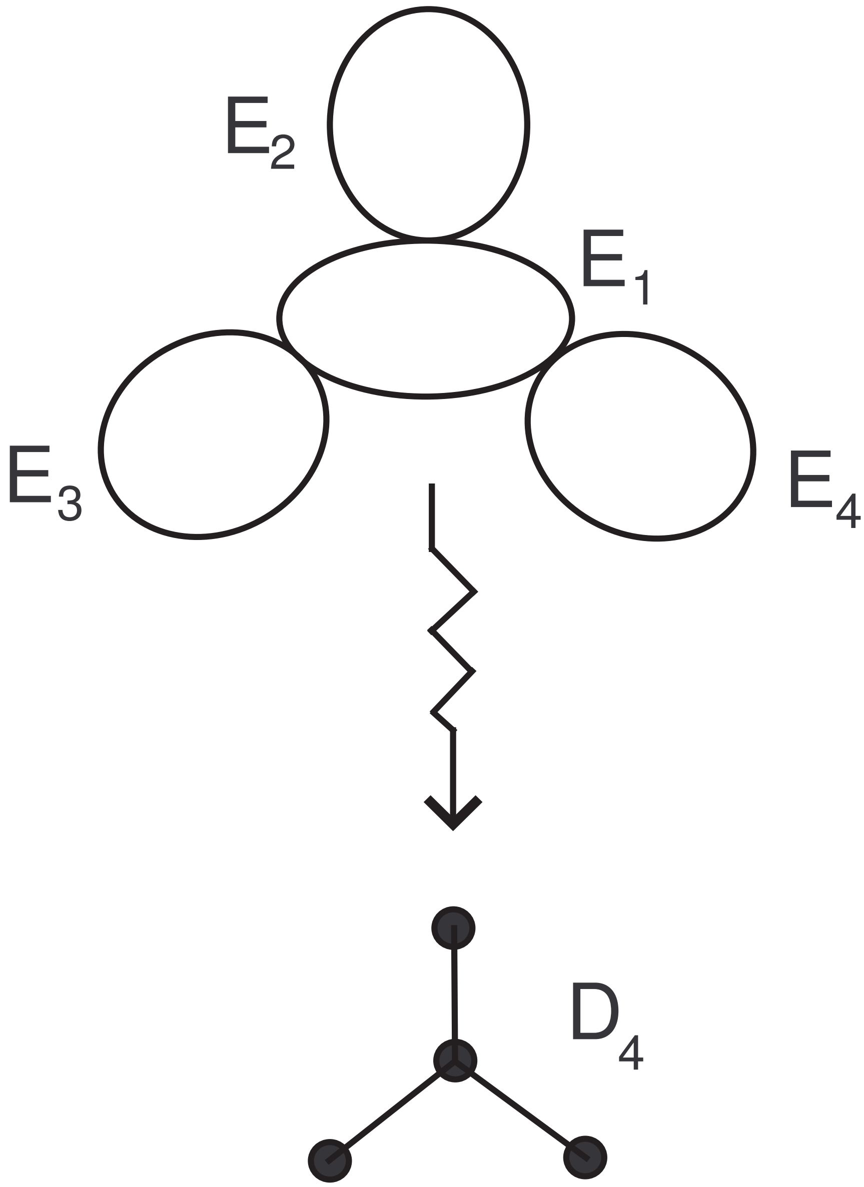


Figure 2

2. Mc Kay's observation

In 1979, McKay constructed directly via representation theory the resulting bijection between the first and last line of this diagram. In particular, according to this so called McKay *correspondence*, each *(nontrivial) irreducible complex representation* of Γ corresponds uniquely to an *irreducible component* of the exceptional set E.

Recall the construction of the McKay *quiver* associated to a binary polyhedral group and - more generally - to a finite *small* subgroup

$$\Gamma \subset \mathrm{GL}(2, \mathbb{C}) .$$

Let $$\mathrm{Irr}\,\Gamma := \{ \rho_0, \rho_1, \ldots, \rho_r \}$$
denote the set of *irreducible complex representations* of Γ, ρ_0 the *trivial* one, and

$$\mathrm{Irr}^0\,\Gamma := \{ \rho_1, \ldots, \rho_r \}$$

the set of its *nontrivial* irreducible complex representations.

Let c denote the natural representation on $\mathbb{C}^2$ given by the inclusion $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$. Then,

$$\rho_i \otimes c^* = \sum_j a_{ij}\, \rho_j$$

where c^* denotes the *dual* representation of c (of course, $c^* = c$ for $\Gamma \subset \mathrm{SL}\,(2, \mathbb{C})$). The McKay *quiver* is formed in the following way: Associate to each representation a vertex and join the ith vertex with the jth vertex by a_{ij} arrows. - McKay's observation may be formulated in the following way:

For every finite subgroup $\Gamma \subset \mathrm{SL}\,(2, \mathbb{C})$ *, one has* $a_{ji} = a_{ij} \in \{\, 0, 1 \,\}$ *. Replacing each* double arrow *by a line, one finds exactly the (extended) CDW diagrams of correct type*.

In particular, there is a canonical bijection

$$\mathrm{Irr}^0\, \Gamma \longrightarrow \mathrm{Irr}\, E = \{\, E_1, \ldots, E_r \,\} \,.$$

For the *binary tetrahedral* group $\mathbb{T}$, that is the preimage of the symmetry group $T \subset$ SO $(3, \mathbb{R})$ of a *regular tetrahedron* under the canonical group epimorphism

$$\mathrm{SU}\,(2, \mathbb{C}) \longrightarrow \mathrm{SO}\,(3, \mathbb{R})\ ,$$

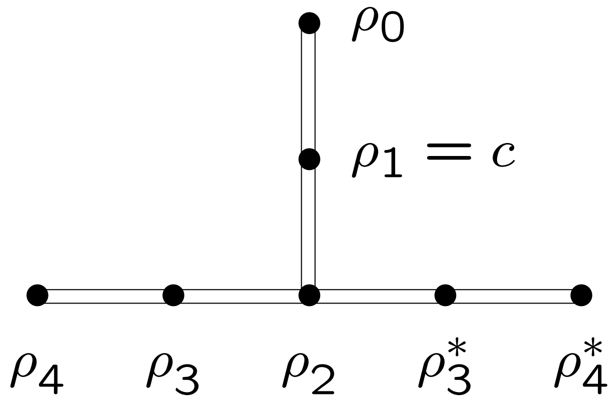
the irreducible representations are easily calculated. The isometry group $T = \mathfrak{A}_4 \in \mathrm{SO}\,(3, \mathbb{R})$ has obviously an irreducible representation of order 3 and 3 representations of order 1, which induce representations ρ_0 (= trivial representation), ρ_4, ρ_4^* (1–dimensional) and ρ_2 (3–dimensional) of $\mathbb{T}$. Of course, by our realization of $\mathbb{T}$ as a subgroup of $\mathrm{SL}\,(2, \mathbb{C})$, there is a canonical 2–dimensional representation which we call $c = \rho_1$. Finally, we define $\rho_3 = c \otimes \rho_4$ and $\rho_3^* = c \otimes \rho_4^*$.

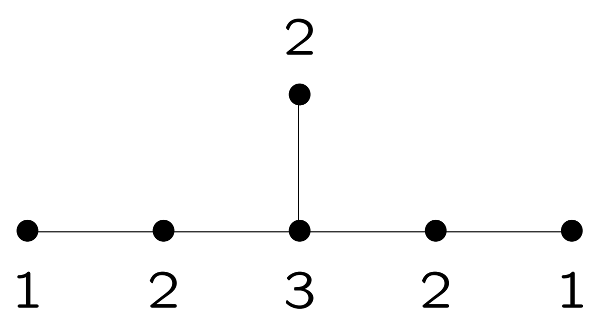
Using the (obvious) character table of $\mathfrak{A}_4$, we get the following character table for $\mathbb{T}$:

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ρ_0	1	1	1	1	1	1	1
$c = \rho_1$	2	-2	0	-1	-1	1	1
ρ_2	3	3	-1	0	0	0	0
ρ_3	2	-2	0	$-\zeta_3$	$-\zeta_3^2$	ζ_3	ζ_3^2
ρ_3^*	2	-2	0	$-\zeta_3^2$	$-\zeta_3$	ζ_3^2	ζ_3
ρ_4	1	1	1	ζ_3	ζ_3^2	ζ_3	ζ_3^2
ρ_4^*	1	1	1	ζ_3^2	ζ_3	ζ_3^2	ζ_3

Since the orthogonality relations are satisfied, we have constructed all irreducible representations of $\mathbb{T}$.

The resulting quiver looks as below where a subgraph $\bullet\!=\!=\!\bullet$ stands for a *double arrow*, i. e. two arrows in opposite direction. Replacing such subgraphs by a simple line $\bullet\!-\!-\!\bullet$, forgetting ρ_0 and inserting the ranks of the corresponding representations, yields the other diagram below which, in fact, is not only the CDW diagram of type E_6 but also represents the *fundamental cycle* Z of the singularity $\mathbb{C}^2/\,\mathbb{T}$.





3. The geometric McKay correspondence for quotient surface singularities

Of course, geometers wanted to understand this phenomenon *geometrically*, and the first who succeeded in this attempt were Gonzales-Sprinberg and Verdier in 1983. They associated to each nontrivial irreducible representation of Γ a vector bundle $\mathcal{F}$ on $\widetilde{X}_\Gamma$ whose first Chern class $c_1(\mathcal{F})$ hits precisely one component of E transversally. Their proof was not completely satisfying since they had to check the details case by case. But in 1985, Artin and Verdier gave a conceptual proof using only standard facts on rational singularities, and in combination with the so called *multiplication formula* contained in the paper of Hélène Esnault and Knörrer from the same year it became clear how to understand the *full strength* of the correspondence, i. e. how to reconstruct the dual graph of $E \subset \widetilde{X}_\Gamma$ from the representations of Γ completely in geometrical terms.

I would like to discuss this construction from the beginning in the more general setting of *quotient surface singularities*, or in other terms: for *small* finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ (instead of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$) in more detail (work of ESNAULT, WUNRAM, RIEMENSCHNEIDER).

Recall that a finite subgroup $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ is called *small*, if it acts freely on $\mathbb{C}^2 \setminus \{0\}$, or equivalently, if its (normal) subgroup generated by (pseudo–) *reflections* is trivial.

By a well known result of GOTTSCHLING and PRILL, every quotient $\mathbb{C}^2/\Gamma$ is complex–analytically isomorphic to a quotient by a small group, and two quotients by small subgroups are complex–analytically isomorphic if the subgroups are *conjugate* in $\mathrm{GL}(2, \mathbb{C})$. Hence, the classification of quotient surface singularities consists in the determination of the conjugacy classes of finite small subgroups in $\mathrm{GL}(2, \mathbb{C})$.

This classification has been carried out by BRIESKORN.

Recall that *cyclic* quotient surface singularities of $\mathbb{C}^2$ are determined by two natural numbers n, q with $1 \le q < n$ and $\gcd(n, q) = 1$. The cyclic group $C_{n,q}$ acting is generated by the linear map with matrix

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix},$$

$\zeta_n = \exp(2\pi i/n)$ an n^{th} *primitive root of unity*.

Lemma *Two cyclic quotients $\mathbb{C}^2/\,C_{n,q}$ and $\mathbb{C}^2/\,C_{n',q'}$ are isomorphic if and only if*

$$n' = n$$

and

$$q' = q \quad \text{or} \quad q\,q' \equiv 1 \bmod n\,.$$

For the general case, notice that we have a surjective group homomorphism

$$\psi : \mathsf{ZGL}_2 \times \mathsf{SL}(2, \mathbb{C}) \longrightarrow \mathsf{GL}(2, \mathbb{C})$$

(ZGL_2 denoting the center of $\mathsf{GL}(2, \mathbb{C})$ consisting of all multiples aE, $a \neq 0$, of the unit matrix) defined by multiplication. It is not difficult to convince oneself that the following is true:

Lemma *Each noncyclic finite subgroup* Γ *of* $\mathsf{GL}(2, \mathbb{C})$ *may be obtained from a quadruple* $(G_1, N_1; G_2, N_2)$*, where*
(a) $G_1 \subset \mathsf{ZGL}_2$ *and* $G_2 \subset \mathsf{SL}(2, \mathbb{C})$ *are finite subgroups,* G_2 *not cyclic,*
(b) $N_1 \subset G_1$ *and* $N_2 \subset G_2$ *are normal subgroups such that there exists an isomorphism*

$$\varphi : G_2/N_2 \xrightarrow{\sim} G_1/N_1 \, ,$$

by the following construction:

$$\Gamma := \psi(G_1 \times_\varphi G_2) \ ,$$

where

$$G_1 \times_\varphi G_2 := \{(g_1, g_2) \in G_1 \times G_2 : \overline{g}_1 = \varphi(\overline{g}_2)\} \ ;$$

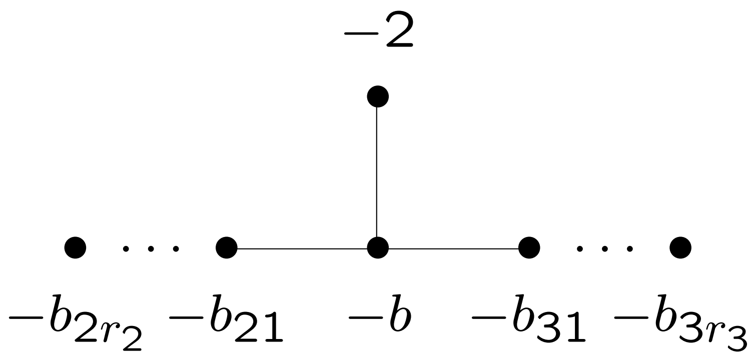
(*here,* $\overline{g}_i$ *denotes the residue class of* g_i *in* G_i/N_i , $i = 1, 2$).

Remark The conjugacy class of Γ in $\mathrm{GL}(2, \mathbb{C})$ does not depend on the specific isomorphism φ . Therefore, we use the symbol

$$(G_1, N_1; G_2, N_2)$$

also as a name for the *conjugacy class* containing the groups $\psi\,(G_1 \times_\varphi G_2)$.

BRIESKORNS classification can now be given in form of the following table ($\mathbf{Z}_\ell$ denotes the group $\langle \zeta_\ell E \rangle$), where the symbol $(b; n_1, q_1, n_2, q_2, n_3, q_3)$ encodes the dual resolution graph. (Of course, the pair $n_1 = 2$, $q_1 = 1$ belongs to the upper (short) arm).



where

$$\frac{n_i}{q_i} = b_{i1} - \frac{1}{b_{i2}} - \cdots - \frac{1}{b_{ir_i}}\,, \qquad i = 2, 3\,.$$

$\Gamma = (G_1, N_1; G_2, N_2)$	$(b; n_1, q_1, n_2, q_2, n_3, q_3)$
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{D}_n, \mathbb{D}_n)$ $(\mathbb{Z}_{4m}, \mathbb{Z}_{2m}; \mathbb{D}_n, C_{2n})$	$(b; 2, 1, 2, 1, n, q) \quad m = (b-1)n - q = \begin{cases} \text{odd} \\ \text{even} \end{cases}$
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{T})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{T})$ $(\mathbb{Z}_{6m}, \mathbb{Z}_{2m}; \mathbb{T}, \mathbb{D}_2)$	$\begin{matrix} (b; 2, 1, 3, 2, 3, 2) \\ (b; 2, 1, 3, 1, 3, 1) \\ (b; 2, 1, 3, 1, 3, 2) \end{matrix} \quad m = 6(b-2) + \begin{cases} 1 \\ 5 \\ 3 \end{cases}$
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{O}, \mathbb{O})$	$\begin{matrix} (b; 2, 1, 3, 2, 4, 3) \\ (b; 2, 1, 3, 1, 4, 3) \\ (b; 2, 1, 3, 2, 4, 1) \\ (b; 2, 1, 3, 1, 4, 1) \end{matrix} \quad m = 12(b-2) + \begin{cases} 1 \\ 5 \\ 7 \\ 11 \end{cases}$
$(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$ $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; \mathbb{I}, \mathbb{I})$	$\begin{matrix} (b; 2, 1, 3, 2, 5, 4) \\ (b; 2, 1, 3, 2, 5, 3) \\ (b; 2, 1, 3, 1, 5, 4) \\ (b; 2, 1, 3, 2, 5, 2) \\ (b; 2, 1, 3, 1, 5, 3) \\ (b; 2, 1, 3, 2, 5, 1) \\ (b; 2, 1, 3, 1, 5, 2) \\ (b; 2, 1, 3, 1, 5, 1) \end{matrix} \quad m = 30(b-2) + \begin{cases} 1 \\ 7 \\ 11 \\ 13 \\ 17 \\ 19 \\ 23 \\ 29 \end{cases}$

Remarks 1. The quotient surface singularities are characterized by several *finiteness conditions*: They are the only surface singularities

- having a finite *fundamental group*
- carrying only finitely many isomorphism classes of indecomposable reflexive modules.

2. It is conjectured that they are also exactly the *deformation finite* surface singularities. A proof of this would imply a positive answer to the old conjecture that there are no *rigid* (normal) surface singularities.

Let ρ be a representation of Γ on the vector space $V = V_\rho$. Γ operates on $\mathbb{C}^2 \times V$ via the natural representation c and ρ, and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/\,\Gamma$ whose (locally free) sheaf of holomorphic sections extends to a *reflexive sheaf* M_ρ on $\mathbb{C}^2/\,\Gamma = X_\Gamma$:

$$M_\rho := \mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_{\rho^*})^\Gamma \,,$$

where μ denotes the canonical projection $\mathbb{C}^2 \to X_\Gamma$ and ρ^* is the *dual* representation. M is indecomposable if and only if ρ is irreducible. - In fact, one gets all reflexive modules M on X_Γ in this manner:

Theorem (Esnault) *There exists a one–to–one correspondence between*

- $\{$ (indec.) reflexive modules M on $(X, 0)$ $\}$
- $\{$ (Γ–indecomposable) free modules $\widehat{M}$ on $(\mathbb{C}^2, 0)$ with a Γ–action $\}$

and

- $\{$ (irreducible) representations ρ of Γ $\}$.

For the step invoking the minimal resolution, we can study more generally any *rational* surface singularity X and an arbitrary reflexive module M on it. Let $\pi : \widetilde{X} \longrightarrow X$ be a minimal resolution, and put $\widetilde{M} := \pi^* M/\,\text{torsion}$. Such sheaves on $\widetilde{X}$ were baptized *full sheaves* by ESNAULT. By *local duality*, one has the following

Theorem (ESNAULT) *A sheaf $\mathcal{F}$ on $\widetilde{X}$ is full if and only if the following conditions are satisfied*:

1. $\mathcal{F}$ *is* locally free, *i. e. (the sheaf of holomorphic sections in) a* vector bundle,
2. $\mathcal{F}$ *is* generated by global sections, *in particular* $H^1(\widetilde{X}, \mathcal{F}) = 0$,
3. $H^1(\widetilde{X}, \mathcal{F}^* \otimes \omega_{\widetilde{X}}) = 0$, *where* $\omega_{\widetilde{X}}$ *denotes the* canonical sheaf *on* $\widetilde{X}$.

Under these assumptions, $M = \pi_* \mathcal{F}$ *is reflexive and* $\mathcal{F} = \widetilde{M}$. *Moreover*, $M^* = \pi_*(\mathcal{F}^*)$ (*but* $\mathcal{F}^*$ *is, in general, not a* full *sheaf*).

Remark For quotient surface singularities X_Γ which are not KLEIN singularities one has always

$$\# \operatorname{Irr} E_\Gamma < \# \operatorname{Irr}^0 \Gamma \ .$$

So we can't expect MCKAY's correspondence literally true in this situation.

Definition A full sheaf $\widetilde{M}$ /
a reflexive module M /
a representation ρ /
is called *special* (perhaps better *exceptional*), if and only if $H^1(\widetilde{X}, (\widetilde{M})^*) = 0$ (where $M := M_\rho$ in case of a representation ρ).

Special full sheaves have been characterized by WUNRAM, special reflexive modules and representations by RIEMENSCHNEIDER. Notice that in former articles, we associated the module $\mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_\rho)^\Gamma$ to a representation ρ instead of $\mu_*(\mathcal{O}_{\mathbb{C}^2} \otimes V_{\rho^*})^\Gamma$. Dealing with the *dual* representations fits better into the framework of the ITO-NAKAMURA construction to be discussed later.

Theorem
1) $\widetilde{M}$ special $\iff$ *the canonical map*

$$\widetilde{M} \otimes \omega_{\widetilde{X}} \to [(M \otimes \omega_X)^{**}]\widetilde{}$$

is an isomorphism.
2) M special $\iff$ $M \otimes \omega_X /$ torsion *is reflexive*.
3) ρ special $\iff$ *the canonical map*

$$(\Omega^2_{\mathbb{C}^2,0})^\Gamma \otimes (\mathcal{O}_{\mathbb{C}^2,0} \otimes V_{\rho^*})^\Gamma \to (\Omega^2_{\mathbb{C}^2,0} \otimes V_{\rho^*})^\Gamma$$

is surjective.

Here, of course, two stars denote the double dual *(or* reflexive hull *) of a coherent analytic sheaf*, Ω^m_X *is the sheaf of* Kähler m–forms *and* $\omega_X := (\Omega^2_X)^{**}$ *the* dualizing sheaf *on a complex analytic surface* X .

As a *Corollary* to the next Theorem of Wunram, one obtains the McKay *correspondence* since for the Klein singularities one has $\omega_X \cong \mathcal{O}_X$ (Gorenstein property) and $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$. Or in other words: *For Klein singularities all reflexive modules etc. are special*.

Let $\mathcal{F} = \widetilde{M}$ be of rank r and full. Then one can construct an exact sequence

$$0 \longrightarrow \mathcal{O}^r_{\widetilde{X}} \longrightarrow \mathcal{F} \longrightarrow N \longrightarrow 0$$

with $D := \operatorname{supp} N$ a divisor in a neighborhood of the *exceptional* set E which cuts E transversally at regular points only. We call D the CHERN *divisor* $c_1(\mathcal{F})$.

Theorem (WUNRAM, 1987 - 1988) *There is a bijection*

$$\{\textit{special nontrivial indecomposable reflexive modules}\}$$
$$\updownarrow$$
$$\{\textit{irreducible components } E_j \textit{ of } E\}$$

via

$$M \longmapsto c_1(\widetilde{M})\, E_k = \delta_{jk} \, .$$

The rank of M_j equals the multiplicity r_j of the curve E_j in the fundamental cycle $Z = \sum r_j \, E_j$.

4. The construction of ITO and NAKAMURA

In 1996 YUKARI ITO and IKU NAKAMURA constructed in joint work the minimal resolution $\widetilde{X}_\Gamma$ in the case of finite subgroups of the *special* linear group $\mathrm{SL}(2, \mathbb{C})$ by invariant theory of Γ acting on a certain HILBERT scheme. They were able, again by checking case by case, to produce the correct representations from the irreducible components of $E \subset \widetilde{X}_\Gamma$ (and even more). Two years later, NAKAMURA lectured in 1998 on this topic in Hamburg; I soon became aware of how one should generalize the statement to (small) subgroups of the *general* linear group $\mathrm{GL}(2, \mathbb{C})$ and developed some vague ideas how to prove this without too many calculations.

This conjecture could be checked in the case of *cyclic* quotients by a simple computation which depended on the concrete results in the doctoral thesis of RIE KIDOH, written in Sapporo under the supervision of NAKAMURA.

I gave some lectures on this topic in Japan during September 1999 and learned from AKIRA ISHII in August 2000 that he succeeded in proving the conjecture via rephrasing the *multiplication formula* of WUNRAM in terms of a functor between certain derived categories. Besides the general proof of A.ISHII which uses much heavier machinery there exists now another independend proof in the cyclic case via *toric geometry* by Y. ITO; she doesn't use KIDOH's explicit construction but my characterization of special representations.

Two years ago I published a manuscript on this and the general theory of the so called *special representations* at the *Hokkaido Mathematical Journal*.

Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the HILBERT scheme of all 0–dimensional subschemes on $\mathbb{C}^2$ of colength n. It is well known that the canonical HILBERT–CHOW–*morphism*

$$\mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n / \mathfrak{S}_n$$

is a resolution of singularities (FOGARTY), and $\mathrm{Hilb}^n(\mathbb{C}^2)$ carries a holomorphic symplectic structure (BEAUVILLE). Let $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$ be a finite small subgroup of order $n = \mathrm{ord}\,\Gamma$, and take the invariant part of the natural action of Γ on $\mathrm{Hilb}^n(\mathbb{C}^2)$. The resulting space $\mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma$ is smooth and maps under the canonical mapping

$$\mathrm{Hilb}^n(\mathbb{C}^2)^\Gamma \longmapsto \mathrm{Sym}^n(\mathbb{C}^2)^\Gamma \cong \mathbb{C}^2 / \Gamma$$

to X_Γ. It may a priori have several components, but there is exactly one which maps *onto* X_Γ and thus constitutes a *resolution* of X_Γ which will be denoted by

$$Y_\Gamma = \mathrm{Hilb}^\Gamma(\mathbb{C}^2)\,.$$

In fact, $\mathsf{Hilb}^{\Gamma}(\mathbb{C}^2)$ is equal to the open subset of so–called Γ–invariant n–*clusters* in $\mathbb{C}^2$, and the resolution is *minimal* (A. ISHII). The last fact has been known before in the case $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ by ITO-NAKAMURA and for cyclic subgroups of $\mathrm{GL}(2, \mathbb{C})$ by KIDOH; it has been conjectured for general finite small subgroups $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ by GINZBURG-KAPRANOV.

In particular, a point on the exceptional set E of Y_{Γ} may be regarded as a Γ–invariant ideal $I \subset \mathcal{O}_{\mathbb{C}^2}$ with support in 0. Now, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, $\mathfrak{m}_X$ that of $\mathcal{O}_{X,0} = \mathcal{O}^{\Gamma}_{\mathbb{C}^2,0}$ and $\mathfrak{n} = \mathfrak{m}_X \mathcal{O}_{\mathbb{C}^2,0}$. Put

$$V(I) := I / (\mathfrak{m} I + \mathfrak{n}) .$$

This is a (finite–dimensional) Γ–module.

For a (nontrivial) irreducible representation $\rho \in \operatorname{Irr}^0 \Gamma := \operatorname{Irr} \Gamma \setminus \{ \rho_0 \}$ with representation space V_ρ put

$$E_\rho = \{ I : V(I) \quad \text{contains} \quad V_\rho \} .$$

In the case of KLEIN singularities, i. e. for finite subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$, one has the following beautiful result of ITO and NAKAMURA which opened up a new way to understand the MCKAY correspondence completely in terms of the binary polyhedral group Γ.

Theorem (ITO - NAKAMURA) *For* $\rho \in \operatorname{Irr}^0 \Gamma$, $E_\rho \cong \mathbb{P}_1$. *Moreover*, $E_\rho \cap E_{\rho'}$ *is empty or consists of exactly one point for* $\rho \neq \rho'$, *and*

$$E = \bigcup_{\rho \in \operatorname{Irr}^0 \Gamma} E_\rho .$$

More precisely, $V(I) = V_\rho$ *for the ideals* $I \in E_\rho$ *corresponding to smooth points of* E, *and*

$$E_\rho \cap E_{\rho'} \ni I \iff V(I) = V_\rho \oplus V_{\rho'} .$$

5. Cyclic quotient surface singularities

The cyclic group $C_{n,q}$ with $\gcd(n, q) = 1$ operates on the polynomial ring $\mathbb{C}[u, v]$ by $(u, v) \longmapsto (\zeta_n u, \zeta_n^q v)$. A monomial $u^\alpha v^\beta$ is invariant under this action if and only if

$$\alpha + q\beta \equiv 0 \mod n\,,$$

e. g. for $(\alpha, \beta) = (n, 0)\,,\ (n - q, 1)\,,\ (0, n)\,.$

The HIRZEBRUCH - JUNG *continued fraction*

$$\frac{n}{n-q} = a_1 - \frac{1}{a_2 - 1/\cdots} = a_1 - 1 \rceil a_2 - \cdots - 1 \rceil a_m$$

with $a_\mu \geq 2$ gives a strictly decreasing sequence

$$\alpha_0 = n > \alpha_1 = n - q > \alpha_2 = a_1\alpha_1 - \alpha_0 > \cdots$$

stopping with $\alpha_{m+1} = 0$, and a strictly increasing sequence

$$\beta_0 = 0 < \beta_1 = 1 < \beta_2 = a_1\beta_1 - \beta_0 < \cdots$$

stopping with $\beta_{m+1} = n\,.$

It is well known that the monomials

$$u^{\alpha_\mu}\, v^{\beta_\mu}\,, \quad \mu = 0, \ldots, m+1$$

generate the invariant algebra

$$A_{n,q} := \mathbb{C}\,\langle u,\, v\rangle^{C_{n,q}} = \mathcal{O}_{\mathbb{C}^2/\,C_{n,q},0}$$

minimally. In particular, $\operatorname{embdim} A_{n,q} = m+2$, hence, $\operatorname{mult} A_{n,q} = m+1$. The numbers a_μ are exponents in canonical equations for $A_{n,q}$.

On the other hand, the continued fraction expansion

$$\frac{n}{q} = b_1 - \underline{1 \rceil\, b_2} - \cdots - \underline{1 \rceil\, b_r}\,,\ b_k \geq 2$$

gives invariants for the minimal resolution of $\mathbb{C}^2/\,C_{n,q}$ whose exceptional divisor consists of a string of rational curves with *selfintersection numbers* $-b_k$.

Define correspondingly the decreasing sequence

$$i_0 = n > i_1 = q > i_2 = b_1 i_1 - i_0 \\ > \cdots > i_r = 1 > i_{r+1} = 0$$

and

$$j_0 = 0 < j_1 = 1 < j_2 = b_1 j_1 - j_0 \\ < \cdots < j_{r+1} = n.$$

Theorem (Kidoh) *Let* (n, q) *be given. Then,* $\mathrm{Hilb}^{C_{n,q}}(\mathbb{C}^2)$ *consists of the* $C_{n,q}$*–invariant ideals*

$$I_k(s_k, t_k)$$

of colength $n = \mathrm{ord}\, C_{n,q}$ *, which are generated by the elements*

$$u^{i_{k-1}} - s_k v^{j_{k-1}},\ v^{j_k} - t_k u^{i_k},\ u^{i_{k-1}-i_k} v^{j_k - j_{k-1}} - s_k t_k \,.$$

Here, $1 \le k \le r+1$ *, and the parameters* $(s_k, t_k) \in \mathbb{C}^2$ *are arbitrary.*

Remarks. 1. These are in fact $C_{n,q}$–invariant ideals, since $i_k \equiv q\, j_k \bmod n$ and the functions $u^{i_{k-1}-i_k}\, v^{j_k-j_{k-1}}$ are invariant.

2. The $(r+1)$ copies of $\mathbb{C}^2$ patch together to form the *minimal resolution* of $\mathbb{C}^2/\,C_{n,q}$, i. e. $I_k\,(s_k,\, t_k) = I_{k+1}\,(s_{k+1},\, t_{k+1}) \Longleftrightarrow s_{k+1}\, t_k = 1$ and $t_{k+1} = t_k^{b_k}\, s_k$.

3. The exceptional divisor E equals

$$I_1\,(0,\, t_1) \cup \bigcup_{k=2}^{r} \{I_k\,(s_k,\, t_k) : s_k\, t_k = 0\} \\ \cup I_{r+1}\,(s_{r+1},\, 0)\,.$$

4. It is not difficult to deduce Kidoh's result by induction using the well known partial resolution of cyclic quotient singularities constructed by Fujiki.

What about the representations of $C_{n,q}$ on the $V\,(I_k)$? For $I_1\,(0,\,t_1)$ the first generator $u^{i_0} = u^n$ is an invariant. The third is such in all cases anyway. So, $C_{n,q}$ acts on $V\,(I_1\,(0,\,t_1)) \cong \mathbb{C}$ as the *one dimensional representation* χ_{i_1} where

$$\chi_i \,:\, z \longmapsto \zeta_n^i z$$

(recall that $qj_k \equiv i_k \bmod n$). This remains automatically true for $I_2\,(s_2,\,t_2)$ with $t_2 = 0\,,\; s_2 \neq 0\,$. The first normal crossing point of the exceptional set is the ideal $I_2\,(0,\,0)$ which is generated by $u^{i_1},\,v^{j_2}$ and an invariant. Therefore, the corresponding representation is the sum

$$\chi_{i_1} \oplus \chi_{qj_2} = \chi_{i_1} \oplus \chi_{i_2}\,.$$

The ideal $I_2(0, t_2)$, $t_2 \neq 0$, is generated by u^{i_1}, $v^{j_2} - t_2 u^{i_2}$ and the invariant $u^{i_1 - i_2} v^{j_2 - j_1}$. Now,

$$\begin{aligned} t_2 u^{i_1} &= v^{j_1} (u^{i_1 - i_2} v^{j_2 - j_1}) - u^{i_1 - i_2} (v^{j_2} - t_2 u^{i_2}) \\ &\in \mathfrak{m} I_2(0, t_2) \,. \end{aligned}$$

Therefore, the representation is just the one–dimensional

$$\chi_{i_2} = \chi_{q j_2} \,.$$

It should be clear how this game goes on: We get precisely the r representations χ_{i_k}, $k = 1, \ldots, r$, resp. the correct sum of two of them at the intersection points.

Due to a result of WUNRAM these are precisely the *special* representations of the group $C_{n,q}$. Hence, this gives a hint how the result of ITO–NAKAMURA should be generalized to arbitrary quotient surface singularities.

The precise result of WUNRAM is the following.

Lemma *For a given number $i \in \mathbb{N}$ between* 0 *and* $n-1$, *there exist uniquely determined nonnegative integers $d_1, \ldots, d_r$ with*

$$\begin{aligned} i &= d_1\, i_1 + t_1\,, & 0 &\le t_1 < i_1\,, \\ t_k &= d_{k+1}\, i_{k+1} + t_{k+1}\,, & 0 &\le t_{k+1} < i_{k+1}\,, \\ & & 1 &\le k \le r-1\,. \end{aligned}$$

Then, the CHERN divisor *of the full sheaf associated to the one dimensional representation χ_i is*

$$\sum_{k=1}^{r} d_k\, E_k\ .$$

In particular, if $i = i_k$, then this CHERN divisor *is equal to* $1 \cdot E_k$.

6. Interpretation in terms of derived categories

McKay correspondence may also be understood as an equivalence of *derived categories*. This has been worked out by Kapranov and Vasserot for $\mathrm{SL}(2,\mathbb{C})$ and by Bridgeland, King, Reid in dimension 3. The last paper led A. Ishii to study more closely the canonical functor

$$\Psi : D_c^\Gamma(\mathbb{C}^2) \longrightarrow D_c(Y_\Gamma)$$

where $D_c^\Gamma(\mathbb{C}^2)$ denotes the derived category of Γ–equivariant coherent analytic sheaves with compact support on $\mathbb{C}^2$, Γ a finite small subgroup of $\mathrm{GL}(2,\mathbb{C})$, and $D_c(Y_\Gamma)$ the derived category of coherent analytic sheaves on $Y_\Gamma = \mathrm{Hilb}^\Gamma(\mathbb{C}^2)$ with compact support.

The main ingredient of his proof is WUNRAM's *multiplication formula* which generalizes the one of ESNAULT and KNÖRRER. We denote by M a reflexive module on $X = \mathbb{C}^2/\Gamma$, its AUSLANDER-REITEN *translate*, i. e. the module $(M \otimes \omega_X)^{**}$, by $\tau(M)$, and finally, we write $N_M = (M \otimes \Omega^1_X)^{**}$. Then we have:

Theorem (WUNRAM)

$$c_1(\widetilde{N_M}) - c_1(\widetilde{M}) - c_1(\widetilde{\tau(M)})$$
$$= \begin{cases} E_j\ , & M = M_j \ \text{special}\ ,\ j \neq 0\ , \\ Z\ , & M = M_0 := \mathcal{O}_X\ , \\ 0\ , & M \ \text{nonspecial}\ . \end{cases}$$

Here, Z *denotes the* fundamental cycle *of the minimal resolution of* X.

A. ISHII first restates and proves once more WUNRAM's *multiplication formula* in the following form.

Theorem (A. Ishii) *Let ρ be an irreducible representation of $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$ and put $\mathcal{O}_0 = \mathcal{O}_{\mathbb{C}^2,0}/\,\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{C}^2$ at the origin. Then*

$$\Psi\,(\mathcal{O}_0 \otimes V_{\rho^*}) = \begin{cases} \mathcal{O}_{E_j}(-1)[1]\,, & \rho = \rho_j \text{ special}\,,\ j \neq 0\,, \\ \mathcal{O}_Z\,, & \rho = \rho_0\,, \\ 0\,, & \rho \text{ nonspecial}\,. \end{cases}$$

He then explicitly constructs a *right adjoint* Φ to Ψ. The resulting isomorphism

$$\mathrm{Hom}_{Y_\Gamma}(\Psi\,(\Delta), \nabla) \cong \mathrm{Hom}_{D_c^\Gamma(\mathbb{C}^2)}(\Delta, \Phi\,(\nabla))$$

leads to the desired result when applied to $\Delta := \mathcal{O}_0 \otimes V_{\rho^*}\,,\ \nabla := \mathcal{O}_y\,,\ y \in Y_\Gamma\,.$

Theorem (A. Ishii) *The* Ito-Nakamura *construction yields the same result as above also for finite small subgroups $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$ if the set $\mathrm{Irr}^0\Gamma$ of* all *nontrivial irreducible representations is replaced by the subset $\mathrm{Irrspec}^0\Gamma \subset \mathrm{Irr}^0\Gamma$ of* (non-trivial) special *ones*.

7. Searching for a more concrete version

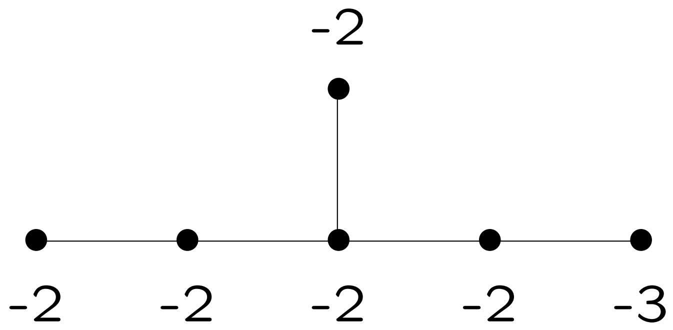
Most questions being answered in a perfect conceptual manner: Why does one need a "more concrete" version?

First of all since mathematical physicists are interested in *non–supersymmetric configurations of D–branes and their evolution via tachyon condensation* (c.f.: HE, YANG–HUI: *Closed String Tachyons, Non–Supersymmetric Orbifolds and Generalized McKay Correspondence*, hep–th or Adv. Theor. Math. Phys 7, 2003). In the Abelian case the special representations are associated to some D–brane charges sitting on the HIGGS branch.

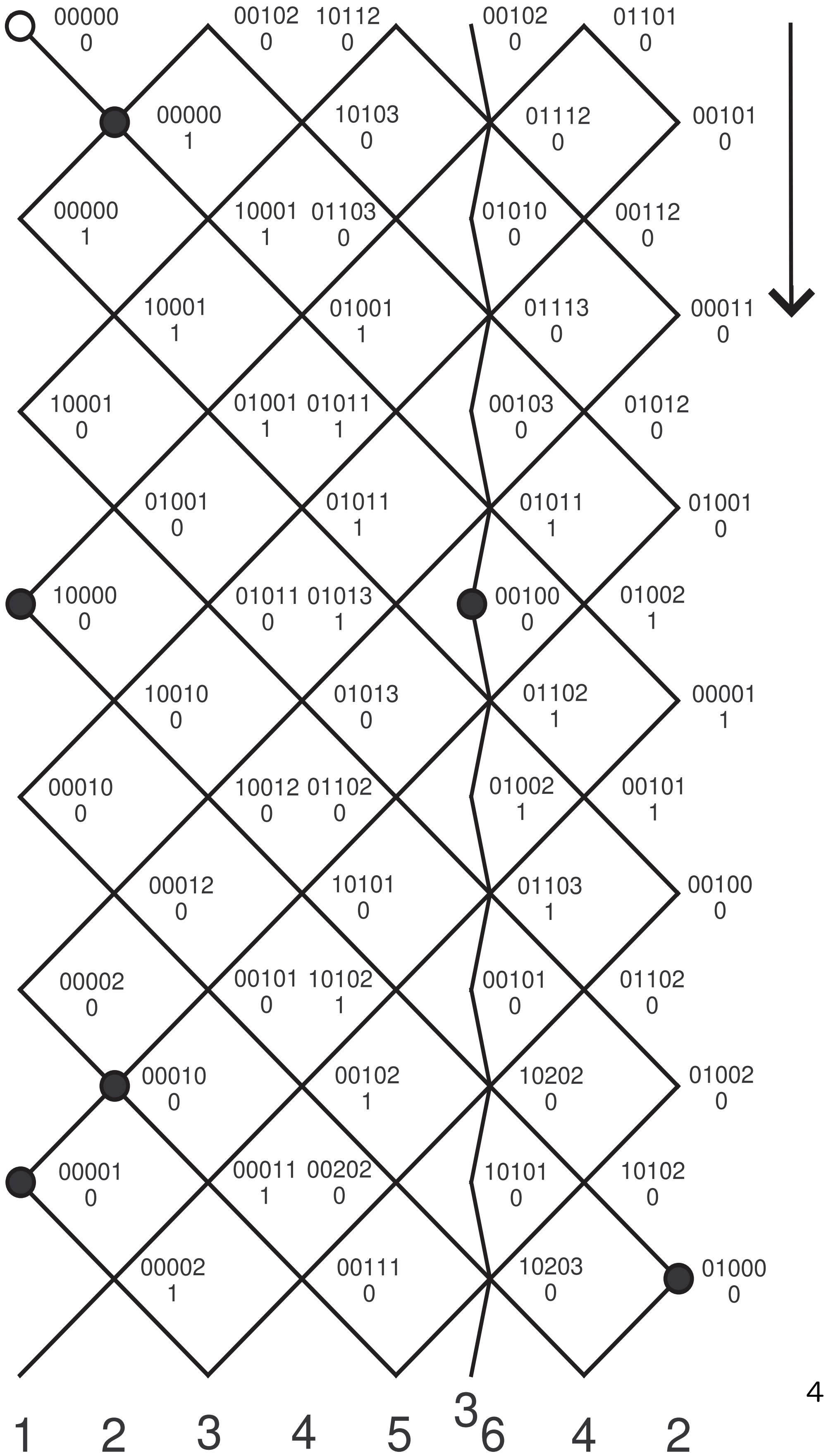
Problem Determine explicitly the special representations for a given small subgroup $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$ and attach them to the vertices in the dual resolution graph of $\widetilde{X}_\Gamma$.

WUNRAM has this task carried out in full detail only for cyclic quotients; for the remaining cases he describes a method how one can in principle compute the CHERN divisors and detect the special representations in the MCKAY–quiver.

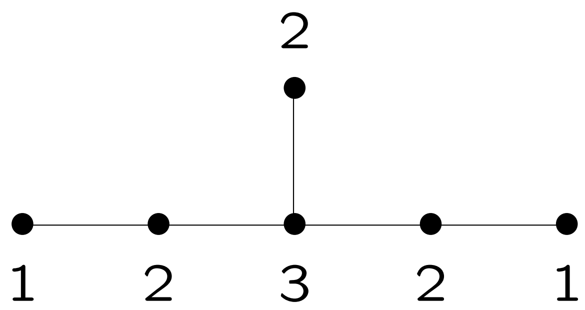
He finds for the group $(\mathbb{Z}_{14}, \mathbb{Z}_{14}; \mathbb{I}, \mathbb{I})$, i. e. for the quotient surface singularity with resolution graph



the following MCKAY–quiver and the CHERN divisors as indicated.


00000 0
00102 0
10112 0
00102 0
01101 0
00000 1
10103 0
01112 0
00101 0
00000 1
10001 1
01103 0
01010 0
00112 0
10001 1
01001 1
01113 0
00011 0
10001 0
01001 1
01011 1
00103 0
01012 0
01001 0
01011 1
01011 1
01001 0
10000 0
01011 0
01013 1
00100 0
01002 1
10010 0
01013 0
01102 1
00001 1
00010 0
10012 0
01102 0
01002 1
00101 1
00012 0
10101 0
01103 1
00100 0
00002 0
00101 0
10102 1
00101 0
01102 0
00010 0
00102 1
10202 0
01002 0
00001 0
00011 1
00202 0
10101 0
10102 0
00002 1
00111 0
10203 0
01000 0
1 2 3 4 5 3 6 4 2

Notice that the fundamental cycle in this example is the following:



Remark This example shows that the irreducible reflexive modules are not determined by their Chern divisor and their rank. This, however, is always true for the special objects (Esnault).

A. ISHII's result may be used to compute them *directly* via invariant theory! Of course, one has to determine $\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$ in all cases and to identify these spaces with the resolution $\widetilde{X}_{\Gamma}$, which might be tedious, but not so difficult.

Due to the construction, $\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$ carries a natural *tautological bundle* T with

$$\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2) \ni I \longmapsto \mathcal{O}_{\mathbb{C}^2}/\, I =: T_I$$

with a canonical Γ–action (which is very simple for $0 \notin \mathrm{supp}\, I$). By the so called *normal basis theorem*, to each irreducible representation ρ of Γ of rank r_ρ there exist $\mathcal{O}_X$–submodules $M_\rho^{(1)} \cong \cdots \cong M_\rho^{(r_\rho)}$ of $\mu_* \mathcal{O}_{\mathbb{C}^2}$ such that

$$\mu_* \mathcal{O}_{\mathbb{C}^2} = \bigoplus_{\rho \in \mathrm{Irr}\, \Gamma} (M_\rho^{(1)} \oplus \cdots \oplus M_\rho^{(r_\rho)})\, .$$

This decomposition given, there is an associated decomposition of T_I for each $I \in \mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$ which fits together to a decomposition of the vector bundle T:

$$T \cong \bigoplus_{\rho \in \mathrm{Irr}\,\Gamma} (\widetilde{M}_\rho^{(1)} \oplus \cdots \oplus \widetilde{M}_\rho^{(r_\rho)})$$

with

$$\widetilde{M}_\rho \cong \widetilde{M}_\rho^{(1)} \cong \cdots \cong \widetilde{M}_\rho^{(r_\rho)} \,.$$

Corollary *One can describe the vector bundles* $\widetilde{M}_\rho$ *via invariant theory as subbundles of* T *on* $\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$. *In particular, the* CHERN *divisor of* $\widetilde{M}_\rho$ *can be constructed in these terms, such leading to a concrete description of the opposite direction of the* MCKAY*–correspondence, i.e. associating to a nontrivial special representation a generic ideal in the exceptional set* E.

A final remark

I am completely aware of the fact that the main trend in what is nowadays called McKay correspondence was to treat the *higher dimensional* case $\Gamma \subset \mathsf{SL}(n, \mathbb{C})$ under the slogan: If X_Γ has a *crepant* resolution $\widetilde{X}_\Gamma$, i. e. if the canonical sheaf of $\widetilde{X}_\Gamma$ is trivial, then there should be a bijection

$$\mathrm{Irr}^0\Gamma \longleftrightarrow \text{basis of } H^*(\widetilde{X}_\Gamma, \mathbb{Z})$$

which - in case $n = 2$ - is just a rephrasing of the result for finite subgroups in $\mathsf{SL}(2, \mathbb{C})$. (C. f. some notes of Reid).

But this is a completely different story and needs at least two more hours (and a speaker who is much more familiar with this subject than I am).

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