The Abdus Salam
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United Nations
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International Atomic
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SMR 1673/23

AUTUMN COLLEGE ON PLASMA PHYSICS

5 - 30 September 2005

Transonic Flows in Two-Fluid Plasmas

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September 16, 2005

Abstract.
Transonically rotating toroidal plasmas occur at all scales in the plasma universe and, recently, also in laboratory tokamak plasmas. This offers great opportunities for new insights of the effects of transonic transitions on the background equilibrium flows, and on the waves and instabilities excited.

Transfer of knowledge and computational methods on MHD and two-fluid waves and instabilities in magnetically confined laboratory fusion plasmas to space and astrophysical plasmas is seriously hampered though by two related difficulties:
(1) in contrast to laboratory plasmas, astrophysical plasmas always have sizeable plasma flows so that they can never be described as a static equilibrium;
(2) these flows are usually 'transonic', i.e. surpass one of the critical speeds related to the different flow regimes with quite different physical characteristics.

Based on previously obtained MHD results on the stationary states and instabilities of transonically rotating accretion disks about compact objects, the extension to two-fluid plasmas is initiated: A variational principle for the computation of two-fluid stationary states is constructed which involves seven fields determining the different physical variables, and six arbitrary stream functions that should be determined by spatially resolved astrophysical observations. It exhibits all the intricacies due to the electron and ion flow excursions from the magnetic flux surfaces. New hyperbolic flow regimes are found with quite different properties than the MHD ones.

Keywords: accretion disks, transonic flows, two-fluid plasma theory

1. Introduction

In the textbook *Principles of Magnetohydrodynamics* (Goedbloed and Poedts, 2004) the main lines of the magnetohydrodynamic (MHD) description of laboratory and astrophysical plasmas are presented together with some excursions into microscopic theories, in particular the two-fluid theory. It is pointed out that the common description of plasmas in the laboratory and astrophysics is possible due to one of the most powerful properties of MHD, viz. its *scale-independence*. That scale-independence breaks down on the microscopic scales of two-fluid dynamics, which are relevant, e.g., for observed electron inertial waves in the magnetosphere (Stasiewicz, 2004). In the present paper, the consequences of this for the wave propagation and stationary states of transonic two-fluid plasmas are presented.

Basic questions concerning transonic MHD flows originated in an attempt to generalize *MHD spectroscopy of tokamaks* (Goedbloed et al., 1993) to astrophysical plasmas. The omnipresence of transonic flows there forces one to consider the issue of what we have recently termed the *transonic enigma* (Goedbloed and Keppens, 2005): At 'transonic' transitions (marked by one of the three characteristic MHD speeds), the flow changes from elliptic to hyperbolic, standard equilibrium solvers diverge, and signatures of the transitions appear in the spectrum of waves and instabilities. As a consequence, computation of stationary states becomes extremely complex (Beliën et al., 2002), but exciting new *trans-slow continuum instabilities* appear (Goedbloed et al., 2004) that may be a source of anomalous dissipation and increased angular momentum transport in rotating astrophysical plasmas like accretion disks (Frank, King, and Raine, 2002). In contrast to the well-know magneto-rotational instability (Velikhov, 1959; Chandrasekhar, 1960; Balbus and Hawley, 1991), the transonic instability operates at arbitrary values of $\beta \equiv 2p/B^2$ and just requires that the poloidal speed surpasses the critical value of the slow magnetosonic speed.

Our MHD model is shown in Fig. 1: An axisymmetric configuration of nested magnetic / flow surfaces with the magnetic field indicated by the vectorial Alfvén speed $\mathbf{b}$ and the plasma velocity indicated by $\mathbf{v}$, both having toroidal and poloidal components, surrounds a compact object of mass M_* in the origin.

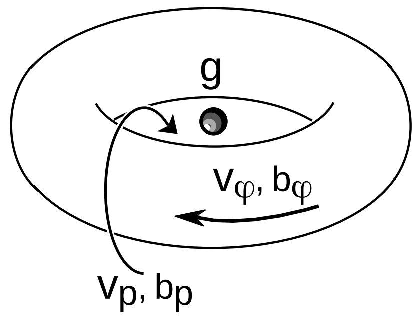

Figure 1. Transonically rotating magnetized 'thick' disk about a compact object: flow and magnetic surfaces coincide in the MHD model. [Goedbloed et al. (2004)]

In order to permit stationary MHD equilibrium, the accretion speeds should be much smaller than the rotation speeds of the disk. In that case, the flow is confined to the magnetic surfaces and the stationary states are described by just *two fields:* the poloidal flux ψ and the square of the poloidal Alfvén Mach number $M^2 \equiv \rho v_p^2/B_p^2 \equiv \chi'^2/\rho$, where $\chi(\psi)$ is the poloidal stream function.

The stationary states are determined by minimization of the *MHD Lagrangian,*

$$\delta \int \mathcal{L}_{\mathrm{MHD}}(\psi, \nabla\psi, M^2; R, Z)\, dV = 0\,, \tag{1}$$

with cylindrical coordinates R, Z, φ, and involving five arbitrary flux functions $\Lambda_i(\psi)$, fixed by whatever observational evidence is available. The Euler equations are a 'Grad-Shafranov' type PDE for $\psi(R, Z)$,

$$R^2\, \nabla \cdot \left(\frac{M^2 - 1}{R^2}\, \nabla\psi \right) = \cdots\,, \tag{2}$$

and an algebraic Bernoulli equation for $M^2(R, Z)$. The solution of the latter equation is of the form $M^2 = M^2(\nabla\psi, \cdots)$, so that insertion into Eq. (1) yields a non-trivial criterion for hyperbolicity:

$$\Delta \sim \frac{(M^2 - 1)^2 (M^2 - M_c^2)}{(M^2 - M_s^2)(M^2 - M_f^2)} \quad \begin{cases} < 0 : \ \text{elliptic} \\ \geq 0 : \ \textit{hyperbolic!} \end{cases}\,,$$

where the values $1, M_c^2, M_s^2$, and M_f^2 correspond to the critical poloidal Alfvén, cusp, slow and fast magnetosonic speeds, respectively.

The transonic enigma, due to the slow and fast hyperbolic flow regimes and the Alfvén singularity $M^2 = 1$, is dealt with in the mentioned papers by Beliën et al. (2002) and Goedbloed et al. (2004), which also contain expositions of the associated numerical programs FINESSE (stationary states) and PHOENIX (transonic instabilities).

Turning now to the *two-fluid picture*, we will demonstrate that, far from resolving the transonic enigma (as implied by McClements and Thyagaraja (2001)), new hyperbolic flow regimes appear there together with a much more intricate dynamical structure (Goedbloed, 2004). [Also see the comment of McClements and Thyagaraja (2005) and the reply by Goedbloed (2005)]. The conclusion will be that two-fluid theory is not to be used to ameliorate problems of MHD analysis, but for its own sake, viz. the refined physics associated with the freedom of the electron and ion motions relative to the magnetic surfaces. These additional freedoms complicate the analysis even more.

2. Waves in two-fluid plasmas

To obtain the transonic transitions in two-fluid theory, the characteristics associated with the large wave number (small-scale) asymptotics should be studied. This can be one in full generality from the general *two-fluid dispersion equation,* which is presented in Fig. 2 together with

$$
\begin{aligned}
&\alpha_{60}\ \bar\omega^{12} \\
&+ (\alpha_{50} + \alpha_{51}\bar k^2)\,\bar\omega^{10} \\
&+ (\alpha_{40} + \alpha_{41}\bar k^2 + \alpha_{42}\bar k^4)\,\bar\omega^{8} \\
&+ (\alpha_{30} + \alpha_{31}\bar k^2 + \alpha_{32}\bar k^4 + \alpha_{33}\bar k^6)\,\bar\omega^{6} \\
&\qquad + (\alpha_{21}\bar k^2 + \alpha_{22}\bar k^4 + \alpha_{23}\bar k^6 + \alpha_{24}\bar k^8)\,\bar\omega^{4} \\
&\qquad\qquad + (\alpha_{12}\bar k^4 + \alpha_{13}\bar k^6 + \alpha_{14}\bar k^8)\,\bar\omega^{2} \\
&\qquad\qquad\qquad + \alpha_{03}\bar k^6 + \alpha_{04}\bar k^8 \quad = \ 0\,.
\end{aligned}
$$

cold

(3) local, HF

(1) cutoff

(4) global, MHD

(2) resonance

Figure 2. Dispersion equation for an ideal two-fluid plasma with asymptotic limits. [Goedbloed and Poedts (2004)]

the asymptotics. The coefficients α_{mn} are explicitly known in terms of the dimensionless parameters $e \equiv \omega_{pe}/\omega_p\,, E \equiv \Omega_e/\omega_p\,, v \equiv v_e/c\,, i \equiv \omega_{pi}/\omega_p\,, I \equiv \Omega_i/\omega_p\,, w \equiv v_i/c\,,$ where $v_{e,i}$ are the respective thermal speeds, and the direction $k_\parallel/k$ of the wave vectors. This dispersion equation is a polynomial of twelfth degree in $\bar\omega \equiv \omega/\omega_p$ and of eighth degree in $\bar k \equiv k\delta$, where δ is the skin depth. The solution, together with the asymptotics, is depicted in Fig. 3 for a particular case. The important point to notice is that, compared to the three asymptotes of MHD, there are two asymptotes now with characteristic speed c and two asymptotes (corresponding to the electron and ion sound waves) that may give rise to hyperbolicity when the electron or ion flow speed surpasses the respective wave speed: see Eq. (9) below.

3. Stationary states in two-fluid plasmas

The much more involved part of the transonic enigma in two-fluid plasmas is the determination of the stationary states. This has not yet been solved to the same degree of completeness (together with numerical implementation) as in MHD. Here, we only outline the theoretical framework of the variational principle, analogous with the MHD expression (1), and indicate the kinds of numerical problems that are to be faced.

The general structure of the stationary states in two-fluid plasmas is governed by the *deviations of the electron and ion flow speeds* $\mathbf{u}_{e,i}$

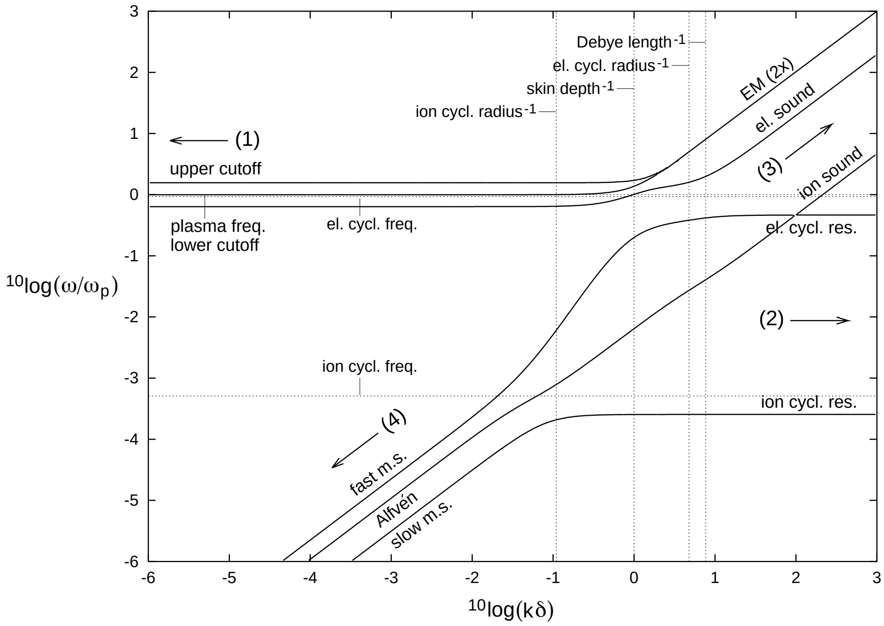

Figure 3. Dispersion diagram for the oblique waves ($k_\parallel/k = 0.5$) of an ideal two-fluid plasma with the asymptotic limits: (1) $k \to 0$ (cutoff), (2) $k \to \infty$ (resonance), (3) $\omega \,\&\, k \to \infty$, ω/k finite (local, HF), (4) $\omega \,\&\, k \to 0$, ω/k finite (global, LF). The latter limit is also the local, high frequency limit of the MHD model. Dashed vertical and horizontal lines indicate characteristic length and time scales of the different waves. [Goedbloed and Poedts (2004)]

from the magnetic surfaces. Further analysis shows that the two-fluid variables in axi-symmetry are derived from *seven fields*, viz.:

– the *poloidal stream functions* χ_α determining the electron/ion flow speeds:

$$\mathbf{u}_\alpha = (\rho_\alpha R)^{-1}\mathbf{e}_\varphi \times \nabla\chi_\alpha + u_{\alpha\varphi}\mathbf{e}_\varphi \quad (\alpha = e, i)\,,$$

– the *poloidal flux function* ψ determining the magnetic field:

$$\mathbf{B} = R^{-1}\mathbf{e}_\varphi \times \nabla\psi + B_\varphi \mathbf{e}_\varphi\,, \qquad B_\varphi = R^{-1} I(\chi_e, \chi_i)\,,$$

– the *electric potential* ϕ etermining the electric field:

$$\mathbf{E} = \nabla\phi\,, \qquad \phi = \phi_* + \widetilde{\phi}\,, \quad \phi_* \equiv Q_*/\Big(4\pi\epsilon_0\sqrt{R^2+Z^2}\Big)\,,$$

– the *gravitational potential* $\mathcal{V}$ determining the gravitational field:

$$\mathbf{g} = \nabla\mathcal{V}\,, \qquad \mathcal{V} = \mathcal{V}_* + \widetilde{\mathcal{V}}\,, \quad \mathcal{V}_* \equiv -GM_*/\sqrt{R^2+Z^2}\,, \tag{3}$$

– the *electron and ion mass densities* ρ_α determined by the Bernoulli functions, as indicated below.

One of the intricate aspects of stationary states is the complete freedom to specify arbitrary functions of the fluxes. In the two-fluid case, this concerns *six (!) arbitrary functions of* χ_e *and* χ_i*:*

$$
\begin{aligned}
&\text{– Bernoulli functions: } H_\alpha(\chi_\alpha) \equiv \tfrac{1}{2}u_\alpha^2 + \tfrac{q_\alpha}{m_\alpha}\phi + \mathcal{V} + \tfrac{\gamma}{\gamma-1}\,\rho_\alpha^{-1}p_\alpha \;\Rightarrow\; \rho_\alpha\,,\\
&\text{– specific entropies: } \quad S_\alpha(\chi_\alpha) \equiv \rho_\alpha^{-\gamma}p_\alpha \;\Rightarrow\; p_\alpha\,,\\
&\text{– specific ang. mom.: } L_\alpha(\chi_\alpha) \equiv R\,u_{\alpha\varphi} - \tfrac{q_\alpha}{m_\alpha}\psi \;\Rightarrow\; u_{\alpha\varphi}\,.
\end{aligned} \tag{4}
$$

Variational principle

Equilibria derive from the *two-fluid Lagrangian* (Goedbloed, 2004),

$$
\delta\int \mathcal{L}_{\rm TF}(\chi_\alpha, \nabla\chi_\alpha, \rho_\alpha, \psi, \nabla\psi, \widetilde{\phi}, \nabla\widetilde{\phi}, \widetilde{\mathcal{V}}, \nabla\widetilde{\mathcal{V}}; R, Z)\, dV = 0\,, \tag{5}
$$

$$
\begin{aligned}
\mathcal{L}_{\rm TF} \;\equiv\;& \tfrac{1}{2R^2}\,\rho_e^{-1}|\nabla\chi_e|^2 + \tfrac{1}{2R^2}\,\rho_i^{-1}|\nabla\chi_i|^2 - \tfrac{1}{2\mu_0 R^2}\,|\nabla\psi|^2 + \tfrac{1}{2}\epsilon_0\,|\nabla\widetilde{\phi}|^2 \\
& - \tfrac{1}{8\pi G}\,|\nabla\widetilde{\mathcal{V}}|^2 + \tfrac{1}{2\mu_0 R^2}\,I^2 + \rho_e\,F_e - \tfrac{1}{\gamma-1}\rho_e{}^\gamma S_e + \rho_i\,F_i - \tfrac{1}{\gamma-1}\rho_i{}^\gamma S_i\,,
\end{aligned}
$$

where the nonlinear terms depend on the composite functions

$$
\begin{aligned}
I \;&\equiv\; RB_\varphi = I_0 + \mu_0\big(\tfrac{e}{m_e}\chi_e - \tfrac{Ze}{m_i}\chi_i\big)\,,\\
F_e \;&\equiv\; H_e - \tfrac{1}{2R^2}\big(L_e - \tfrac{e}{m_e}\,\psi\big)^2 + \tfrac{e}{m_e}\,(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,,\\
F_i \;&\equiv\; H_i - \tfrac{1}{2R^2}\big(L_i + \tfrac{Ze}{m_i}\psi\big)^2 - \tfrac{Ze}{m_i}\,(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,.
\end{aligned} \tag{6}
$$

The associated Euler equations are five PDEs for $\chi_{\rm e,i}$, ψ, $\widetilde{\phi}$, and $\widetilde{\mathcal{V}}$:

$$
\begin{aligned}
\nabla\cdot\Big(\tfrac{1}{\rho_e R^2}\nabla\chi_e\Big) \;&=\; \tfrac{e}{m_e R^2}I + \rho_e\Big[H_e{}' - \tfrac{1}{R^2}\big(L_e - \tfrac{e}{m_e}\psi\big)L_e{}'\Big] - \tfrac{1}{\gamma-1}\rho_e{}^\gamma S_e{}',\\
\nabla\cdot\Big(\tfrac{1}{\rho_i R^2}\nabla\chi_i\Big) \;&=\; -\tfrac{Ze}{m_i R^2}I + \rho_i\Big[H_i{}' - \tfrac{1}{R^2}\big(L_i + \tfrac{Ze}{m_i}\psi\big)L_i{}'\Big] - \tfrac{1}{\gamma-1}\rho_i{}^\gamma S_i{}',\\
\tfrac{R^2}{\mu_0}\nabla\cdot\Big(\tfrac{1}{R^2}\nabla\psi\Big) \;&=\; Rj_\varphi \equiv -\tfrac{e}{m_e}\,\rho_e\big(L_e - \tfrac{e}{m_e}\psi\big)^2 + \tfrac{Ze}{m_i}\,\rho_i\big(L_i + \tfrac{Ze}{m_i}\psi\big)^2,\\
\epsilon_0\nabla^2\widetilde{\phi} \;&=\; -\,\tau \equiv \tfrac{e}{m_e}\rho_e - \tfrac{Ze}{m_i}\rho_i\,,\\
\tfrac{1}{4\pi G}\,\nabla^2\widetilde{\mathcal{V}} \;&=\; \rho \;\equiv \rho_e + \rho_i\,,
\end{aligned} \tag{7}
$$

and two algebraic Bernoulli equations for $\rho_{\rm e,i}$:

$$
\begin{aligned}
&\tfrac{1}{2R^2}|\nabla\chi_e|^2 - \rho_e^2\,F_e + \tfrac{\gamma}{\gamma-1}\,\rho_e{}^{\gamma+1}S_e = 0 \quad \Rightarrow\; \rho_e\,,\\
&\tfrac{1}{2R^2}|\nabla\chi_i|^2 - \rho_i^2\,F_i + \tfrac{\gamma}{\gamma-1}\,\rho_i{}^{\gamma+1}S_i = 0 \quad \Rightarrow\; \rho_i\,.
\end{aligned} \tag{8}
$$

Substitution yields the two-fluid version of the transonic enigma.

Hyperbolicity

Substituting the solutions $\rho_\alpha(|\nabla\chi_\alpha|,\ldots)$ of the Bernoulli equations (8) back into the PDEs (7) a,b for χ_α yields second order derivatives that produce *hyperbolicity when:*

$$\Delta_e \equiv \frac{1}{M_e^2 - 1} \geq 0\,, \quad \text{or} \;\; M_e^2 \equiv \frac{u_{ep}^2}{\gamma p_e/\rho_e} \geq 1 \quad \text{for electrons}\,,$$
$$\Delta_i \equiv \frac{1}{M_i^2 - 1} \geq 0\,, \quad \text{or} \;\; M_i^2 \equiv \frac{u_{ip}^2}{\gamma p_i/\rho_i} \geq 1 \quad \text{for ions}\,. \tag{9}$$

Hence, unmagnetized electron and ion flows, characterized by ordinary (not poloidal!) Mach numbers $M_{e,i}$, determine hyperbolicity in the two-fluid picture. This hyperbolicity is quite different from that in MHD because it originates from the *asymptotic freedom of the electron and ion flows from the magnetic surfaces.* Since this occurs on skin-depth length-scales, computational demands on resolution are enormous.

To establish the two-fluid counterpart of MHD spectroscopy of astrophysical objects, accurate numerical methods to compute the stationary equilibria should be developed. Even if this entirely non-trivial part of the problem is taken for granted, still the incredible logistic problem remains of handling the two Bernoulli equations. First, one has to specify the six arbitrary functions H_α, L_α, S_α of χ_α. The only meaningful approach appears to be the exploitation of *spatially resolved astronomical observations.* Obviously, this part of the problem still lies in a remote future. Next, one has to solve the Bernoulli equations on each grid point and for each step of the nonlinear iteration. This may result in the four following possibilities:

(1) no solution is found with the specified input $\Rightarrow$ one has to restart the iteration with a different choice of the input functions;
(2) one solution is found $\Rightarrow$ one may proceed;
(3) two or more solutions are found $\Rightarrow$ one should select one by some physical identification *and keep it* for all grid points and iterations;
(4) the solution may have entered a hyperbolic regime $\Rightarrow$ an entirely different solver is needed.

One could summarize this by saying that the computation of two-fluid stationary states involves the transonic and Bernoulli nightmares!

4. Conclusions

(a) The computation of transonic equilibria of accretion disks about compact objects is a very complex problem because hyperbolic flow regimes upset the standard equilibrium solvers. Nevertheless, a large class of highly localized MHD instabilities was found that become explosive for large central mass (Goedbloed et al., 2004) and, hence, constitute very viable candidates to produce the anomalous dissipation required for transport of angular momentum in those systems.

(b) In order to generalize the methods of MHD spectroscopy to two-fluid plasmas, a variational principle for the computation of two-fluid stationary states was constructed that exhibits all the intricacies (including hyperbolicity) due to the electron and ion flow excursions from the magnetic flux surfaces (Goedbloed, 2004).

(c) The computation of transonic two-fluid stationary states should only be pursued for its own sake, to describe details of electron and ion motions, not because it would alleviate hyperbolicity problems in MHD [see the reply by Goedbloed (2005) to a comment of McClements and Thyagaraja (2005)].

Acknowledgments This work was performed as part of the research program of the Euratom-FOM Association Agreement, with support from the Netherlands Science Organization (NWO).

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