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Wave-particle Interactions in Collisionless Plasmas How Good is the Quasilinear Theory ?

J. BUECHNER

Max-Planck-Institut for Solar System Research
Lindau, Germany

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Wave-particle interactions in collisionless plasmas - how good is the quasilinear theory?

J. Büchner

Max-Planck Institut für Sonnensystemforschung Katlenburg-Lindau, Germany

Acknowledgment of collaboration:

I.Silin (Edmonton), N.Elkina (Mocow)

Outline

- Which plasmas are collisionless ?
- Applicability of the Vlasov equation
- Quasilinear theory -> numerical tests
- Numerical Vlasov solvers from
  - Semi-Lagrange time-splitting methods to
  - Eulerian grid unsplit finite volume
- Example 1 (1D1V): Bunemann instability
- Examples 2-4 (1D1V-2d2V): Ion-acoustic (IA) "effective collisions" for different cases
- Example 5 (2D3V) : current sheets and LHD

Which plasma is collisionless?

- Discrete particles: Mean free path between two particle collisions ->
- Collision frequeny ->

  ... has to be compared with the
- Continuous fluid plasma eigenfrequency ->
- the ratio of the two should vanish -> collisionless description ->

(where $\lambda_D = \omega_{pe}^{-1}\sqrt{\frac{\kappa T_e}{m_e}}$)

$$l \approx n_o^{-1}\left(\frac{\epsilon_o \kappa T_e}{e^2}\right)^2$$

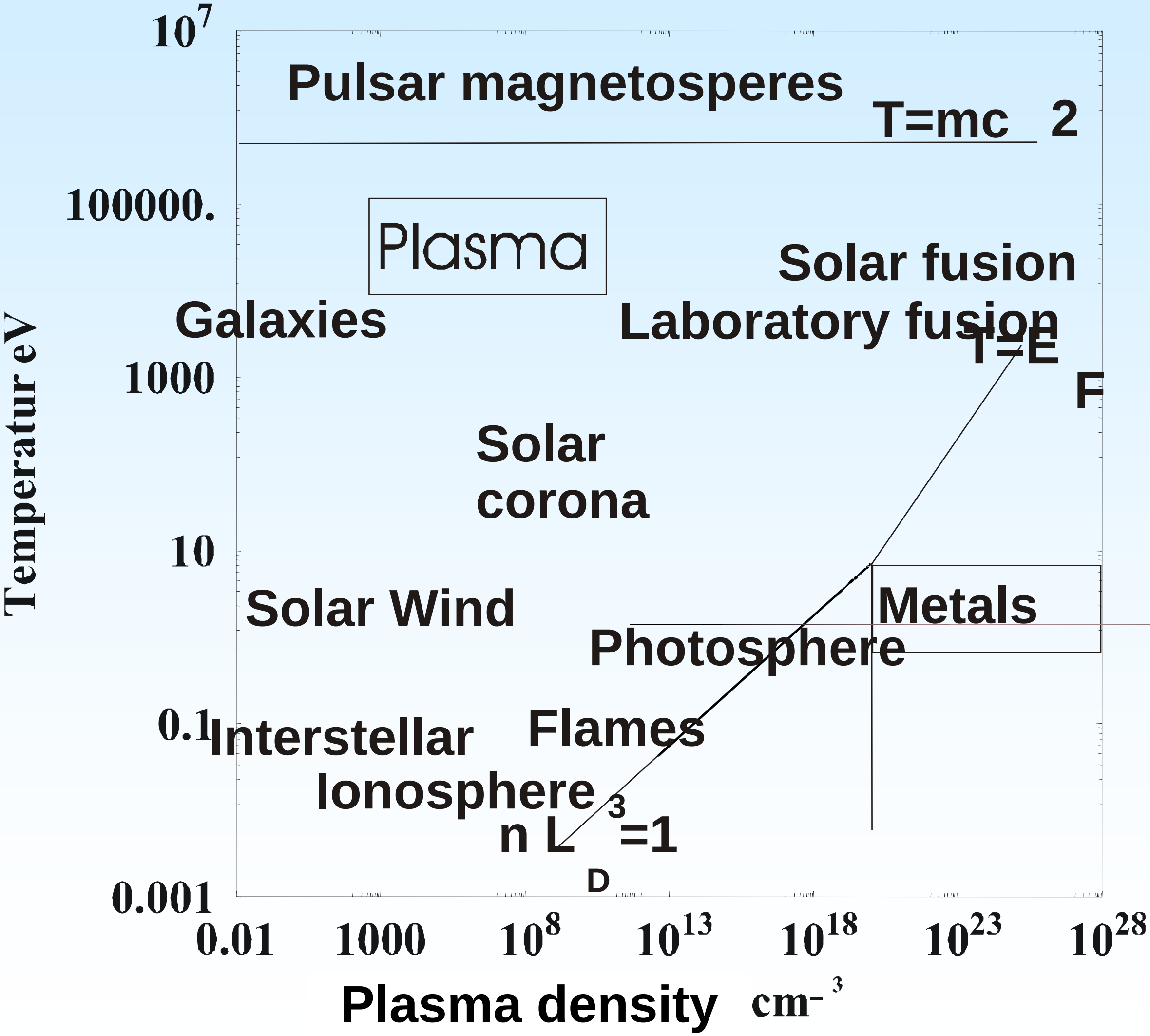
$$\nu \approx l^{-1}\sqrt{\frac{\kappa T_e}{m_e}}$$

$$\omega_{pe} = \sqrt{\frac{n_o e^2}{m_e \epsilon_o}}$$

$$\frac{\nu}{\omega_{pe}} \approx \frac{1}{n_o \lambda_D^3} = g \rightarrow 0$$

„graininess factor“

Loss of course-graining:
-> Vlasov description possible, if ...



Fusion plasmas:
g= 10^{-9}-10^{-7}
Astrophysical plasmas:
g= 10^{-19}-10^{-13}
This is very, very small, though finite.
A description of plasmas as a collisionless fluid is possible for times <<

$$\tau \approx (g \cdot \omega_{pe})^{-1}$$

Vlasov‘s idea: gas description

Т. 8 Журнал экспериментальной и теоретической физики Вып. 3

1938

A.A. Vlasov: „About the vibrational properties of an electron gas“
(J. Exp. Theor. Phys., 8, 291-318, 1938)

О ВИБРАЦИОННЫХ СВОЙСТВАХ ЭЛЕКТРОННОГО ГАЗА

А. А. Власов

1. Постановка задачи. — 2. Исходные уравнения и их упрощение. — 3. Решение линеаризованных уравнений. — 4. Дисперсия продольных волн. — 5. Дисперсия продольных волн в электронном газе с функцией распределения по Ферми. — 6. Дисперсия поперечных волн. — 7. Резюме и заключение.

1. Постановка задачи

Во многих проблемах приходится иметь дело с большой совокупностью

Vlasov's equation

тельно оценка σ, ι, v_{gt} остается ... свойствах допускает
Итак, вопрос о вибрационных
упрощение задачи—можно пренебречь всеми взаимодей-
ствиями посредством „удара“.

Рассматриваемый круг вопросов, связанных с большими частотами, допускает еще одно упрощение в исходных уравнениях — вследствие большой массы ионов в сравнении с электронами можно их перемещением пренебречь, т. е. считать ионы фактически неподвижными. При всех этих условиях система исходных уравнений принимает вид:

$$\frac{\partial f}{\partial t} + \mathrm{div}_r \mathbf{v} f + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{H}]\right)\mathrm{grad}_v f = 0,$$

$$\mathrm{div}\,\mathbf{E} = 4\pi e\left(\int_{-\infty}^{+\infty} f\, d\xi\, d\eta\, d\zeta - N\right); \quad \mathrm{rot}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e}{c}\int_{-\infty}^{+\infty} \mathbf{v} f\, d\xi\, d\eta\, d\zeta,$$

$$\mathrm{div}\,\mathbf{H} = 0, \quad \mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

где f — функция распределения для электронов. Таким образом в рассматри-
ваемой проблеме приходим к системе уравнений, описывающей поведение

... neglect all interactions via „collisions“

... and for high frequency applications neglect ions and describe the electron gas alone ...

Meaning of Vlasov's equation

Advection form, good for non-relativistic plasmas, i.e. velocities are used as independent variables:

$$\frac{\partial f_j}{\partial t} + \vec{v}\frac{\partial f_j}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E} + \vec{v} \times \vec{B}\right)\frac{\partial f_j}{\partial \vec{v}} = 0$$

with E,B being the mean electric and magnetic fields, i.e

Conservative form, mandatory for plasmas, where momenta = dependent variables:

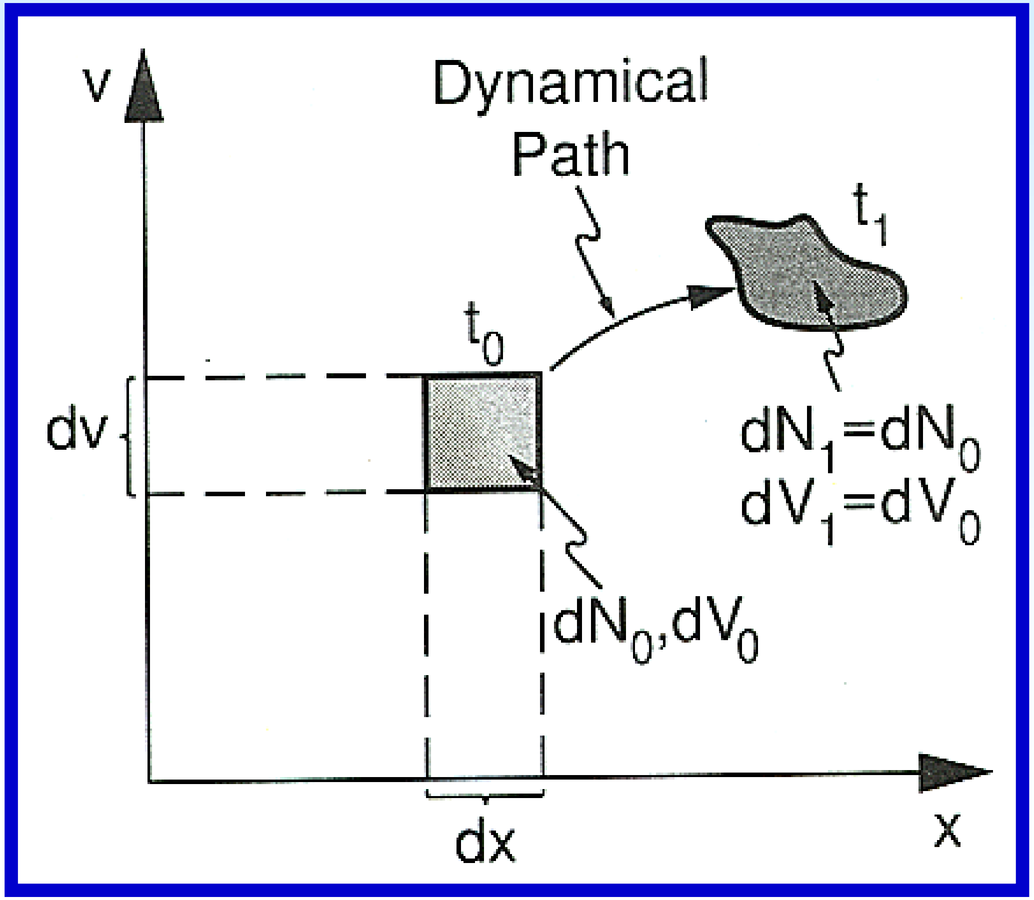
$$\frac{\partial f_j}{\partial t} + \frac{\partial}{\partial \vec{r}}\left(\frac{\vec{p}}{\gamma_j m_j} f_j\right) + e_j\left[\vec{E} + \frac{\vec{p} \times \vec{B}}{\gamma_j m_j}\right]\frac{\partial}{\partial \vec{p}} f_j = 0$$

$$\gamma_j^2 = 1 + \frac{p_x^2 + p_y^2 + p_z^2}{m_j^2 c^2}$$

Vlasov equations are closed via Maxwell's equations -> highly nonlinear!

Properties of Vlasov's equation

Vlasov equation: df/dt =0 means conservation of the phase space density f (Liouville theorem)



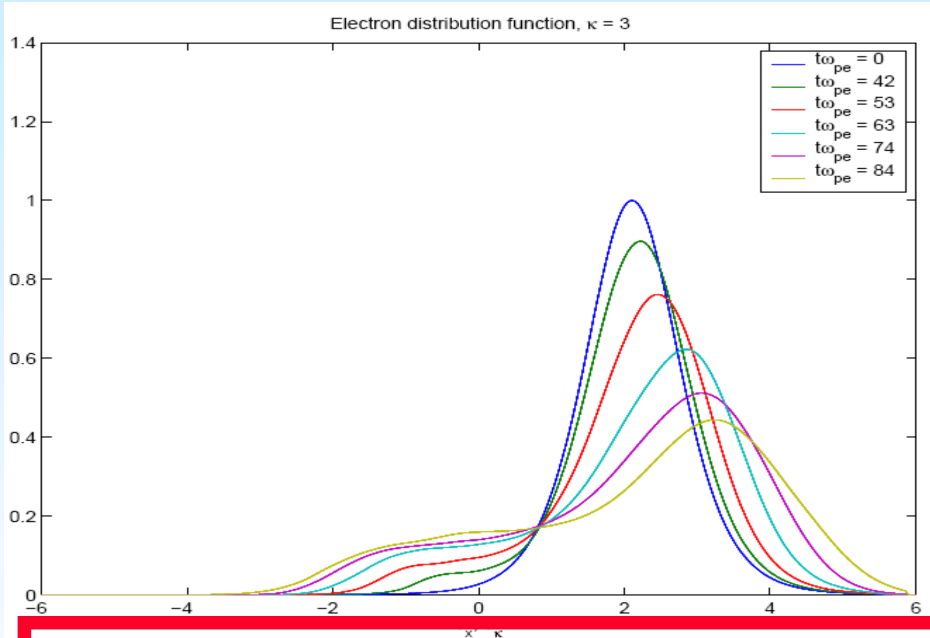
Any volume element becomes deformed under the action of (Lorentz...) forces like in an imcompressible fluid. But its voulme and remains constant like the number of particles contained in it.

Main application of Vlasov's eq.: Description of wave-particle interactions

- Shifted electron distribution ->
- Instability via „inverse Landau damping“ ->
- wave growth, but:

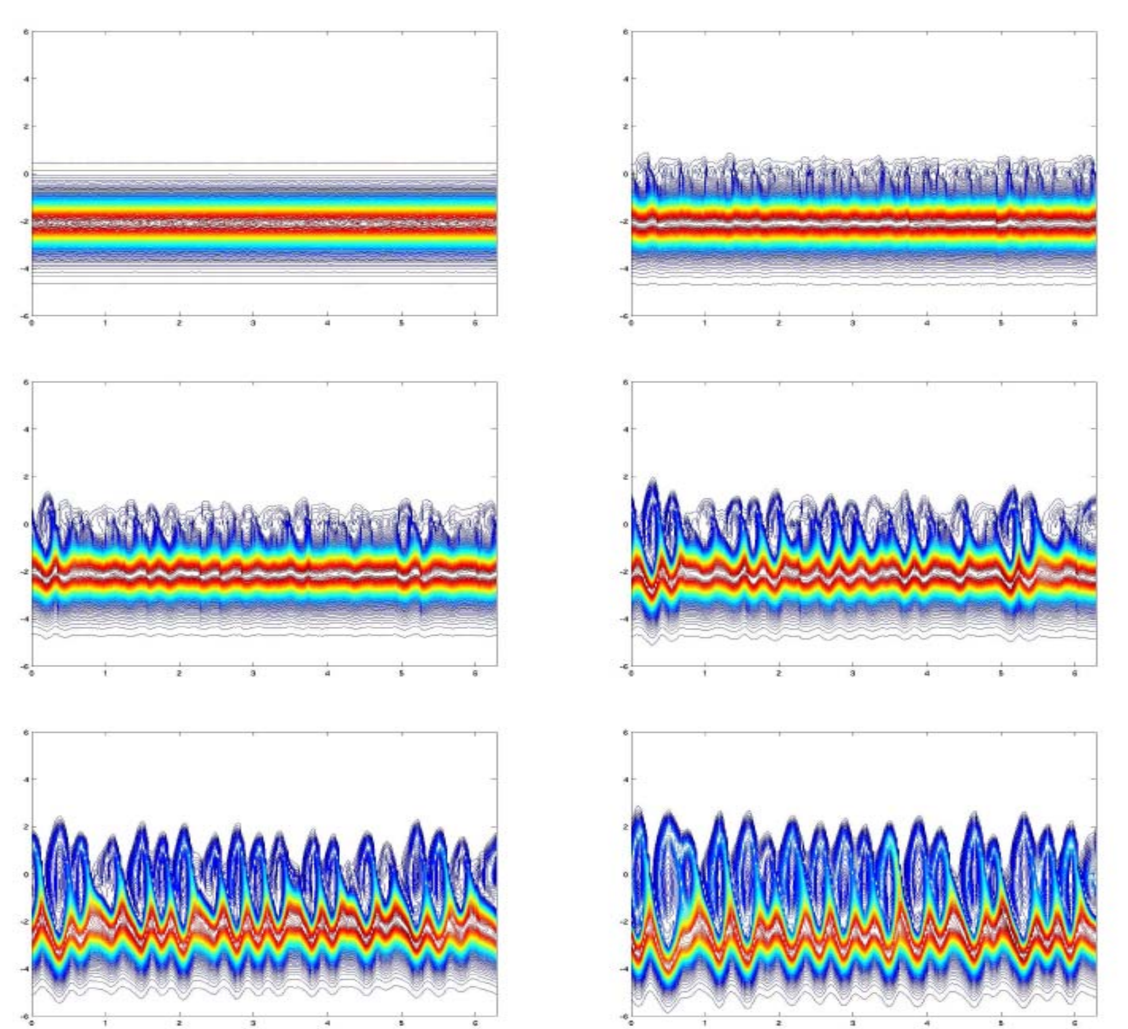
=> Wave saturation amplitude?

- If one neglects the modification of the distribution function:
- 1962: Vedenov, Velikhov, Sagdeev & Drummond, Pines:
- Quasilinear theory, a weak turbulence theory, if not
- large wave amplitudes / coherent structures instead of phase mixing / strongly changed distribution functions



Limits of the neglect of strong nonlinearities and feedbacks on particles can be best investigated by numerical methods!

Vlasov's main numerical challenge: Phase space filamentation



<- **1D Distribution function evolutiondue towave-particle-resonant interaction:**

(Vx vs. X coordinate)

->Challenge for any numerical treatment:

With time the gradient scales become smaller than any realistic mesh scale !

Semi-Lagrangian approach

- A semi-Lagrangian approach uses the Vlasov equations' conservation of the phase space volume in time (Liouville theorem)

  $$df(t+1) = df(t) = const. = df(0)$$

- The new values f(t+1) are then extrapolated to the Eulerian grid, where which the fields are calculated and stored
- This is okay in lower dimensionslike 1D1V, but practially impossible in higher dimensions up to 3D3V or even 3D3P

Time splitting in one dimension

C.Z. Cheng and G. Knorr, 1976:

The Integration of the Vlasov Equation in Configuration Space*†

C. Z. CHENG‡ AND GEORG KNORR

Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242

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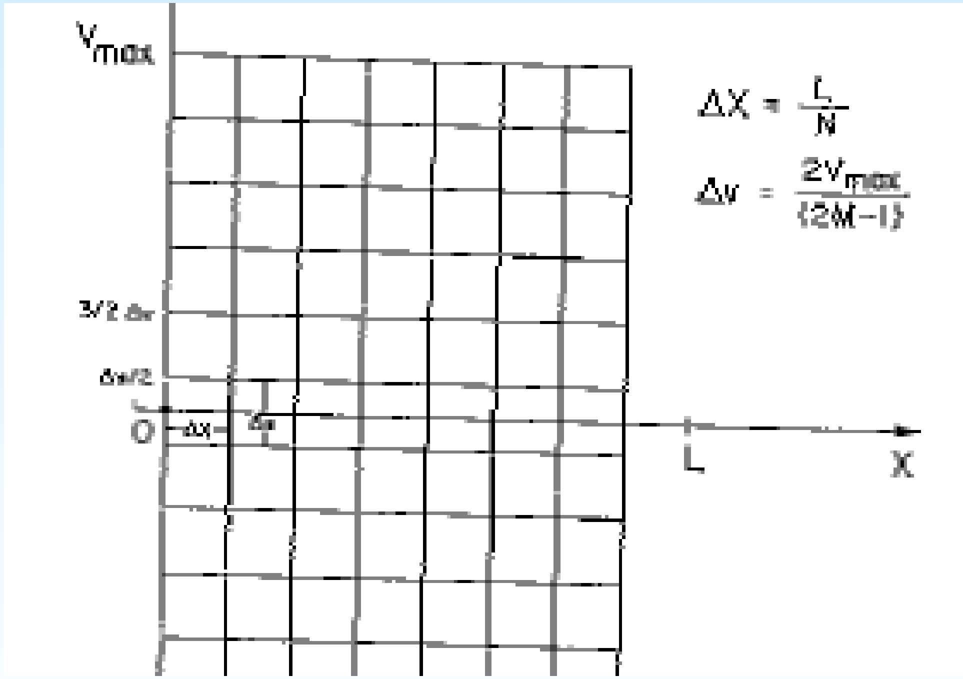


The two main ideas of CK76:

- **A semi-Lagrangian approach, allowing high accuracy**
- **Time splitting of the equation allowing high effectivity**

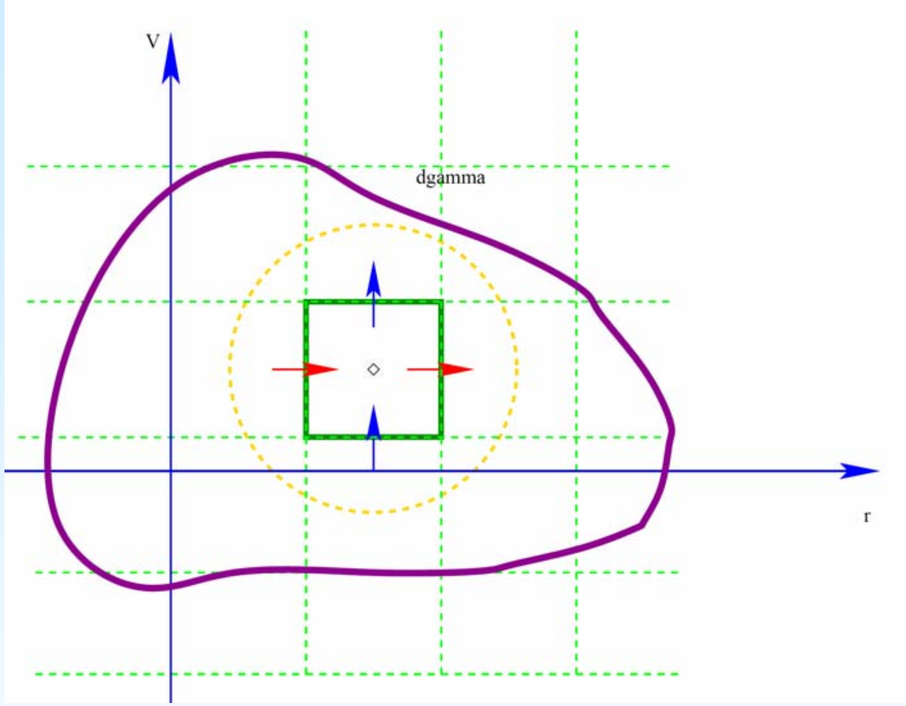
$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0, \qquad \frac{\partial f}{\partial t} + \frac{q}{m} E \cdot \nabla_v f = 0.$$

But: In-accurate due to the necessity to extrapolate, numerical losses, negative distribution functions ...

Finite volume discretization

In the 1D1V case one can write

$$\frac{\partial f}{\partial t} + \frac{\partial \vec{H}}{\partial \vec{R}} = 0 \quad where \quad \vec{H} = \{H^x, H^v\} = \left\{ f\vec{v}, f\frac{\vec{F}}{m} \right\}$$



then for the cell-averaged function

$$f_{i,j} = \frac{1}{|V_{i,j}|} \int_{V_{i,j}} f dV$$

the discrete Vlasov equation becomes

$$\frac{\partial \overline{f}^n_{i,j}}{\partial t} = -\frac{1}{|V_{ij}|} \int_{t_n}^{t_{n+1}} \oint_{\partial V_{ij}} \vec{H}_{i,j} \vec{n}_{i,j} dS dt$$

where the physical fluxes H have to be approximated numerically (by G)

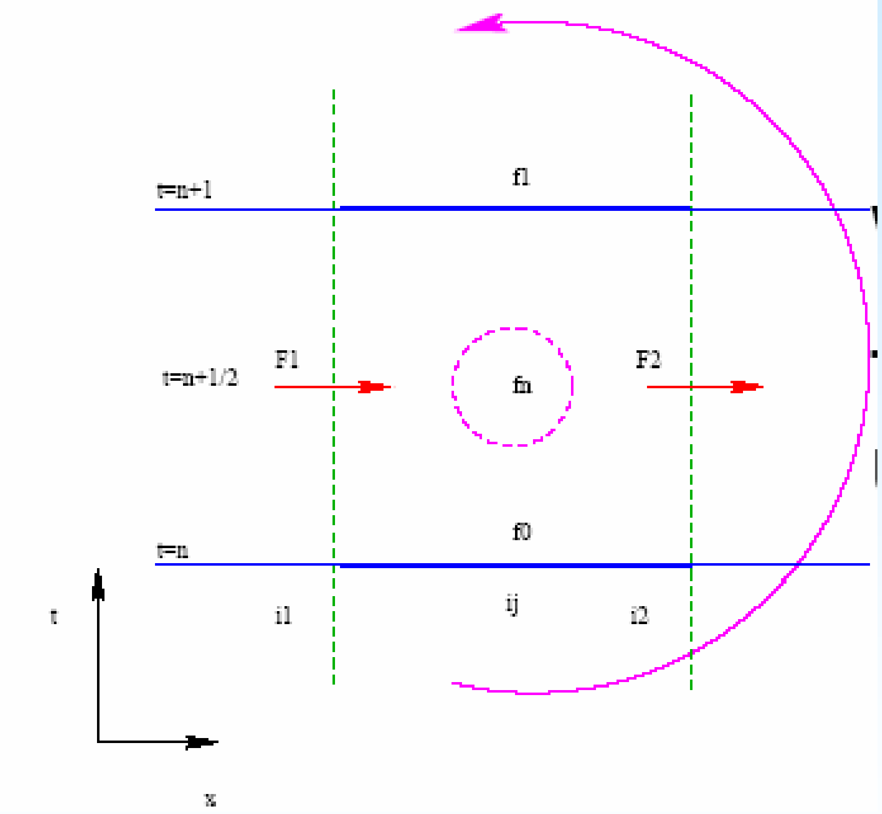
Flux balancing

After turning the volume integrals over H into surface integrals of the fluxes the discrete solution fo t-> n+1 for f can be written as

$$f_{i,j}^{n+1} = f_{i,j}^{n} - \frac{\Delta t}{|V_{i,j}|} \sum_{\beta=1}^{4} G_{i,j,\beta}^{n+\frac{1}{2}}$$

where G are the fluxes through the cell boundaries at t=n+1/2:

$$G_{i,j,\beta}^{n+1/2} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{\Delta S} H(f)\vec{n}_{i,j,\beta} dS dt$$



Since fluxes are balanced-> conservative discretization

Second order flux solution of the Riemann problem and flux limiter

For higher accuracy the fluxes G should be calculated as second order upwind, instead of the usually used for Riemann-problems first order Godunov we suggest

$$G_{i+1/2,j,S} = f_{i+k,j} + \left(\sigma_{i+1/2,j} - \frac{\Delta t}{\Delta x} u^x_{i+1/2,j}\right) H^x_{i+1/2,j} - \frac{\Delta t}{2}\frac{\partial H^v}{\partial v}$$

To avoid oscillations and numerical instability -> introduction, in the in the second order, of a limiter instead of the flux (as in FLT schemes in the Computational Fluid Dynamics (CFD):

$$H^x_{i+1/2,j} = Limiter\left\{Q^C_{i,j}, Q^R_{i,j}, Q^L_{i,j}\right\}$$

e.g.

$$Limiter\left\{Q^C_i, Q^R_i, Q^L_i\right\} = \begin{cases} \min\left\{\frac{1}{2}|Q^C_{i,j}|, 2|Q^L_{i,j}|, 2|Q^R_{i,j}|\right\} & Q^R_{i,j}Q^L_{i,j} > 0 \\ 0 \quad otherwise & \end{cases}$$

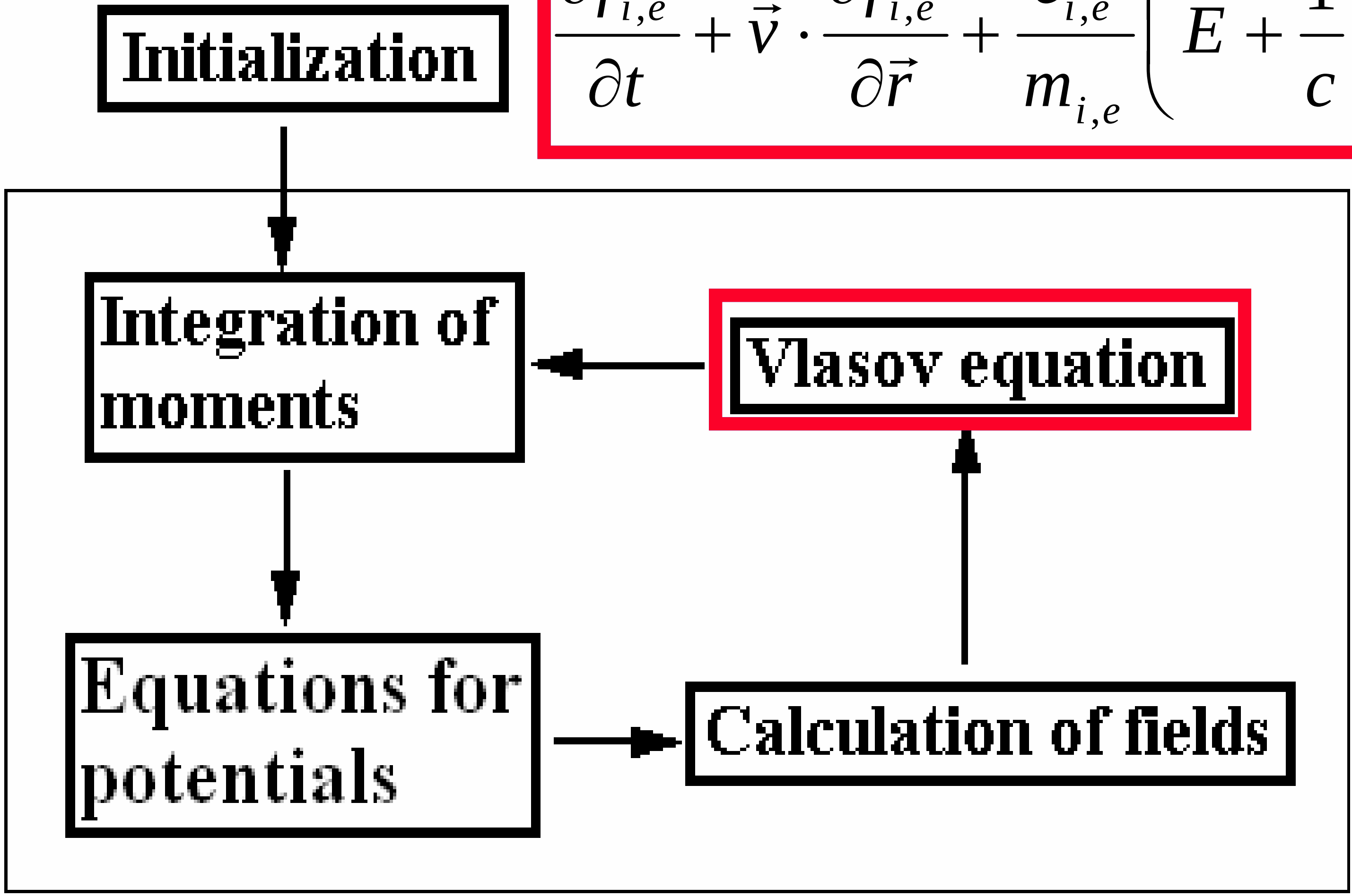
where $Q^C_{i,j} = f_{i+1,j} - f_{i-1,j}, \quad Q^R_{i,j} = f_{i+1,j} - f_{i,j} \quad Q^L_{i,j} = f_{i,j} - f_{i-1,j}$

2nd order upwind unsplit finite volume Vlasov solver with a flux limiter

1. Discretization of the integral form of the Vlasov equation
   -> The conservation laws are satisfied
2. Fluxes are calculated only inside the domain of dependence (upwind scheme)
   -> The causality principle is automatically fulfilled
3. No splitting necessary
   -> Isotropy and symmetry
4. Maximum principle for second-order fluxes (flux limiting)
   -> Numerically caused entropy growth: very slow

From Vlasov solver to code

$$\frac{\partial f_{i,e}}{\partial t} + \vec{v} \cdot \frac{\partial f_{i,e}}{\partial \vec{r}} + \frac{e_{i,e}}{m_{i,e}} \left(\vec{E} + \frac{1}{c} (\vec{v} \times \vec{B}) \right) \cdot \frac{\partial f_{i,e}}{\partial \vec{v}} = 0$$



Simulation cycle

Specifics of Vlasov codes

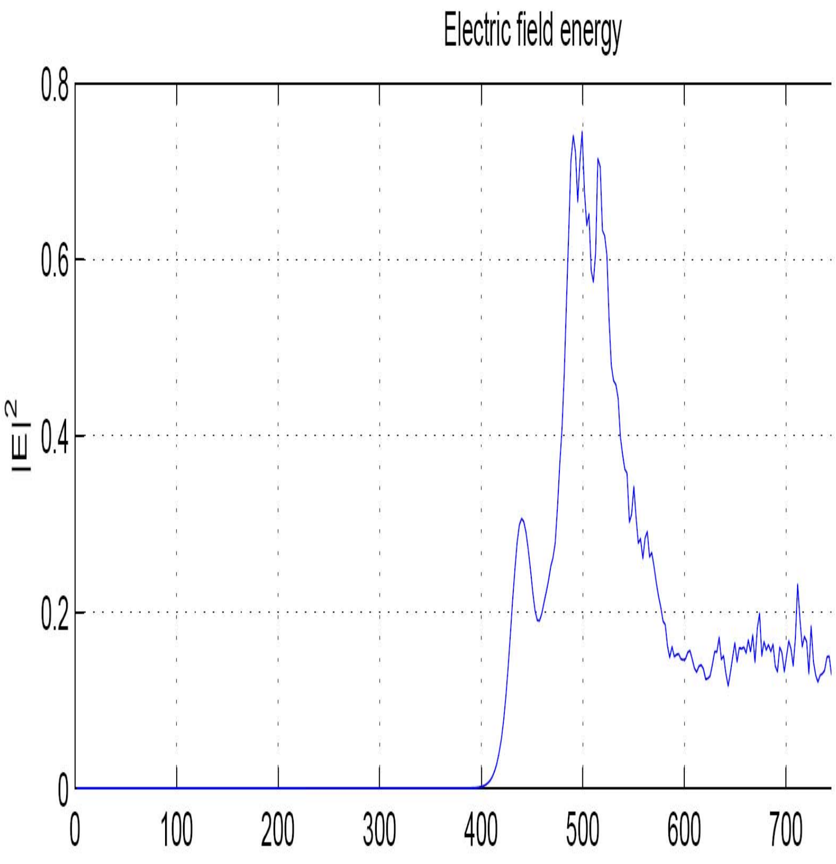
- **Boundary conditions:**
  - **also needed for distribution function / in velocity space, e.g. f=const.**
- **Initial conditions:**
  - **Since Vlasov solvers are noiseless**

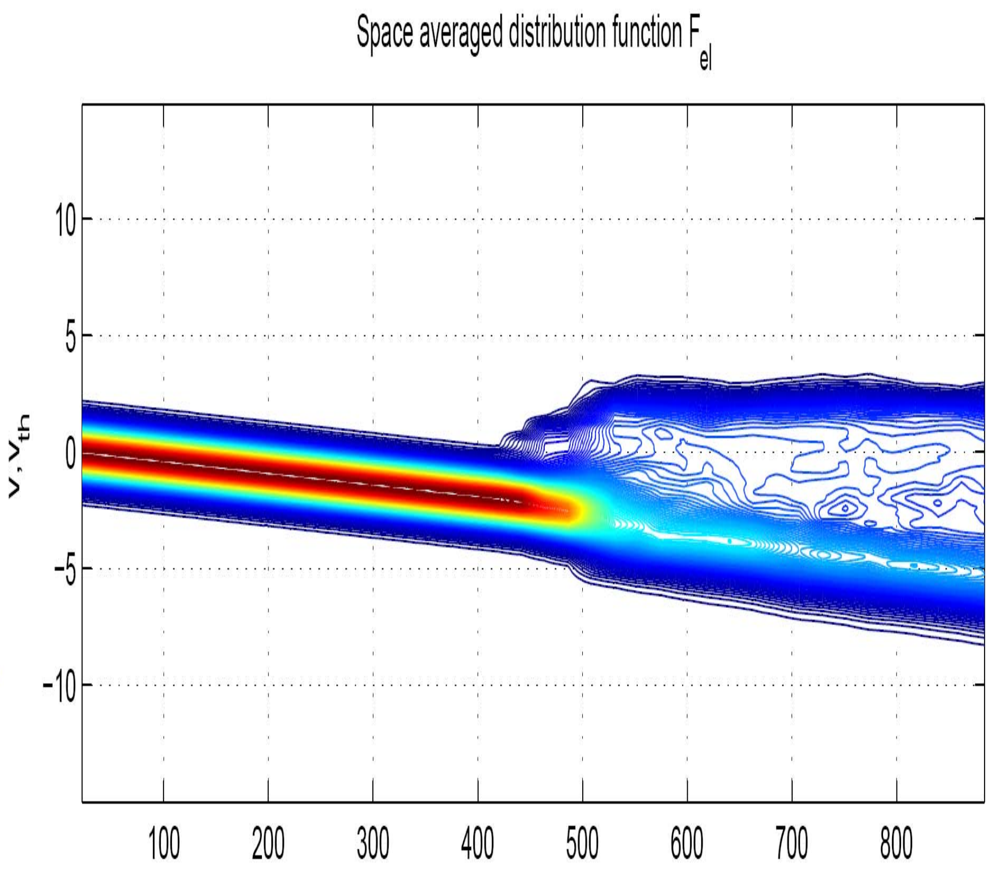
  -> In initial value problems like instability analyses one needs to add noise, e.g.

  (a) of the distribution functions

  (b) of the electromagnetic fields

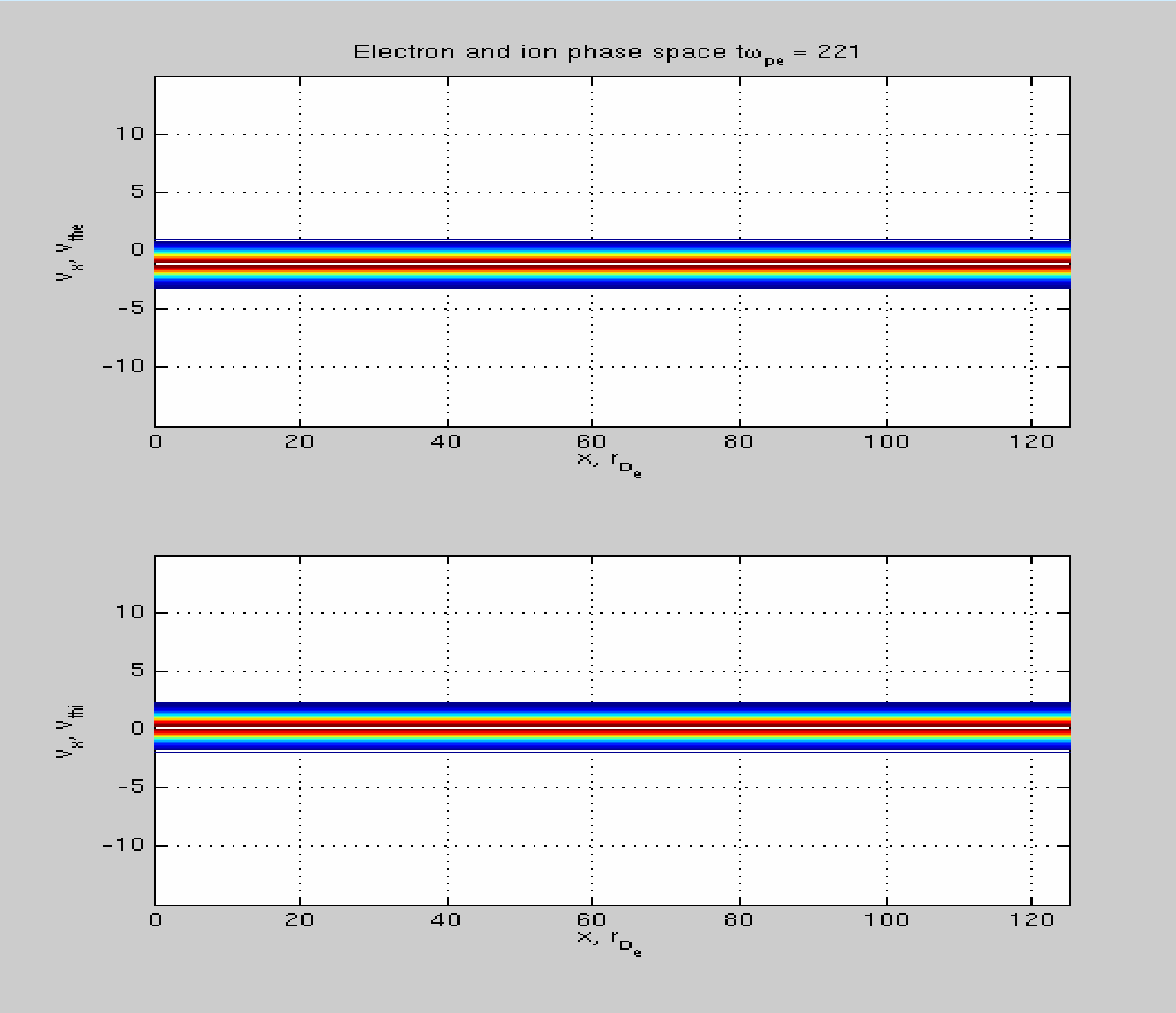
Example 1: Buneman instability





- Te = 3 Ti

Phase space evolution



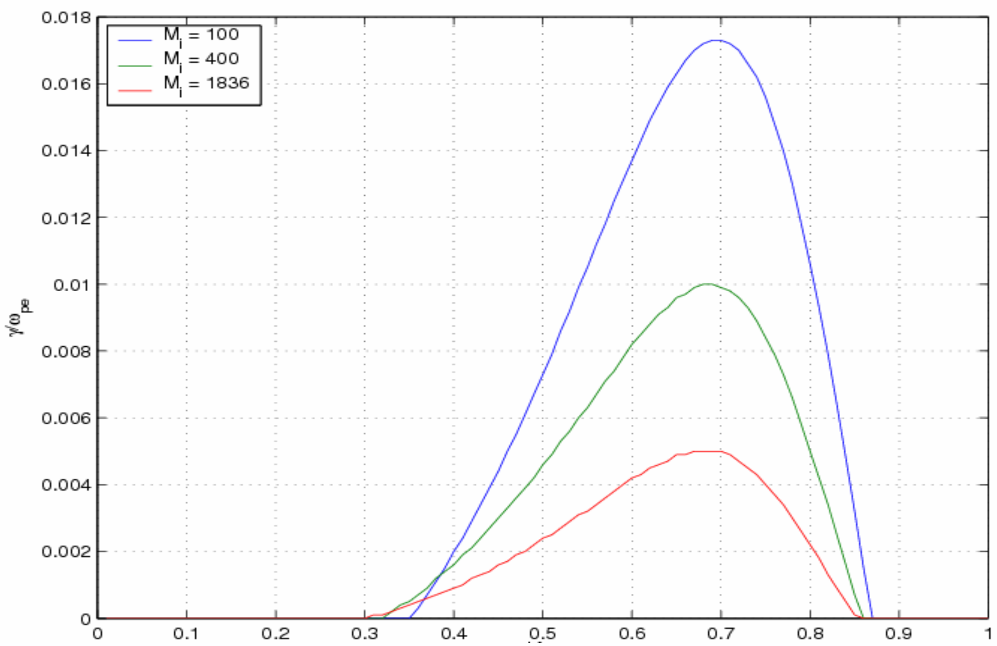
- electrons
- ions

2. Linear instability for Ti<Te

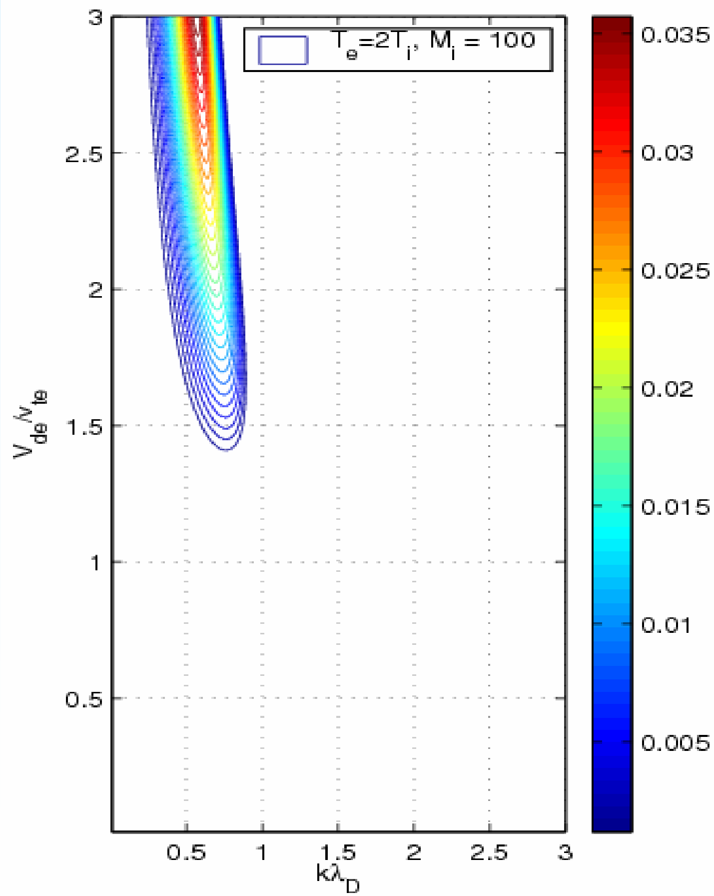
$$f_i = \sqrt{\frac{M}{2\pi T_i}} \exp\left(-\frac{M_i}{2}\frac{v_i^2}{v_{thi}^2}\right)$$

$$f_e = (1 + a(x))\sqrt{\frac{m_e}{2\pi T_e}} \exp\left(-\frac{(v_e - v_{de})^2}{v_{the}^2}\right)$$

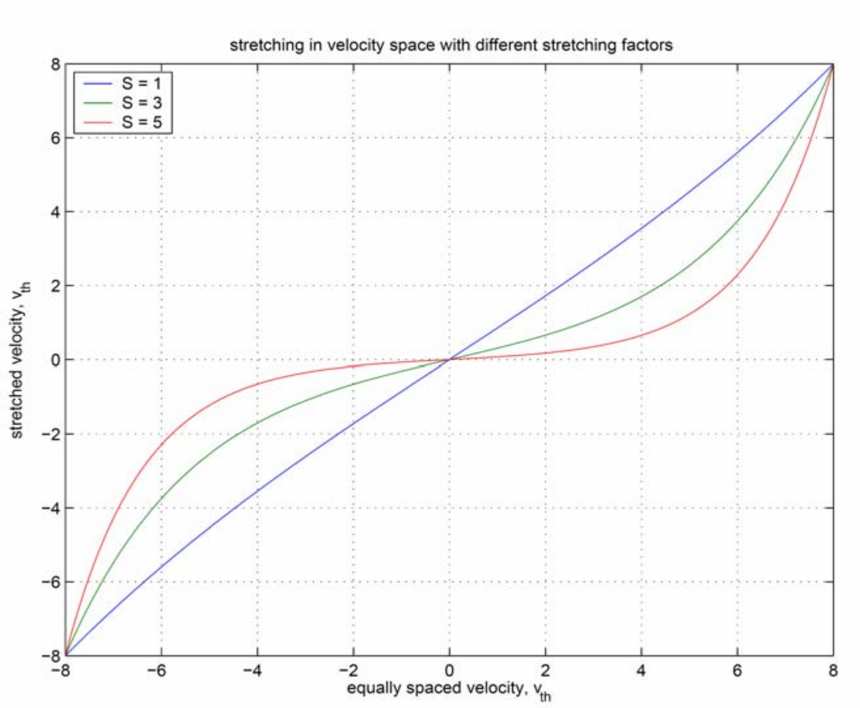
a(x) is noise
Vde the drift

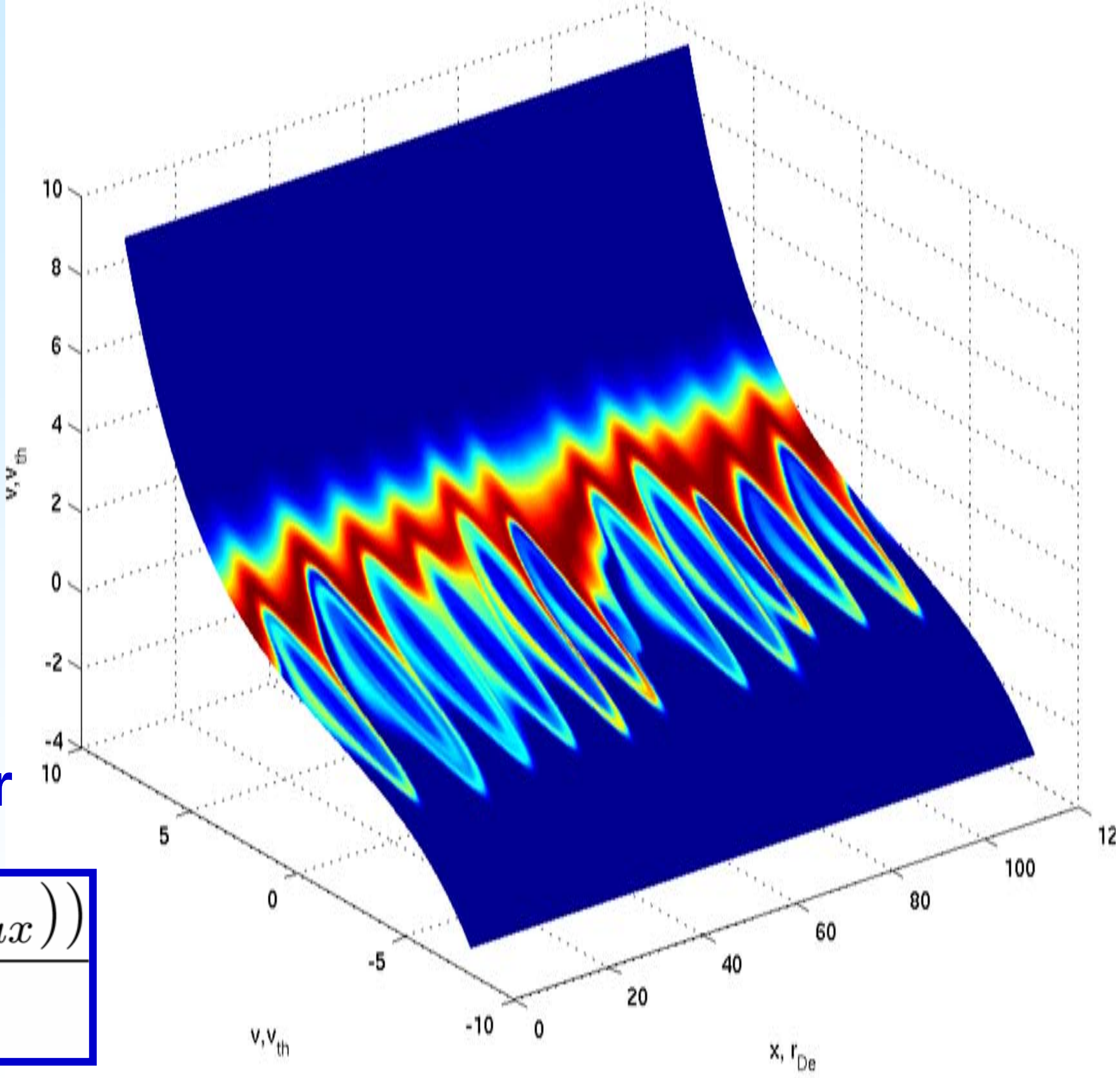


Linear dispersion for Te = 2Ti (not, as usual in theory, Te>> Ti)



Stretched grid in velocity space



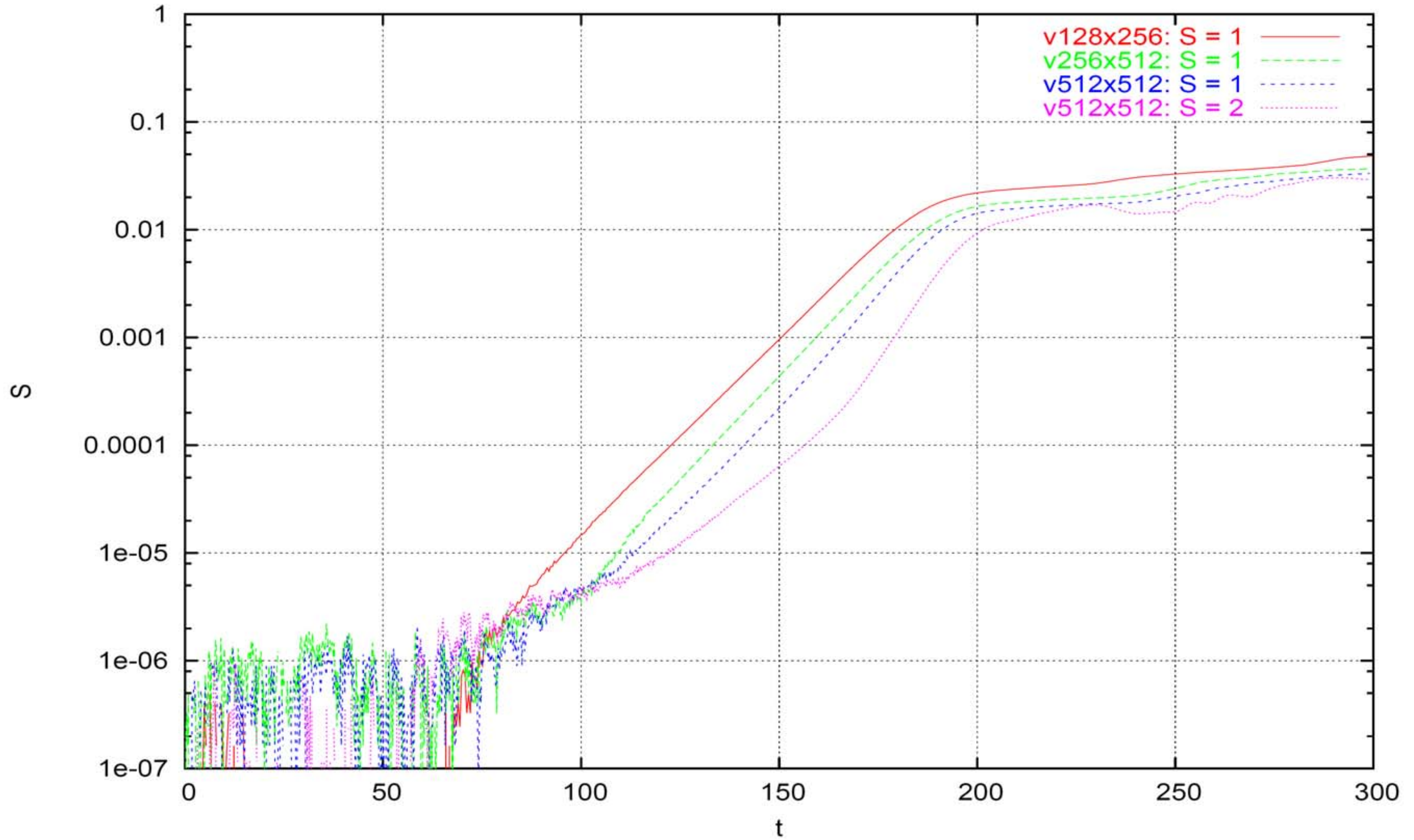


where Str = stretching factor

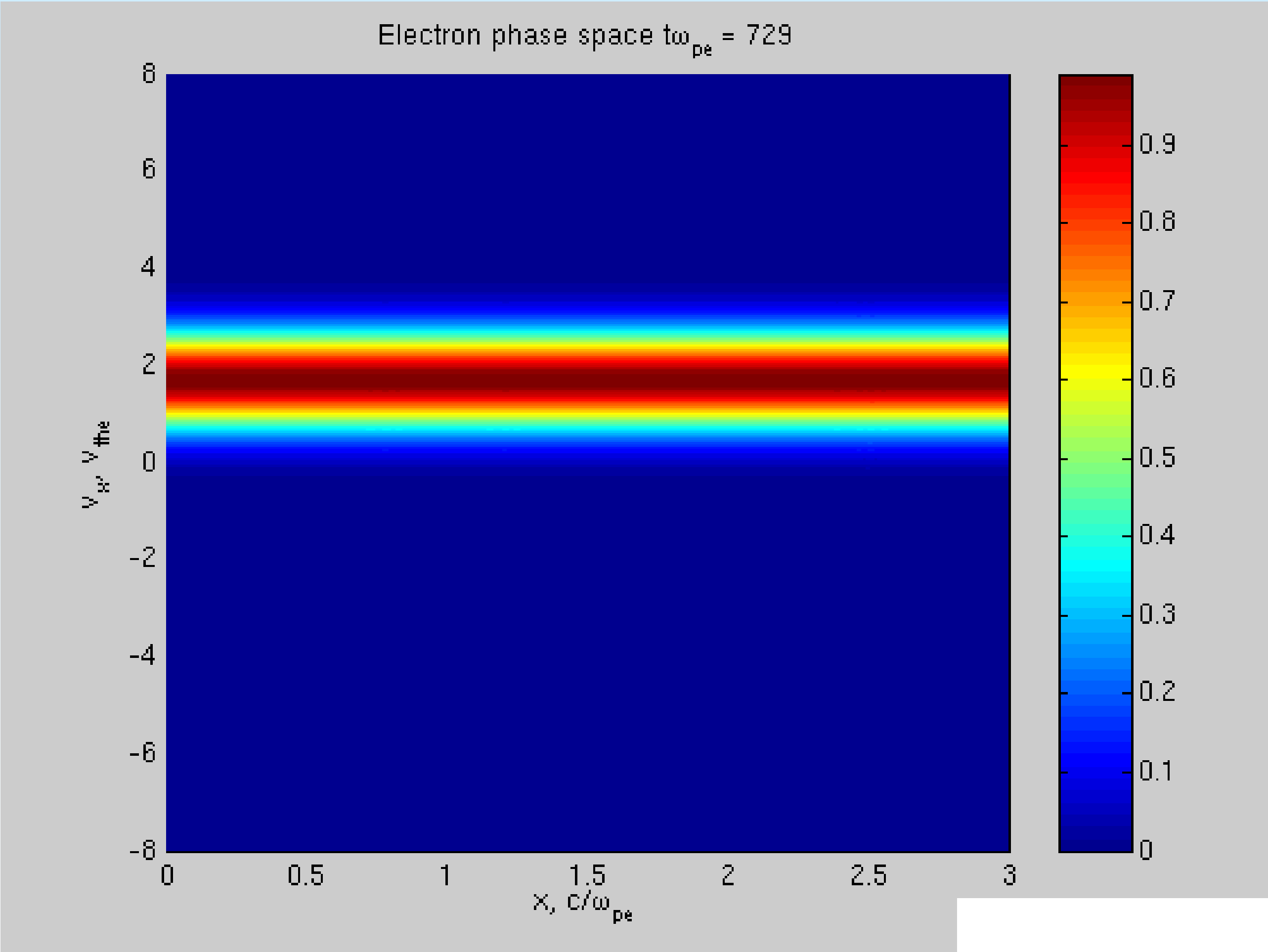
$$V_i^{str} = v^{max} \frac{\sinh\left(v_i(Str/v_{max})\right)}{\sinh(Str)}$$

Grid size & numerical entropy

$$S(t) = -\int f \ln f dv dx \quad S_{rel} = [S(t) - S(0)]S(0)$$



Ion-acoustic instability



Mi/me =1800
Ti = 0.5 Te
Vde= 0.7 Vthe
Vthe(max)
= +- 8 Vte

$$f_i = \sqrt{\frac{M}{2\pi T_i}} \exp\left(-\frac{M_i}{2}\frac{v_i^2}{v_{thi}^2}\right)$$

Vx vs. X

$$f_e = (1 + a(x))\sqrt{\frac{m_e}{2\pi T_e}} \exp\left(-\frac{(v_e - v_{de})^2}{v_{the}^2}\right)$$

„Anomalous“ / effective collision rate

The ensemble averaging of the Vlasov equation for $f_j = f_{0j} + \delta f_j.$

with $\langle \delta f_j \rangle = \left\langle \delta \vec{E} \right\rangle = \left\langle \delta \vec{B} \right\rangle = 0.$

reveals

$$\frac{\partial f_{0j}}{\partial t} + \vec{v} \cdot \frac{\partial f_{0j}}{\partial \vec{r}} + \frac{e_j}{c m_j} \left(\vec{v} \times \vec{B} \right) \cdot \frac{\partial f_{0j}}{\partial \vec{v}} = \left(\frac{\partial f_j}{\partial t} \right)_{an}$$
$$= -\frac{e_j}{m_j} \left\langle \left(\delta \vec{E} + \frac{\vec{v} \times \delta \vec{B}}{c} \right) \cdot \frac{\partial \delta f_j}{\partial \vec{v}} \right\rangle .$$

and

$$\left(\frac{\partial}{\partial t} n_j m_j v_{y,j} \right)_{an} = \left\langle \delta E_y \delta \rho_j + \frac{\delta j_{z,j} \delta B_x - \delta j_{x,j} \delta B_z}{c} \right\rangle .$$

$$\nu_{eff,j} = \frac{1}{\langle n_j m_j v_{y,j} \rangle} \left(\frac{\partial}{\partial t} n_j m_j v_{y,j} \right)_{an} .$$

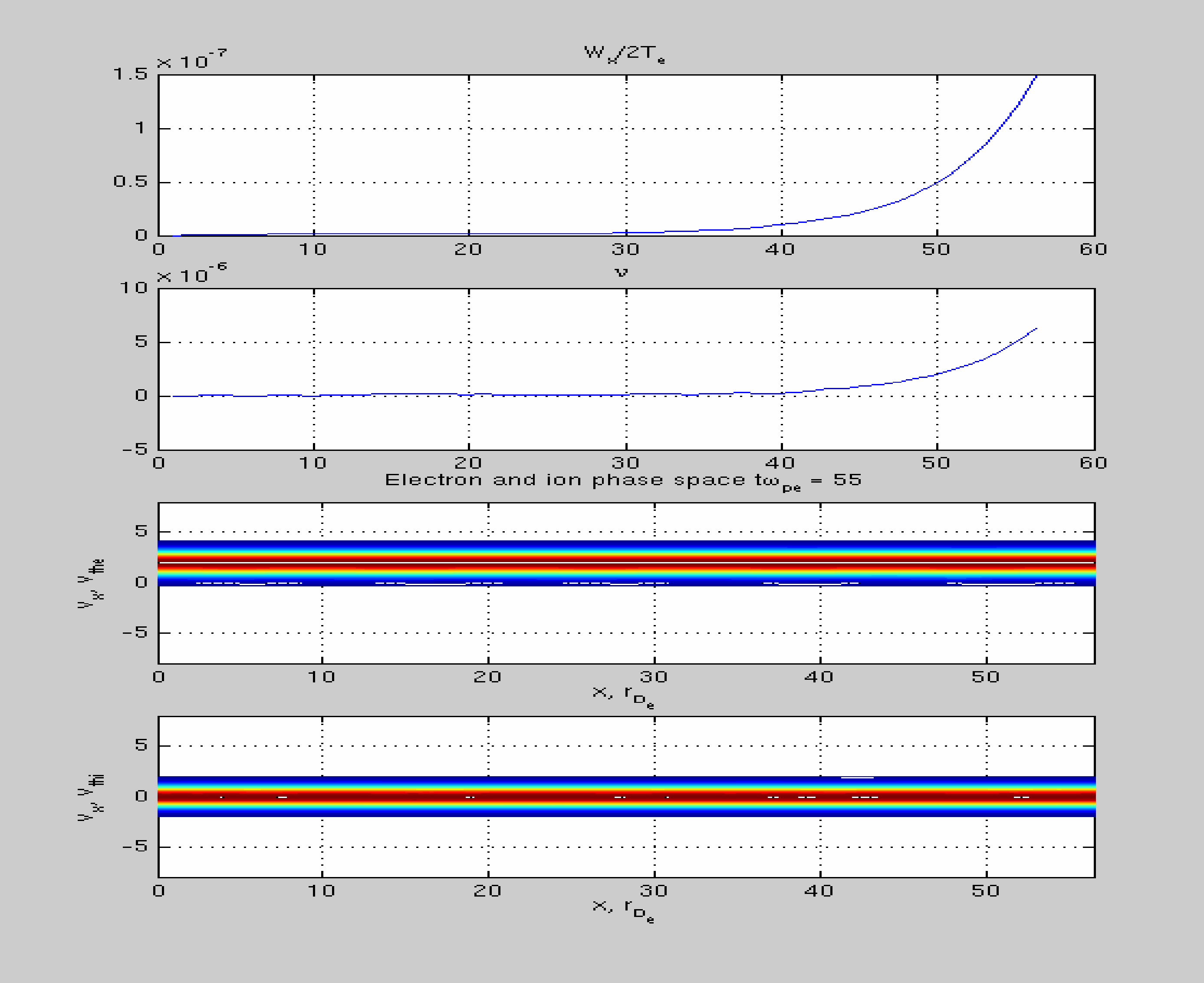
In a simulation one can directly determine the momentum exchange rate

$$\nu(t + \delta t/2) = \frac{2}{\delta t} \frac{p(t + \delta t) - p(t)}{p(t + \delta t) + p(t)}$$

Often used is a theoretical estimate of the anomalous collision frequency based on waves and their dispersion (quasilinear approach):

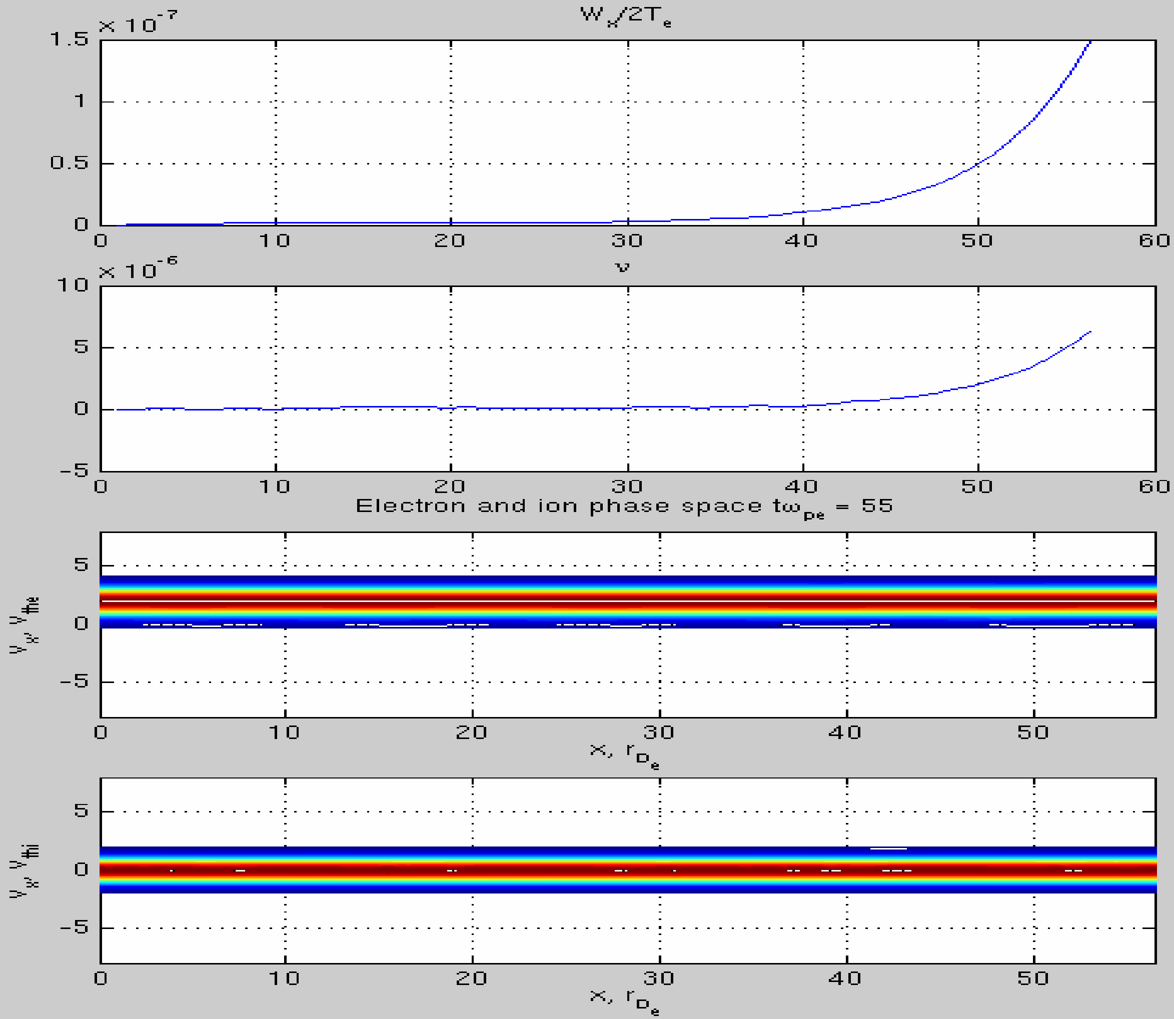
$$\nu = \sum_k \frac{\Delta k |\delta E(k)|^2 \omega_{pe}}{k v_{te}^2 m_e n v_d} Im \xi_e Z(\xi_e)$$

Waves, „collisions“,distributions



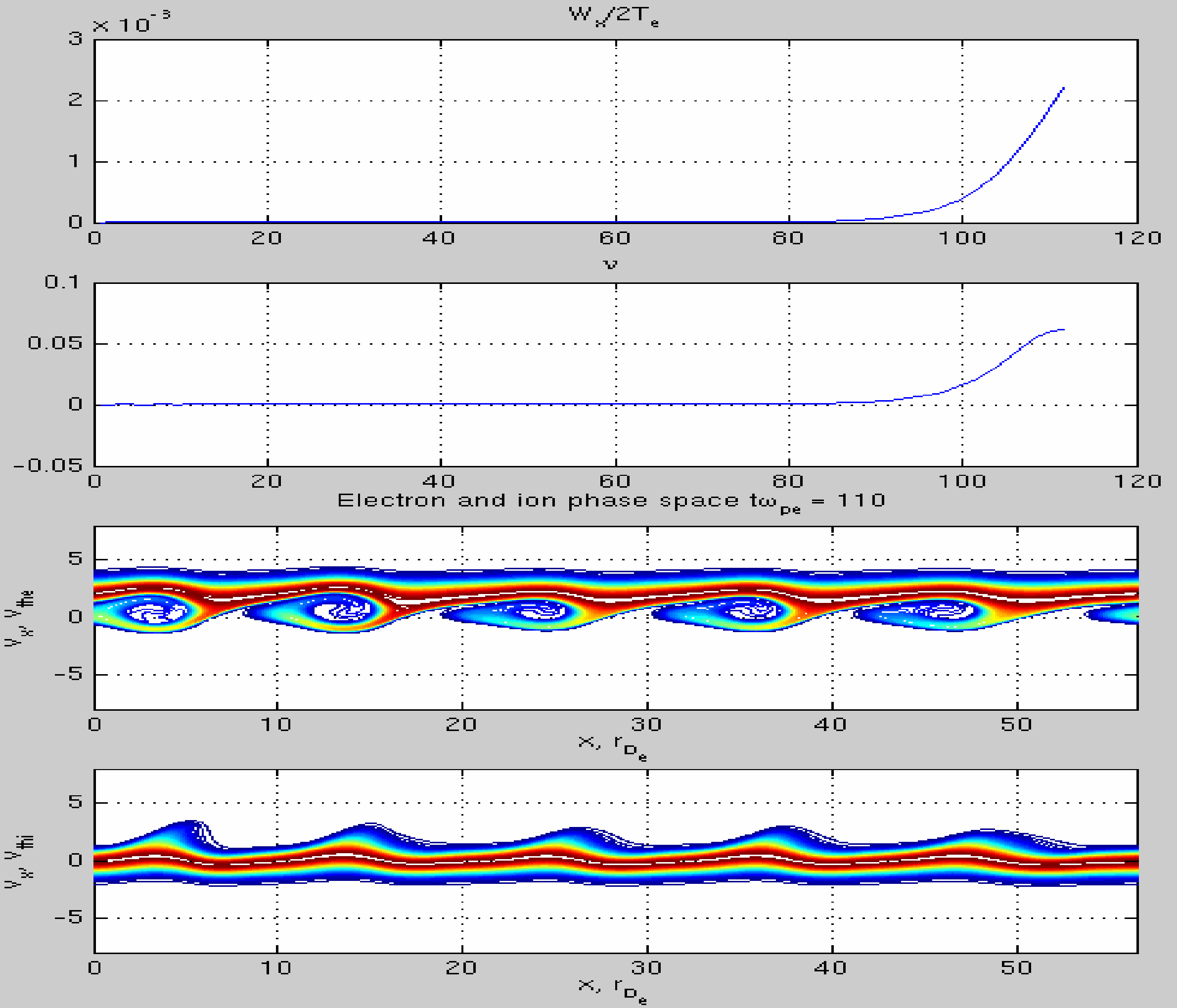
- **wave energy**
- **anomalous collision rate**
- **v <-> x electrons**
- **v <-> x ions**

Linear instability growth



- ← **wave energy starts to grow**
- ← **effective, i.e. collisionless "collision rate"**
- ← **f(Ve) <-> X (electron distribution function)**
- ← **f(Vi) <-> X (ion distribution function)**

Saturation by trapping



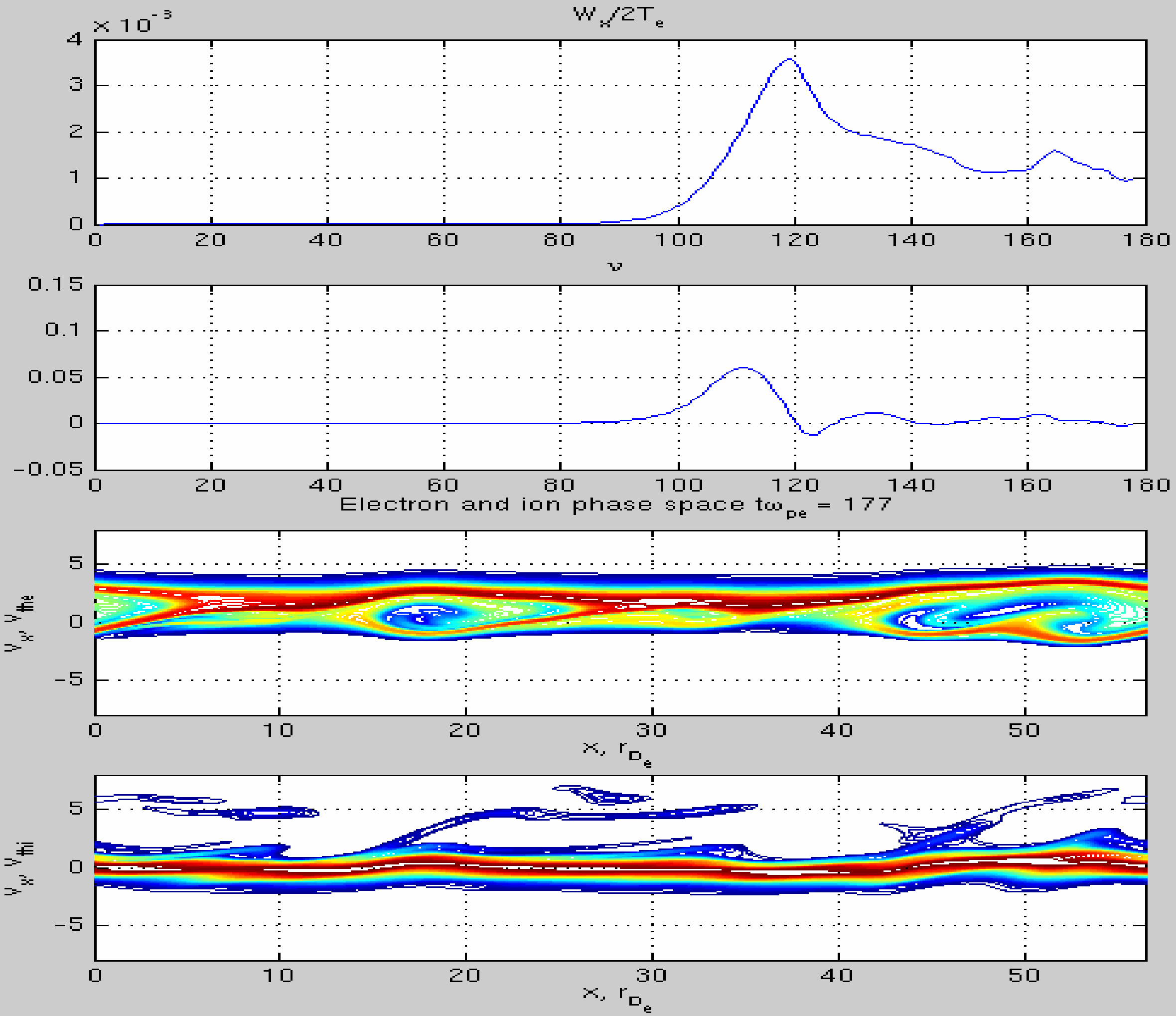
- ← wave energy at its maximum:
- ← E^2 = 0.006 n T
- ← the maximum effective "collision rate" nu = 0.05 ω_{pe}

is still close to Sagdeev's prediction:

$$D = \left(\frac{c}{\omega_e}\right)^2 \omega_e \frac{\epsilon_0 \delta E^2}{2nT}$$

$$D^{th} = 10^{-2}\left(\frac{c}{\omega_e}\right)^2 \omega_i \left(\frac{v_d}{c_s}\right)\left(\frac{T_e}{T_i}\right)$$

Strongly nonlinear islands



- wave energy decreased
- low "collision rate"

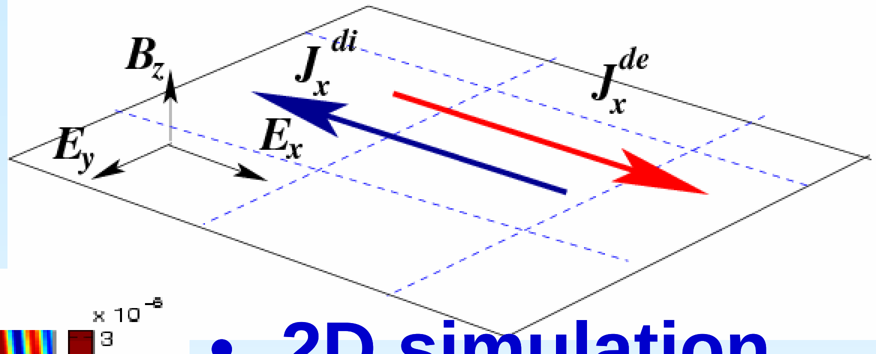
  the electron current is reduced, the free energy is exhausted, islands
- cannot grow any further
- ions heated, saturation at low quasilinear level

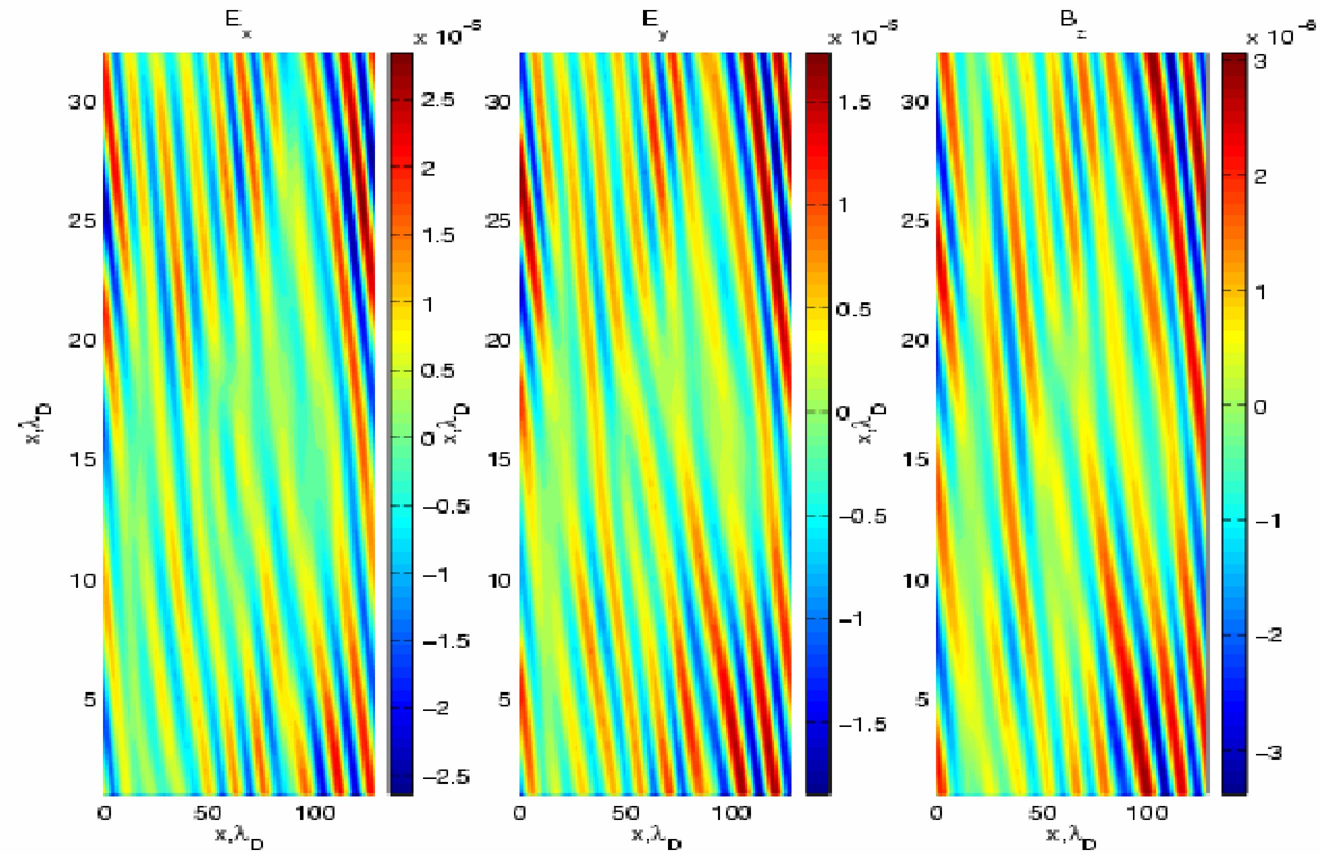
3. Example 2D IA instability

- **Setup:**

$$f_i = \sqrt{\frac{M}{2\pi T_i}} \exp\left(-\frac{M_i}{2}\frac{v_i^2}{v_{thi}^2}\right)$$

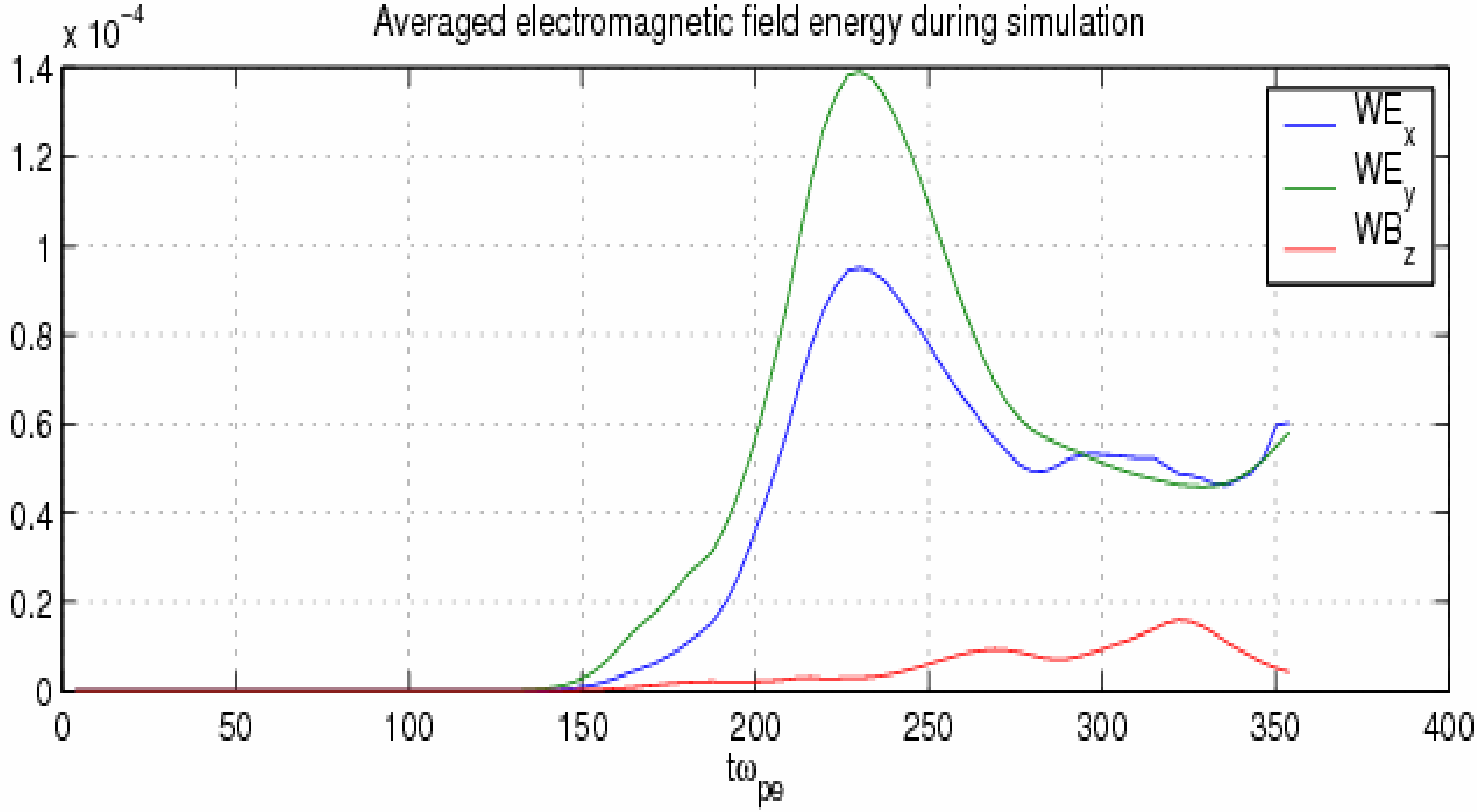
$$f_e = (1 + a(x))\sqrt{\frac{m_e}{2\pi T_e}} \exp\left(-\frac{(v_e - v_{de})^2}{v_{the}^2}\right)$$



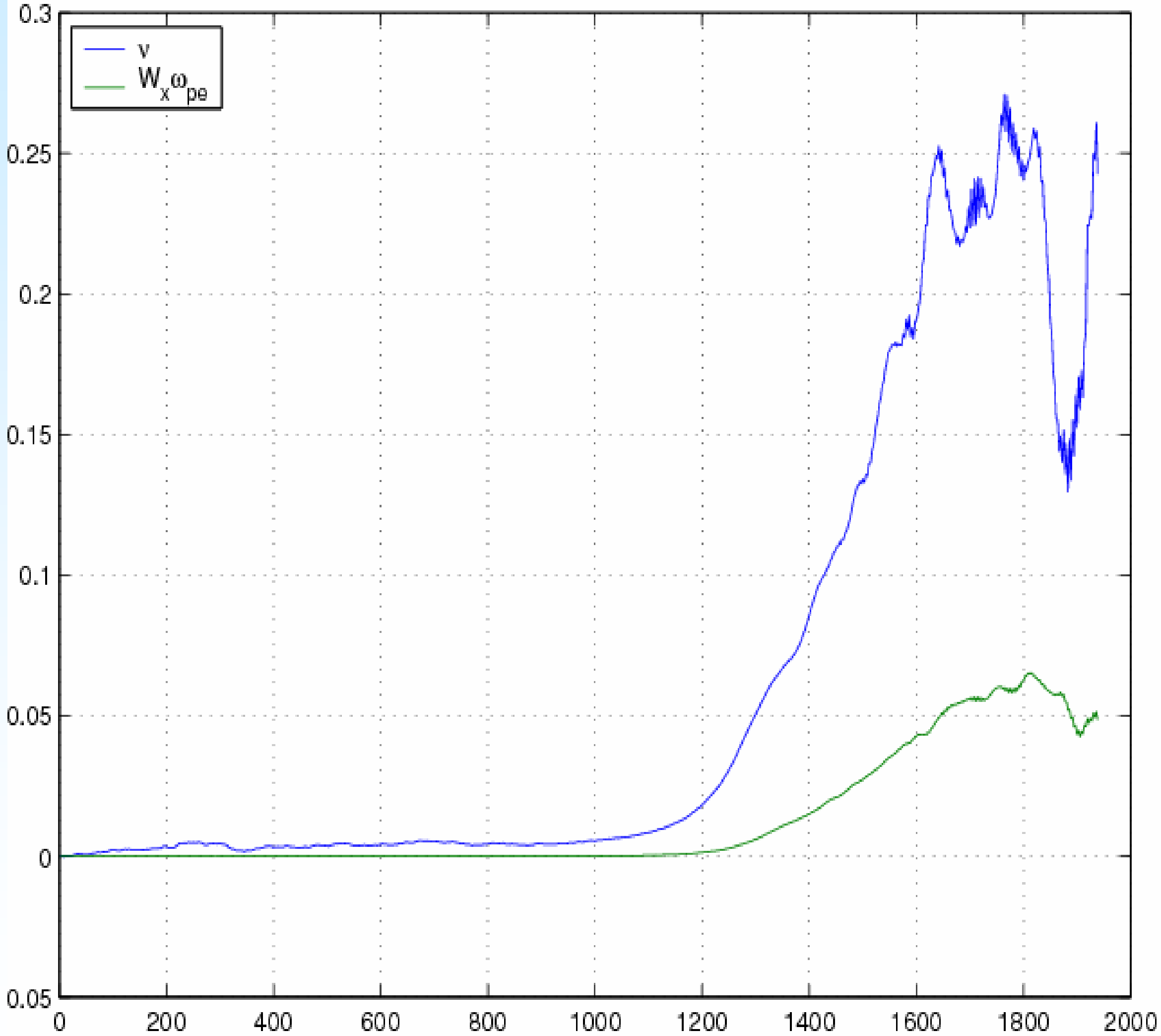


- **2D simulation result:**
- **tω$_{pe}$= 2000**
- **Oblique wave propagation:**
- **angle: 45°**
- **components: Ex,Ey,Bz**

2D wave power



Effective collision frequency



← Blue: momentum exchange rate (simulation result):

$$\nu(t+\delta t/2) = \frac{2}{\delta t}\frac{p(t+\delta t)-p(t)}{p(t+\delta t)+p(t)}$$

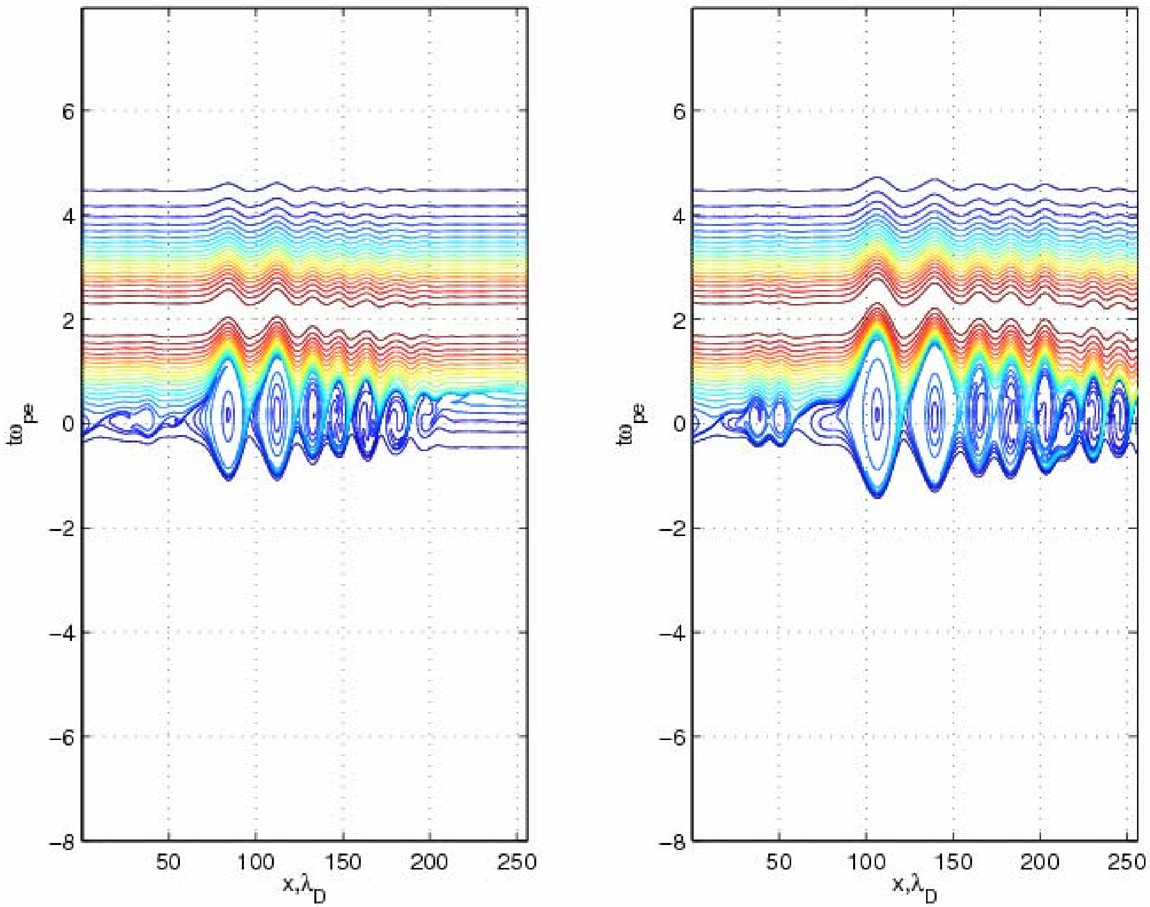
← green:theoretical estimate for E^2

$$\nu = \sum_k \frac{\Delta k |\delta E(k)|^2 \omega_{pe}}{k v_{te}^2 m_e n v_d} Im \xi_e Z(\xi_e)$$

„Sagdeev-formula“:

$$D = (\frac{c}{\omega_e})^2 \omega_e \frac{\epsilon_0 \delta E^2}{2nT}$$

4. Example external current - J=const. – and open boundary:



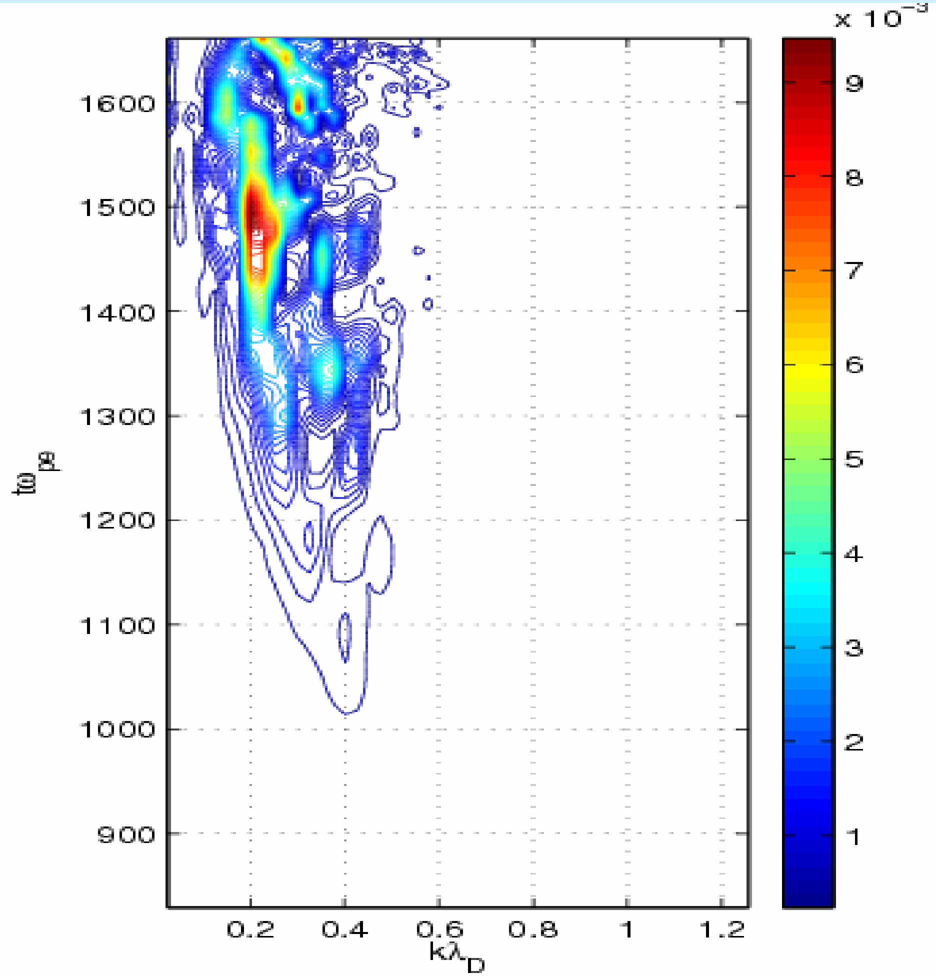
t ω_{pe}= 1940 t ω_{pe} = 2100

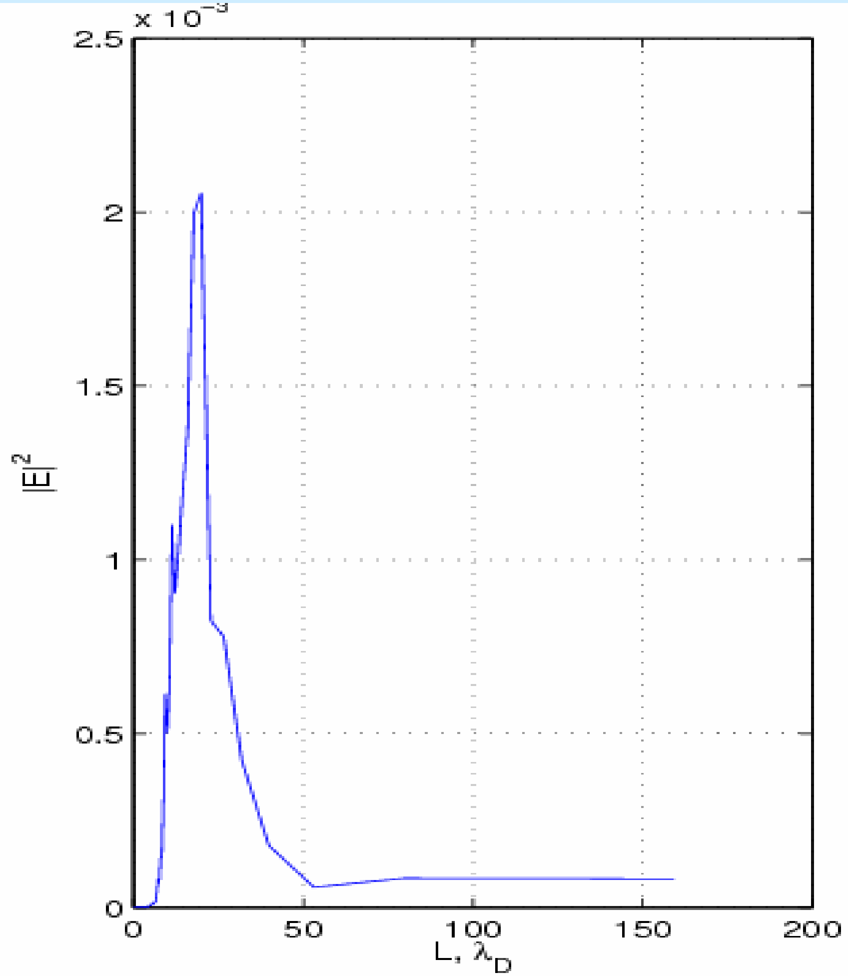
In this more realistic situation the electron distribution function continues to evolve, since free energy is continously supplied

=> transition to a

a strong nonlinear regime, formation of density holes

Width of the density holes

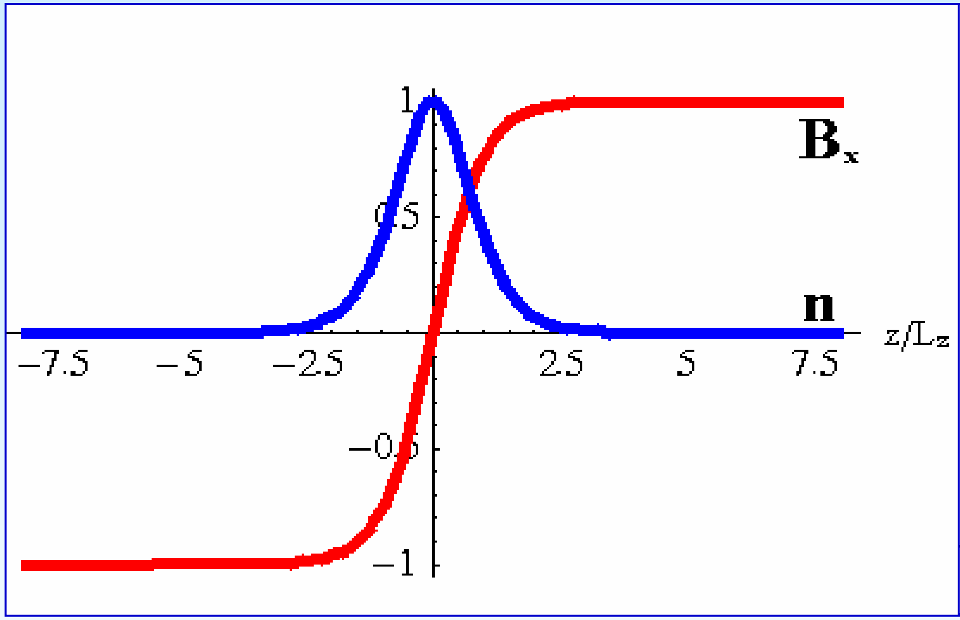


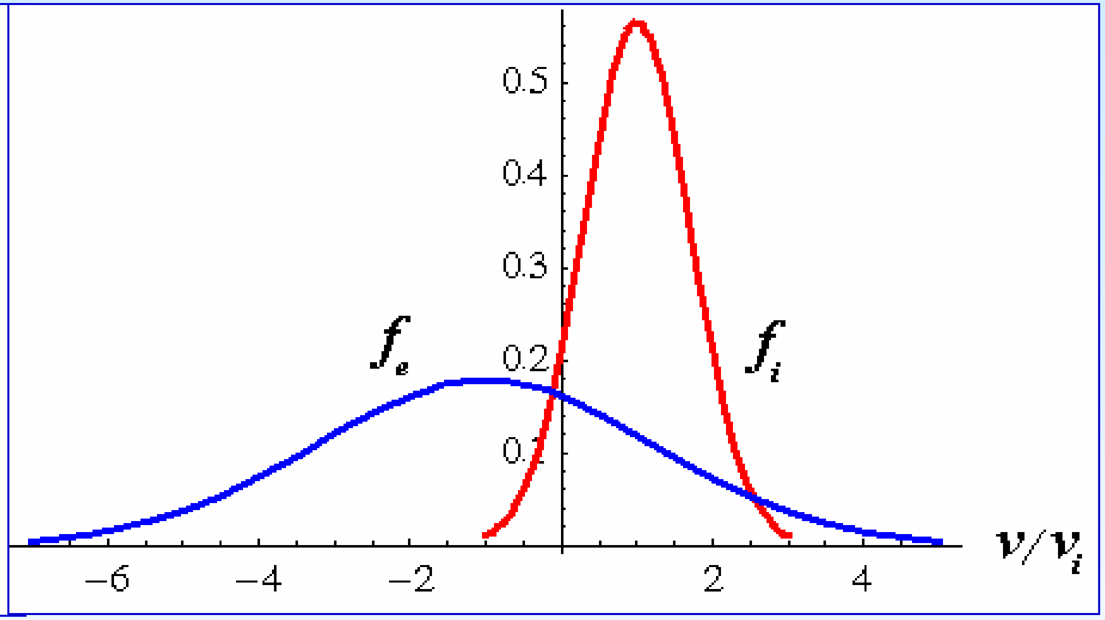


a) evolution of the E-field power spectra

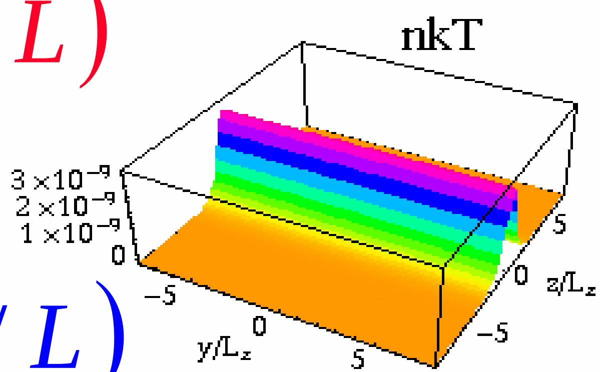
b) spectrum of hole sizes measured in E-field strength

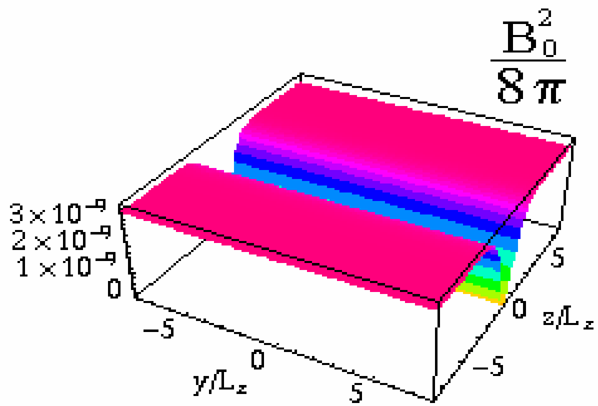
5. Example: Current sheets





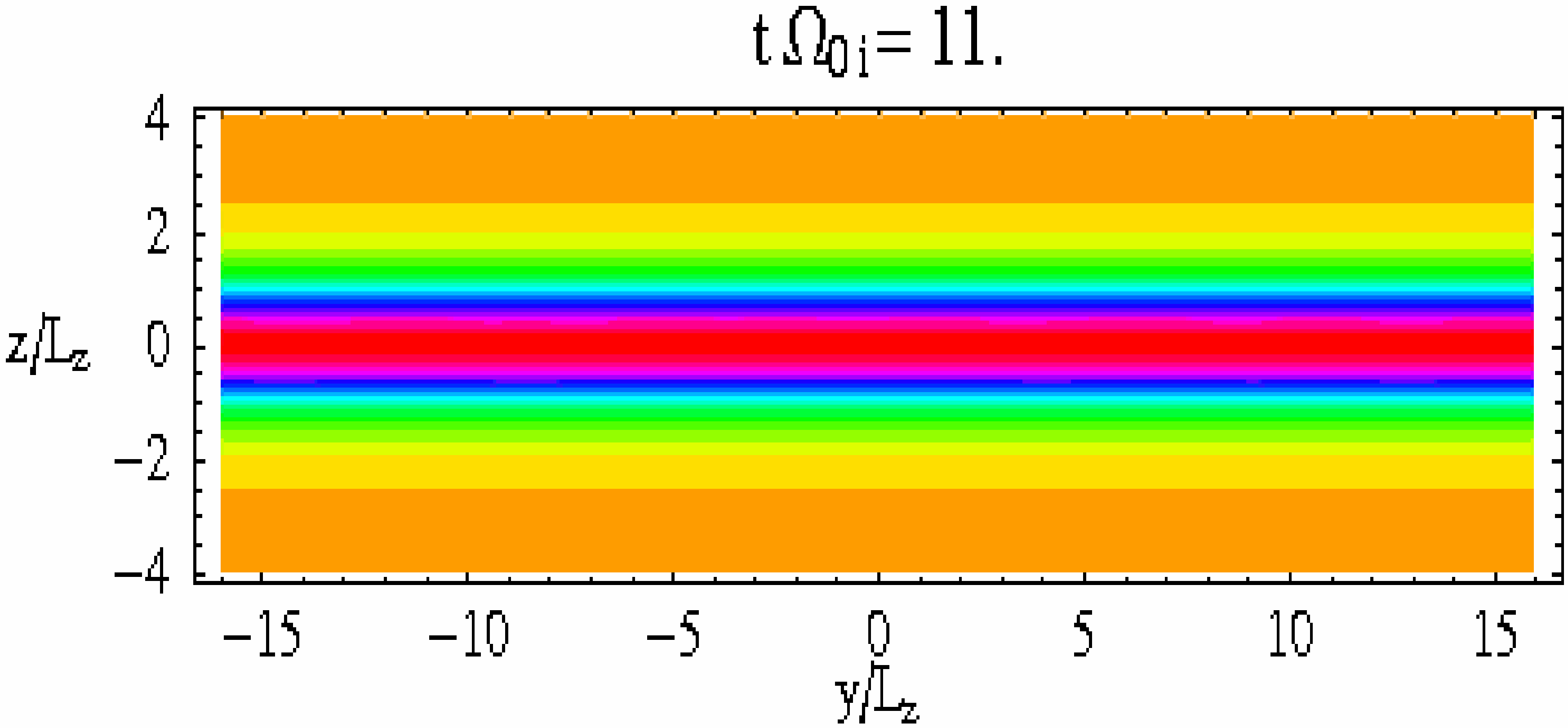
$$B_x = B_0 \tanh\left(z / L\right)$$





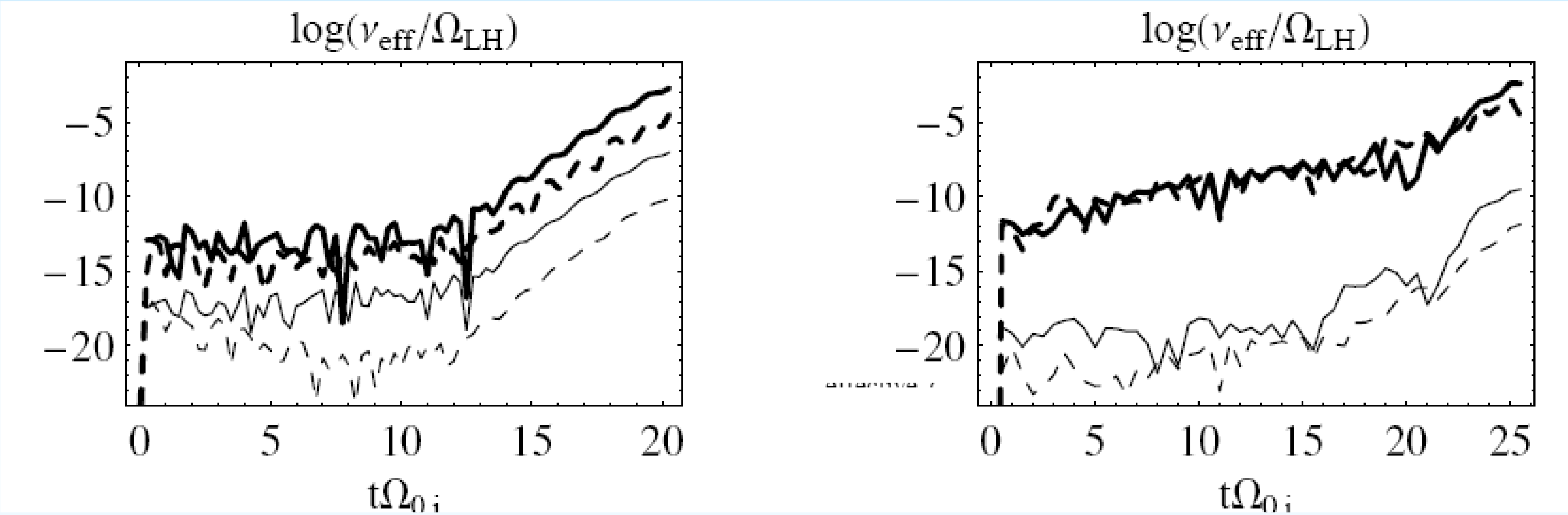
$$n = n_0 \cosh^{-2}\left(z / L\right)$$

Gradient driven lower hybrid drift instability -> drift-kink/sausage



Simulated effective collision rates

M/m = 25 M/m =100



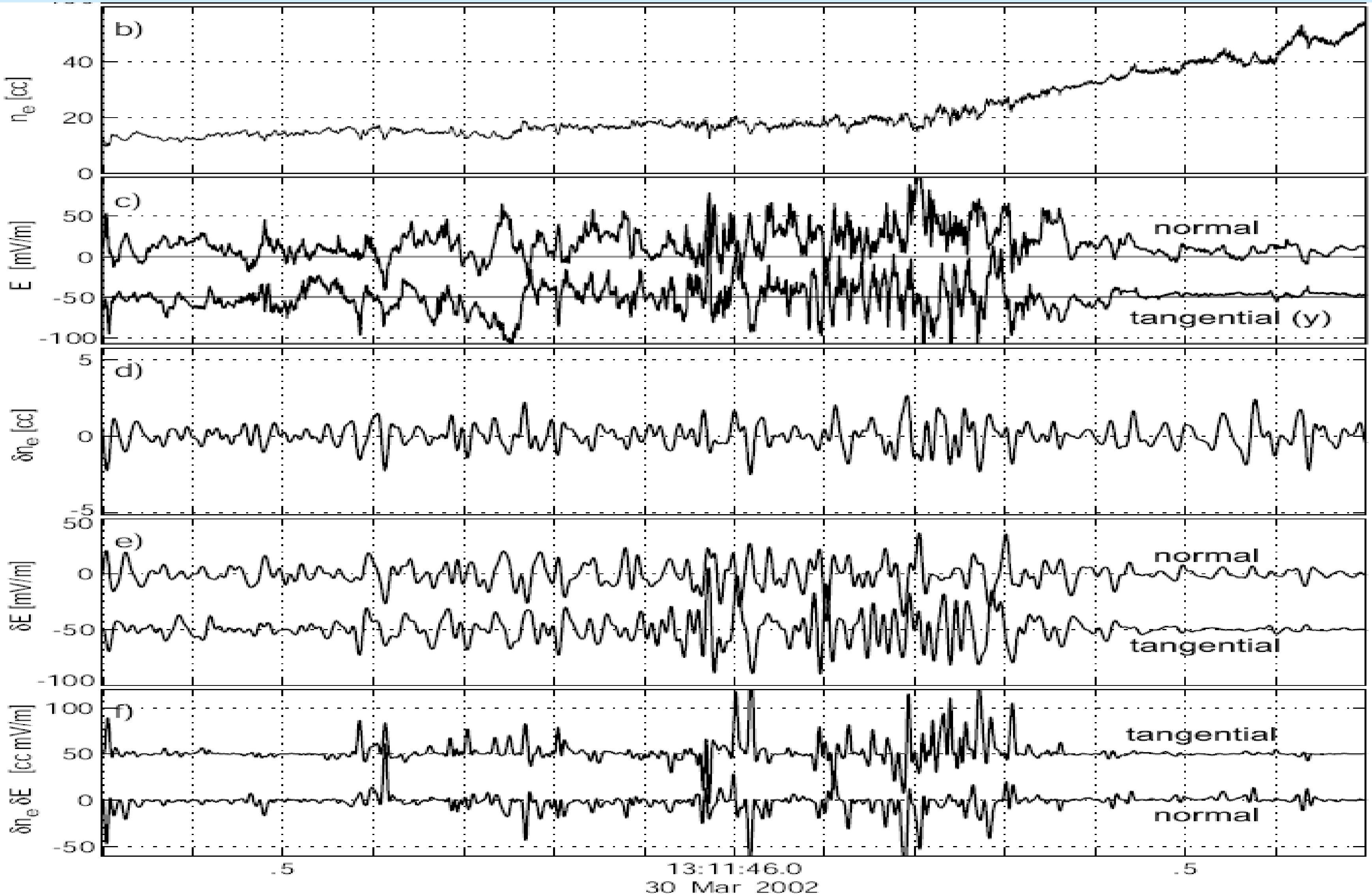
Here $\delta\rho\delta E_y$ and $\delta j \times \delta B$ are shown by solid and dashed lines
thick lines = electron contribution, thin lines = ion contribution

The simulated nu-level and the one measured by CLUSTER (next slides) match well. Both exceed the Huba and Papadopolus (1978) quasilinear estimate by a factor of about 100 ->

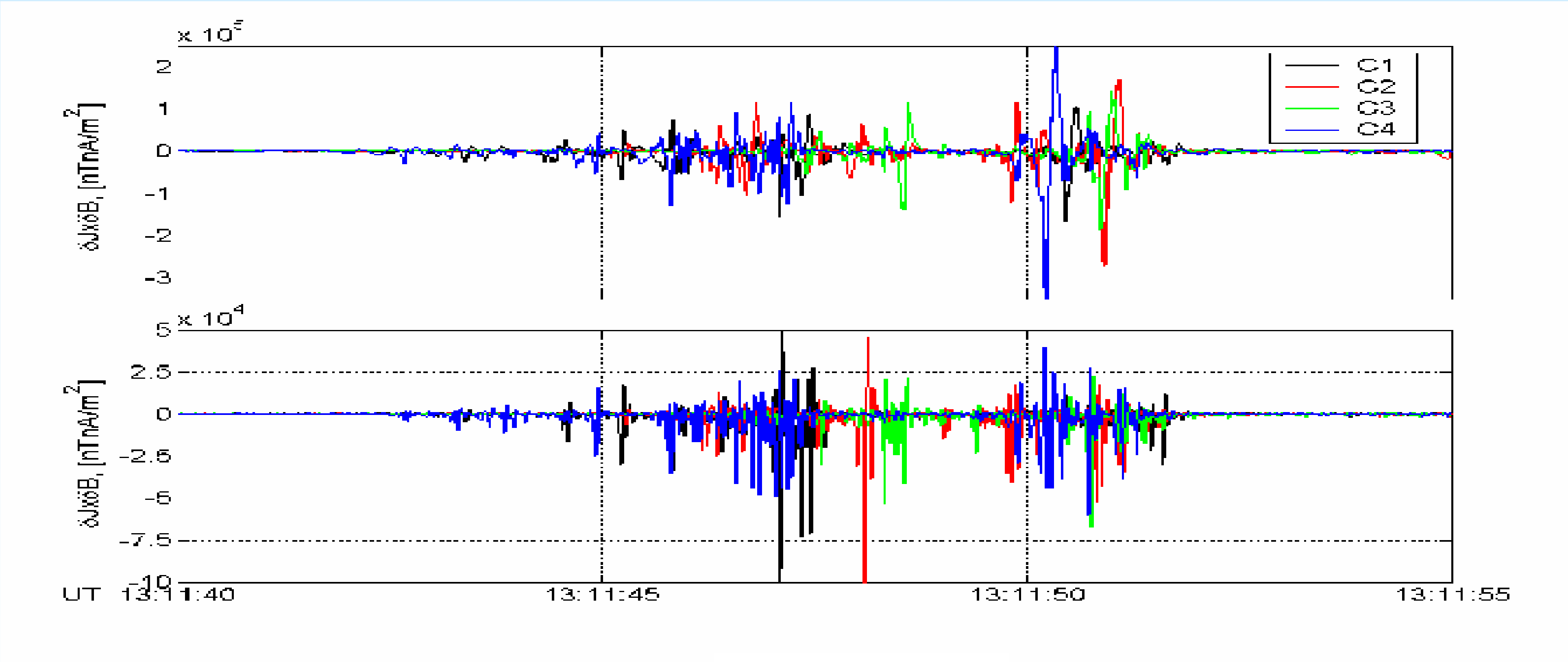
$$D = (\frac{\pi}{8})^{\frac{1}{2}}(\frac{m_i}{m_e})(\frac{T_i}{T_e})\rho_e^2\omega_{\rm LH}\frac{\epsilon_0\delta E^2}{2nT_i}$$

$$D^{th} = 0.5\rho_e^2(\frac{m_i}{m_e})(\frac{v_d}{v_e})^2\omega_{\rm LH}$$

CLUSTER:n-E correlation



+CLUSTER: j - B correlation:



Using ... one obtains ->

$$D_{an} = \frac{\nu_{an}}{\epsilon_0 \mu_0 \omega_{pe}}$$

diffusion: 10^9 m^2/s, explains the boundary layer

$$\nu_{an} = \frac{1}{n_e m_e V}(e\langle \delta n \delta \mathbf{E}\rangle + \langle[\delta \mathbf{j} \times \delta \mathbf{B}]\rangle) \quad \eta_{an} = m_e \nu_{eff}/(n_e e^2) \quad \eta_{an} \sim 10^4 \Omega m$$

Summary

- The consequences of wave-particle interactions in collisionless plasmas like particle transport and scattering, are well described by the Vlasov (1938) equation, despite of Landau's criticism!
- Currently numerical simulations have shown:
  - Quasilinear theory is okay for weak 1D waves
  - But the ql theory must be replaced in cases of
    - higher than 1D problems (current sheets...)
    - strongly excited waves and turbulence,
    - non-local instabilities and
    - phase space structure formation (holes...).