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Dynamics of Rapidly Rotating Laboratory and Astrophysical Plasmas

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Dynamics of rapidly rotating laboratory and astrophysical plasmas

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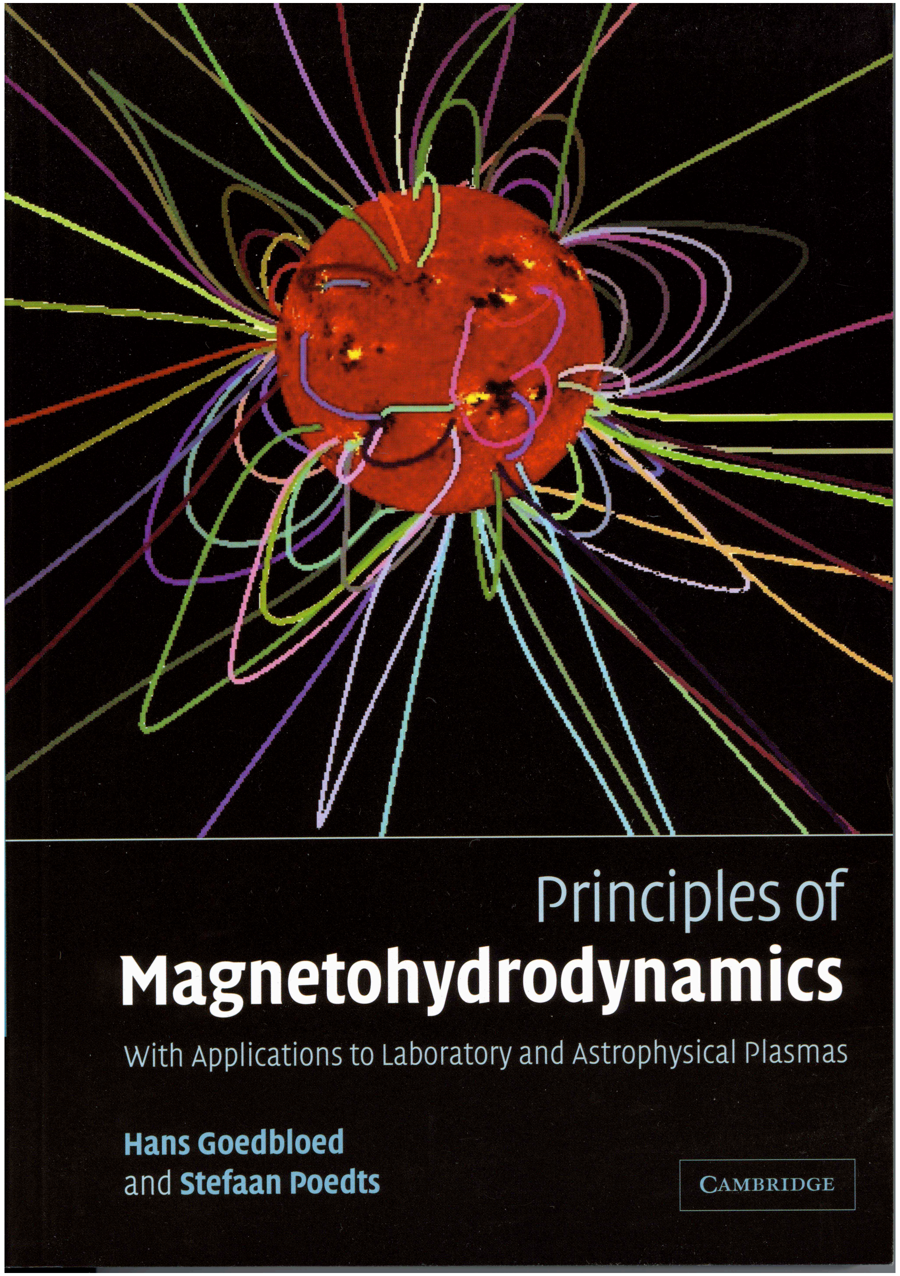


Presents basic principles:

Magnetohydrodynamics
– scale independence

Two-fluid plasmas
– also small-scale dynamics

1. Introduction

- **'Principles of Magnetohydrodynamics': Scale independence of MHD.**
- Example: **Transonic instabilities in accretion disks.** ⇐ Transonic enigma!

2. Waves in two-fluid plasmas

- Small-scale dynamics: **Breaking of scale independence.**
- Dispersion diagram: **Asymptotics.**

3. Stationary states of two-fluid plasmas

- Axisymmetric equilibria: **Variational principle.**
- Transonic enigma returns: **Hyperbolicity.**

4. Conclusions

Ideal MHD equations: Scale independence

- Gas dynamics + pre-Maxwell (neglecting displacement current & electrostatic terms)

  $\Rightarrow$ electric field determined from $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ *($\equiv$ perfect conductivity).*

- Eliminating $\mathbf{E}$ and $\mathbf{j}$ yields the **basic equations of ideal MHD:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \qquad \textit{(conservation of mass)} \qquad (1)$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \qquad \textit{(momentum)} \qquad (2)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,, \qquad \textit{(entropy)} \qquad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,. \qquad \textit{(magnetic flux)} \qquad (4)$$

- **Plasma size (l_0), magnetic field magnitude (B_0), and time scale (t_0) scale out !**

  Time scale due to density (ρ_0) in *Alfvén speed* $v_A \equiv B_0/\sqrt{\mu_0 \rho_0} \;\Rightarrow\; t_0 \equiv l_0/v_A$.

Scales of actual plasmas

	l_0 (m)	B_0 (T)	t_0 (s)
tokamak	20	3	3×10^{-6}
magnetosphere earth	4×10^7	3×10^{-5}	6
solar coronal loop	10^8	3×10^{-2}	15
magnetosphere neutron star	10^6	10^8 *)	10^{-2}
accretion disk YSO	1.5×10^9	10^{-4}	7×10^5
accretion disk AGN	4×10^{18}	10^{-4}	2×10^{12}
galactic plasma	10^{21}	10^{-8}	10^{15}
	$(= 10^5$ ly$)$		$(= 3 \times 10^7$ y$)$

*) Some recently discovered pulsars, called magnetars, have record magnetic fields of 10^{11} T : the plasma universe is ever expanding!

Angle:

Unify laboratory and astrophysical pictures of MHD waves and instabilities

(exploiting scale-independence MHD equations) $\Rightarrow$ **MHD Spectroscopy**

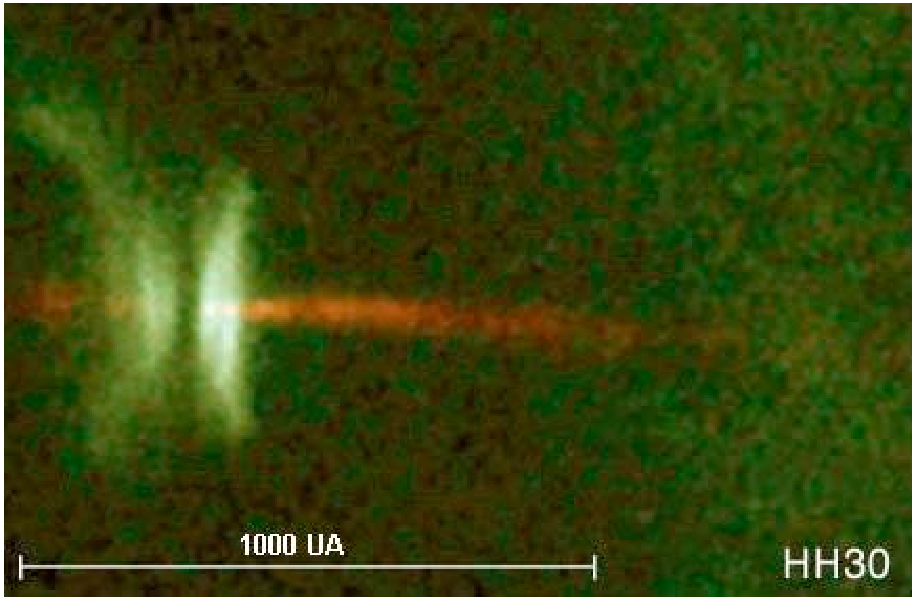
'Historical' development:

- MHD spectral theory, with large-scale numerical computations, since 1970s.
- Laboratory plasmas: **MHD spectroscopy for tokamaks.**
  [Goedbloed, Huysmans, Holties, Kerner, Poedts, PPCF **35**, B277 (1993)]
- Astrophysical plasmas: **Magnetoseismology of accretion disks.**
  [Keppens, Casse, Goedbloed, ApJ **579**, L121 (2002)]
- Accretion-ejection requires **anomalous dissipation** $\equiv$ **small-scale instabilities.**
  [Goedbloed, Beliën, van der Holst, Keppens, PoP **11**, 28 (2004)]

$\Rightarrow$ MHD spectral theory for Transonic Flows (2D)

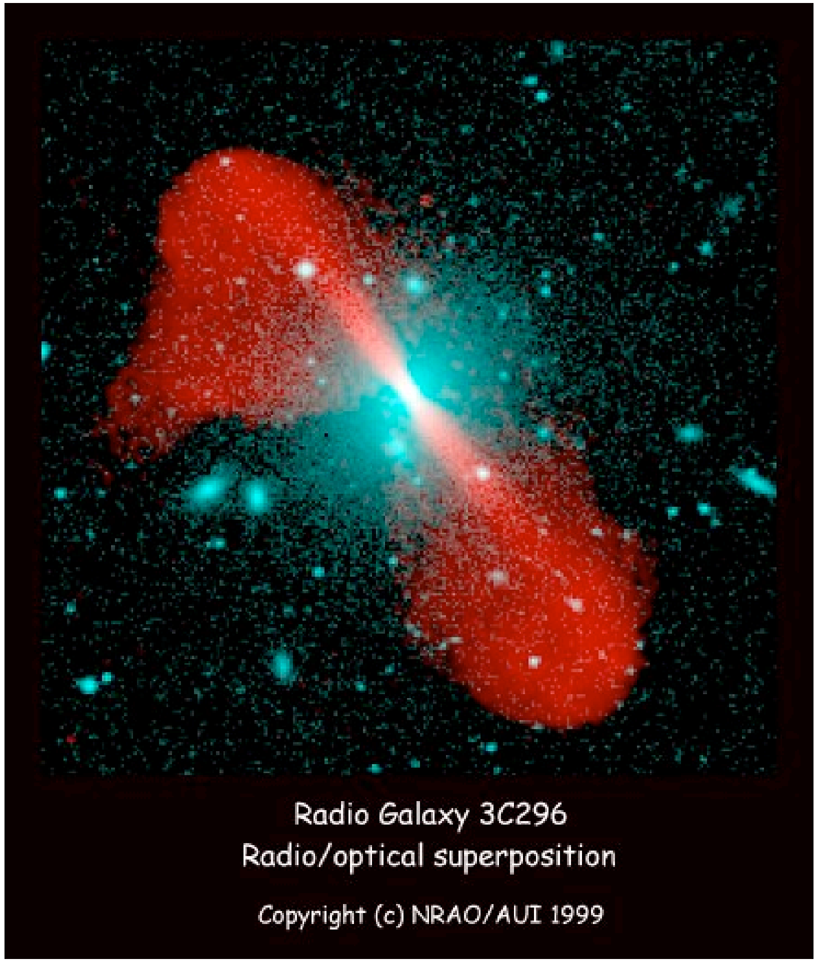
Accretion disk and jets (YSO & AGN)

Young stellar object ($M_* \sim 1 M_\odot$):



disk: dark strip, jets: red.

Active galactic nucleus ($M_* \sim 10^8 M_\odot$):



disk: blue (optical), jets: red (radio).

Magnetized Accretion-Ejection Structure

- Stationary state from simulation with VAC (Versatile Advection Code, Tóth 1996): *accretion-ejection particle trajectory (red).*

  [Casse & Keppens, Ap. J. **601**, 90 (2004)]

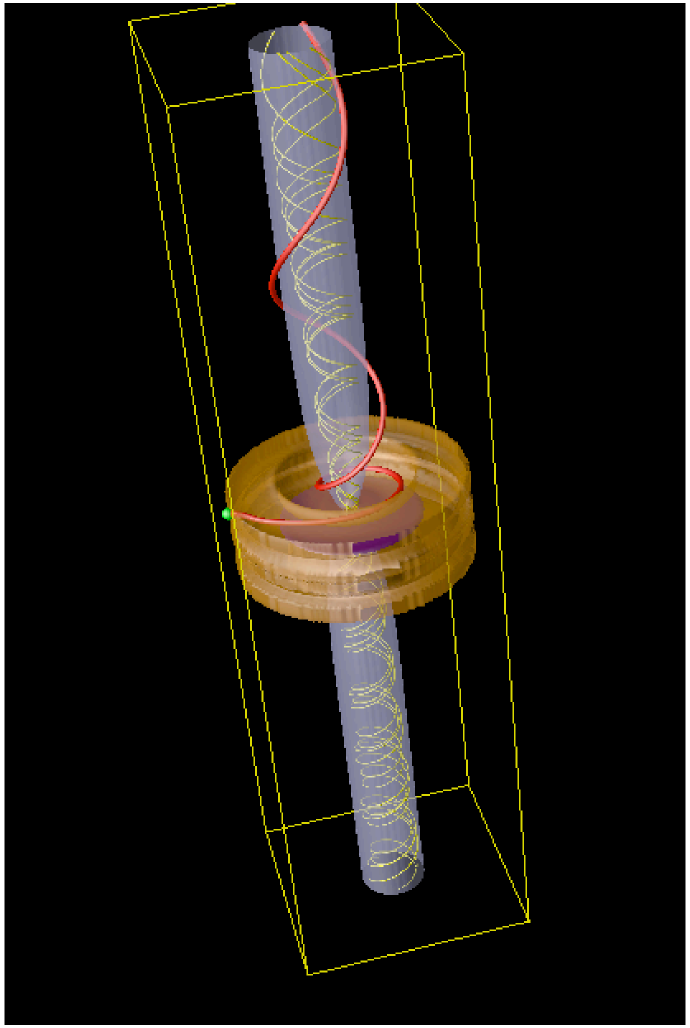
- Requires anomalous resistivity due to **local instabilities:**

  – usual **Magneto-Rotational Instability,** only works for $\beta \;(\equiv 2p/B^2) \gg 1$;

  [Velikhov (1959), Chandra (1960), B & H (1991)]

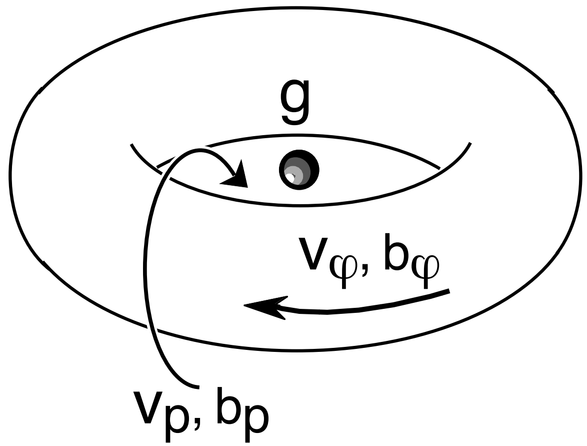
  – new **Trans-slow continuum instability,** works for arbitrary β (including $\ll 1$).

  [Goedbloed, Beliën, van der Holst, Keppens (2004)]



Model

- **Transonically rotating magnetized 'thick' disk about compact object.**



- Accretion speed $\ll$ rotation speeds of the disk
  $\Rightarrow$ **Flow on magnetic surfaces in MHD!**

- Investigated (with FINESSE & PHOENIX codes):
  Stationary 2D equilibrium $+$ local instabilities.

- Axisymmetric MHD equilibria $\Rightarrow$ ρ, p, $\mathbf{v}_p$, v_φ, $\mathbf{B}_p$, B_φ from **two fields:**
  poloidal flux ψ and *poloidal Alfvén Mach number* $M^2 \equiv \rho v_p^2 / B_p^2 \equiv \chi'^2/\rho$.

  – 'Grad-Shafranov' type PDE for ψ:
  $$R^2 \nabla \cdot \left(\frac{M^2 - 1}{R^2} \nabla \psi \right) = \cdots \,, \qquad (5)$$

  – Bernoulli eq. (algebraic) for M^2 $\Rightarrow$ $M^2(\nabla\psi, \cdots)$, inserted into Eq. (5) yields

$$\Delta \sim \frac{(M^2-1)^2(M^2-M_c^2)}{(M^2-M_s^2)(M^2-M_f^2)} \begin{cases} < 0: & \text{elliptic} \\ \geq 0: & \textbf{hyperbolic!} \end{cases} .$$

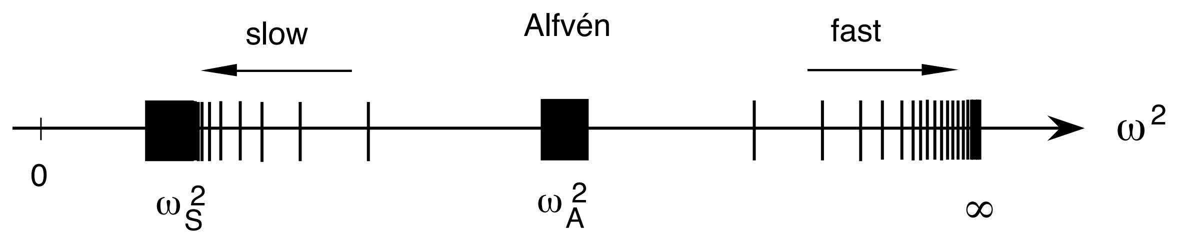
Transonic enigma in MHD

- **At transonic transitions, flows change character from elliptic to hyperbolic.**

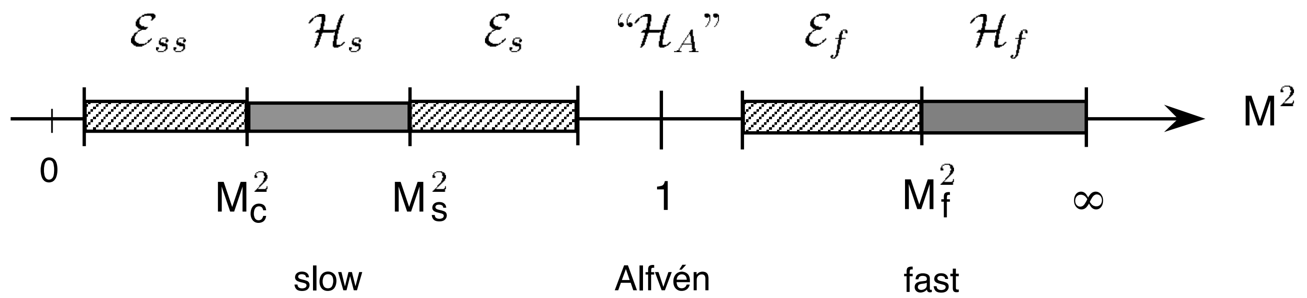
  – Equilibrium solvers diverge in *hyperbolic* regimes $\Rightarrow$ stay in *elliptic* regimes.

- **Linear waves & nonlinear stationary states no longer separate problems!**

  – *Wave spectra* cluster at *continuous spectra* $\{\omega_S^2\}$, $\{\omega_A^2\}$, ∞ $(\equiv \omega_F^2)$:



  – *Hyperbolic flow regimes* delimited by *critical poloidal Alfvén Mach numbers:*

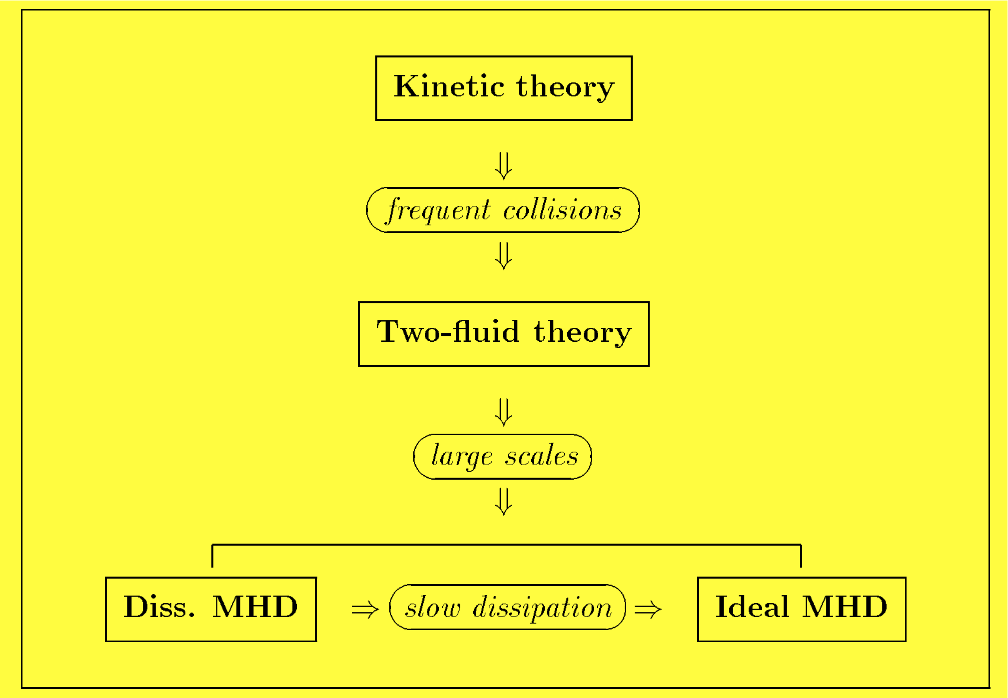


[Goedbloed & Lifschitz, PoP **4**, 3544 (1997)]

- McClements & Thyagaraja: [MNRAS **323**, 733 (2001)] **"Two-fluid model free of singularities . . . and elliptic!" ?**

Two-fluid model

- Intermediate between kinetic theory and MHD:



- Singularity/hyperbolicity problem involves

  – **Asymptotics of the waves** (small scales),

  – **2D stationary states** (deviations of flow and magnetic surfaces).

Dispersion equation

- Derivation needs *“in small proportion, insight, and in large proportion, stamina”* (Stix).
- Abbreviate: $e \equiv \omega_{pe}/\omega_p\,,\quad E \equiv \Omega_e/\omega_p\,,\quad v \equiv v_e/c\,,$
  $i \equiv \omega_{pi}/\omega_p\,,\quad I \equiv \Omega_i/\omega_p\,,\quad w \equiv v_i/c\,,$ and keep going!

$\Rightarrow$ **Polynomial of twelfth degree in $\bar\omega \equiv \omega/\omega_p$ and of eighth degree in $\bar k \equiv k\delta$:**

$$
\begin{aligned}
&\alpha_{60}\,\bar\omega^{12} \\
&+ (\alpha_{50} + \alpha_{51}\bar k^2)\,\bar\omega^{10} \\
&+ (\alpha_{40} + \alpha_{41}\bar k^2 + \alpha_{42}\bar k^4)\,\bar\omega^{8} \\
&+ (\alpha_{30} + \alpha_{31}\bar k^2 + \alpha_{32}\bar k^4 + \alpha_{33}\bar k^6)\,\bar\omega^{6} \\
&+ (\alpha_{21}\bar k^2 + \alpha_{22}\bar k^4 + \alpha_{23}\bar k^6 + \alpha_{24}\bar k^8)\,\bar\omega^{4} \\
&+ (\alpha_{12}\bar k^4 + \alpha_{13}\bar k^6 + \alpha_{14}\bar k^8)\,\bar\omega^{2} \\
&+ \alpha_{03}\bar k^6 + \alpha_{04}\bar k^8 \quad = 0\,.
\end{aligned}
$$

cold

(1) cutoff

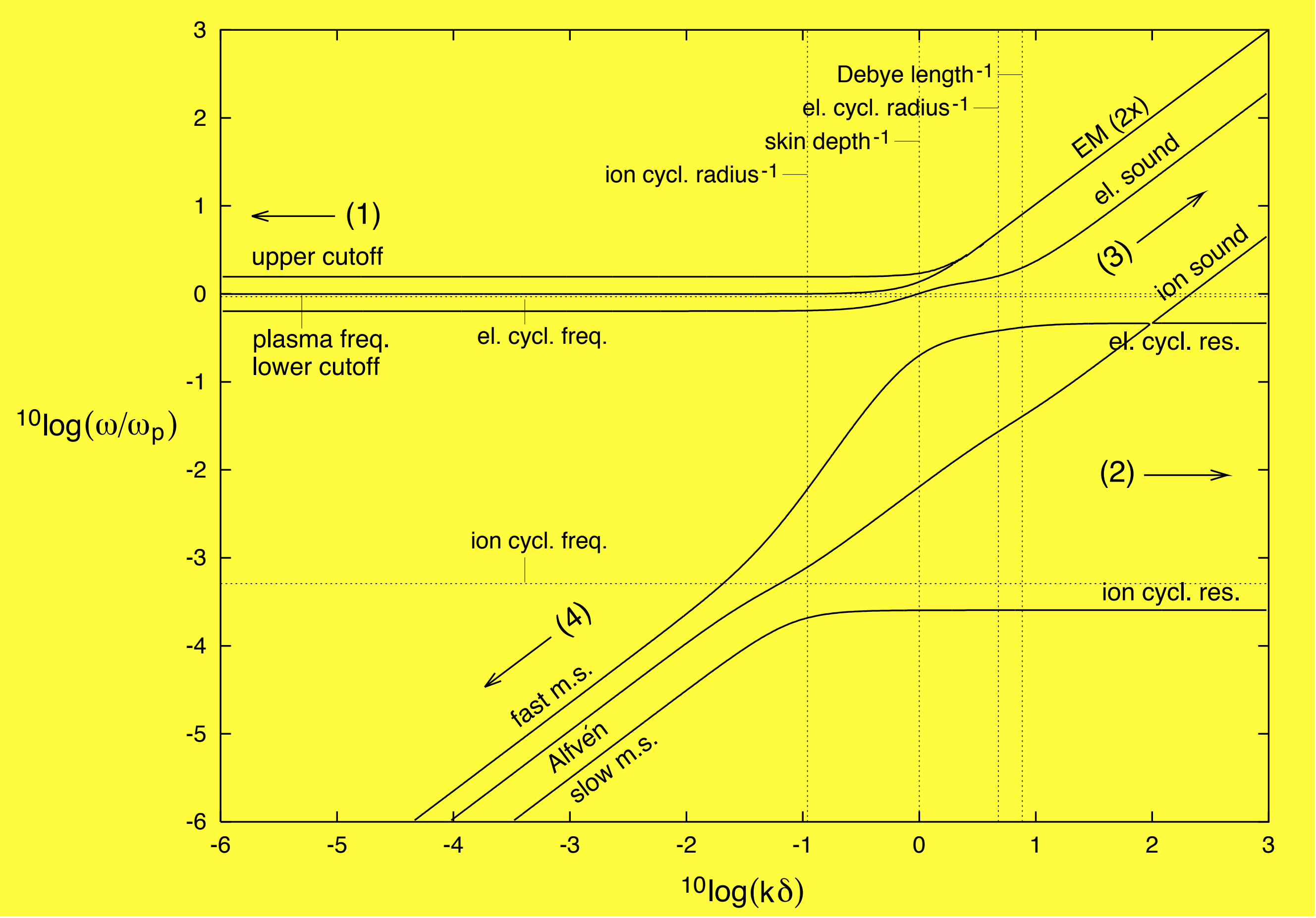
(2) resonance

(3) local, HF

(4) global, MHD

Dispersion diagram

- **Asymptotic limits:** (1) $k \to 0$ *(cutoff)*, (2) $k \to \infty$ *(resonance)*,
  (3) $\omega \,\&\, k \to \infty$, ω/k finite *(local HF)*: 2 EM + 2 sound waves,
  (4) $\omega \,\&\, k \to 0$, ω/k finite *(global LF)*: three MHD waves.



Two-fluid asympt. (3):

⇒ 2 char. speeds $< c$ that may be surpassed by flow speeds u_e or u_i.

MHD asympt. (4):

⇒ 3 char. speeds that may be surpassed by plasma flow speed.

Two-fluid variables in axi-symmetry

(seven fields)

- Flow speeds $\mathbf{u}_e$ and $\mathbf{u}_i$ from *poloidal stream functions χ_α :*

$$\mathbf{u}_\alpha = (\rho_\alpha R)^{-1}\mathbf{e}_\varphi \times \nabla\chi_\alpha + u_{\alpha\varphi}\mathbf{e}_\varphi \quad (\alpha = e, i)\,, \tag{6}$$

magnetic field $\mathbf{B}$ from *poloidal flux function ψ :*

$$\mathbf{B} = R^{-1}\mathbf{e}_\varphi \times \nabla\psi + B_\varphi\mathbf{e}_\varphi\,, \qquad B_\varphi = R^{-1}I(\chi_e, \chi_i)\,, \tag{7}$$

electric field $\mathbf{E}$ from *electric potential ϕ :*

$$\mathbf{E} = \nabla\phi\,, \qquad \phi = \phi_* + \widetilde{\phi}\,, \quad \phi_* \equiv Q_*/(4\pi\epsilon_0\sqrt{R^2+Z^2})\,, \tag{8}$$

gravitational field $\mathbf{g}$ from *gravitational potential $\mathcal{V}$:*

$$\mathbf{g} = \nabla\mathcal{V}\,, \qquad \mathcal{V} = \mathcal{V}_* + \widetilde{\mathcal{V}}\,, \quad \mathcal{V}_* \equiv -GM_*/\sqrt{R^2+Z^2}\,. \tag{9}$$

- **Six (!) arbitrary functions of χ_e and χ_i :**

– *Bernoulli functions:* $H_\alpha(\chi_\alpha) \equiv \frac{1}{2}u_\alpha^2 + \dfrac{q_\alpha}{m_\alpha}\phi + \mathcal{V} + \dfrac{\gamma}{\gamma-1}\dfrac{p_\alpha}{\rho_\alpha}$ $\Rightarrow$ *densities ρ_α ,*

– *spec. entropies:* $S_\alpha(\chi_\alpha) \equiv \rho_\alpha^{-\gamma}p_\alpha$ $\Rightarrow$ *pressures p_α ,*

– *spec. ang. mom.:* $L_\alpha(\chi_\alpha) \equiv Ru_{\alpha\varphi} - \dfrac{q_\alpha}{m_\alpha}\psi$ $\Rightarrow$ *toroidal speeds $u_{\alpha\varphi}$.* (10)

Variational principle

- Stationary equilibria from the **two-fluid Lagrangian:** [Goedbloed, PoP **11**, L81 (2004)]

$$\delta \int \mathcal{L}_{\rm TF}(\chi_\alpha, \nabla\chi_\alpha, \rho_\alpha, \psi, \nabla\psi, \widetilde{\phi}, \nabla\widetilde{\phi}, \widetilde{\mathcal{V}}, \nabla\widetilde{\mathcal{V}}; R, Z)\, dV = 0 ,$$

$$\mathcal{L}_{\rm TF} \equiv \frac{1}{2\rho_e R^2}\,|\nabla\chi_e|^2 + \frac{1}{2\rho_i R^2}\,|\nabla\chi_i|^2 - \frac{1}{2\mu_0 R^2}\,|\nabla\psi|^2 + \tfrac{1}{2}\epsilon_0\,|\nabla\widetilde{\phi}|^2 - \frac{1}{8\pi G}\,|\nabla\widetilde{\mathcal{V}}|^2$$
$$+ \frac{1}{2\mu_0 R^2}\, I^2 + \rho_e\, F_e - \frac{1}{\gamma-1}\rho_e{}^\gamma S_e + \rho_i\, F_i - \frac{1}{\gamma-1}\rho_i{}^\gamma S_i\,. \qquad (11)$$

- Nonlinear terms depend on composite functions,

$$I(\chi_e,\chi_i) \;\equiv\; RB_\varphi = I_0 + \mu_0[(e/m_e)\chi_e - (Ze/m_i)\chi_i]\,,$$

$$F_e(\chi_e,\psi,\widetilde{\phi},\widetilde{\mathcal{V}};R,Z) \;\equiv\; H_e - \frac{1}{2R^2}(L_e - \frac{e}{m_e}\psi)^2 + \frac{e}{m_e}\,(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,,$$

$$F_i(\chi_i,\psi,\widetilde{\phi},\widetilde{\mathcal{V}};R,Z) \;\equiv\; H_i - \frac{1}{2R^2}(L_i + \frac{Ze}{m_i}\psi)^2 - \frac{Ze}{m_i}\,(\phi_* + \widetilde{\phi}) - \mathcal{V}_* - \widetilde{\mathcal{V}}\,,$$

of the stream functions $H_e, L_e, S_e(\chi_e)$, $H_i, L_i, S_i(\chi_i)$, and the potentials ψ, ϕ, $\mathcal{V}$.

Euler-Lagrange equations

- PDEs for $\chi_{\rm e,i}\,,\psi\,,\widetilde{\phi}\,,\widetilde{\mathcal{V}}$:

$$\nabla\cdot\Big(\frac{1}{\rho_e R^2}\nabla\chi_e\Big) = \frac{e}{m_e R^2}I + \rho_e\Big[H_e' - \frac{1}{R^2}\Big(L_e - \frac{e}{m_e}\psi\Big)L_e'\Big] - \frac{\rho_e{}^\gamma}{\gamma-1}S_e'\,, \quad (12)$$

$$\nabla\cdot\Big(\frac{1}{\rho_i R^2}\nabla\chi_i\Big) = -\frac{Ze}{m_i R^2}I + \rho_i\Big[H_i' - \frac{1}{R^2}\Big(L_i + \frac{Ze}{m_i}\psi\Big)L_i'\Big] - \frac{\rho_i{}^\gamma}{\gamma-1}S_i'\,, \quad (13)$$

$$\frac{R^2}{\mu_0}\nabla\cdot\Big(\frac{1}{R^2}\nabla\psi\Big) = Rj_\varphi \equiv -\frac{e}{m_e}\rho_e\Big(L_e - \frac{e}{m_e}\psi\Big)^2 + \frac{Ze}{m_i}\rho_i\Big(L_i + \frac{Ze}{m_i}\psi\Big)^2\,, \quad (14)$$

$$\epsilon_0\nabla^2\widetilde{\phi} = -\,\tau \equiv \frac{e}{m_e}\rho_e - \frac{Ze}{m_i}\rho_i\,, \quad (15)$$

$$\frac{1}{4\pi G}\nabla^2\widetilde{\mathcal{V}} = \rho \equiv \rho_e + \rho_i\,. \quad (16)$$

- Bernoulli equations for $\rho_{\rm e,i}$:

$$\frac{1}{2R^2}|\nabla\chi_e|^2 - \rho_e^2\,F_e + \frac{\gamma}{\gamma-1}\rho_e{}^{\gamma+1}S_e = 0 \quad \Rightarrow \quad \rho_e \;\; (\text{insert } \Uparrow)\,, \quad (17)$$

$$\frac{1}{2R^2}|\nabla\chi_i|^2 - \rho_i^2\,F_i + \frac{\gamma}{\gamma-1}\rho_i{}^{\gamma+1}S_i = 0 \quad \Rightarrow \quad \rho_i \;\; (\text{insert } \Uparrow)\,. \quad (18)$$

Hyperbolicity

- Substituting solutions $\rho_\alpha(|\nabla\chi_\alpha|, \ldots)$ of Bernoulli equations back into PDEs for χ_α yields second order derivatives that produce **hyperbolicity when:**

$$\Delta_e \equiv \frac{1}{M_e^2 - 1} \ge 0\,, \quad \text{or} \quad M_e^2 \equiv \frac{u_{ep}^2}{\gamma p_e/\rho_e} \ge 1 \quad \text{for electrons}\,, \tag{19}$$

$$\Delta_i \equiv \frac{1}{M_i^2 - 1} \ge 0\,, \quad \text{or} \quad M_i^2 \equiv \frac{u_{ip}^2}{\gamma p_i/\rho_i} \ge 1 \quad \text{for ions}\,. \tag{20}$$

$\Rightarrow$ **Unmagnetized fluids (ordinary Mach numbers) determine hyperbolicity.**

- *For specified input H_α, L_α, S_α (to be obtained from spatially resolved astronomical observations!), Bernoulli equations may have:*
  - *no solutions* $\Rightarrow$ restart with different choice of input,
  - *one solution* $\Rightarrow$ OK (proceed),
  - ≥ 1 *solution* $\Rightarrow$ choose one *and keep* for all grid points and iteration steps,
  - *entered a hyperbolic regime* $\Rightarrow$ need entirely different solver.

$\Rightarrow$ **Two-fluid stationary states involve Bernoulli and transonic nightmares.**

- **Transonic equilibria of accretion disks about compact objects exhibit highly localized MHD instabilities that become explosive for large central mass.**

  [Goedbloed, Beliën, van der Holst, Keppens, PoP **11**, 28 (2004)]

- **A variational principle for two-fluid stationary states is obtained exhibiting all intricacies of electron and ion flow excursions from the magnetic flux surfaces.**

  [Goedbloed, PoP **11**, L81 (2004)]

- **Computation of transonic two-fluid stationary states only to be pursued for its own sake, not because it would alleviate hyperbolicity problems in MHD.**

  [McClements & Thyagaraja comment, Goedbloed replies: PoP **12**, June (2005).]